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Deissler Rank Complexity of Powers of Indecomposable Injective Modules

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Abstract Minimality ranks in the style of Deissler are one way of measuring the structural complexity of minimal extensions of first-order structures. In particular, *positive Deissler rank* measures the complexity of the injective envelope of a module as an extension of that module. In this paper we solve a problem of the second author by showing that certain injective envelopes have the maximum possible positive Deissler rank complexity. The proof shows that this complexity naturally reflects the internal structure of the injective extension in the form of the levels of the Matlis hierarchy.

In this paper we present a general and positive solution to a problem raised in Kucera [4]. The problem concerns the structural complexity of injective modules over a commutative Noetherian ring Λ. An injective module is essentially one wherein every formally consistent system of linear equations has a solution. The injective envelope of a module (minimal injective extension) can be constructed by adding solutions to linear systems, in a manner analogous to the construction of the algebraic closure of a field (see, for instance, Kucera [5] for details). Thus it makes sense to analyze the structural complexity of injective envelopes in terms of patterns of solutions of linear systems. There is further support for this idea from the viewpoint of mathematical logic. A *positive primitive formula* (in the first-order language of Λ-modules) is a formula $\varphi(\vec{x})$ of the form

$$\exists \vec{y} \bigwedge_{i=1}^{n} \varphi_i(\vec{x}, \vec{y})$$

where each φ_i is a linear equation with coefficients from Λ. As is well known, every first-order formula in the language of Λ-modules is equivalent to a Boolean combination of positive primitive formulas (see for example Section 2.4 of Prest [8]). The situation is even better for injective modules over Noetherian rings, for in the first-order theory of such a module, every positive primitive formula is equivalent to a finite system of linear equations (see Theorem 3.12 of [2] and see also [4]). Thus in

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such a module, every definable set is a Boolean combination of solution sets of finite systems of linear equations.

One way of carrying out an analysis of the kind desired is by means of the definability rank of Deissler introduced in his [1] for complete first-order theories, and generalized in Kucera [3]. Deissler's rank was studied further in Woodrow and Knight [9]. We use the *positive rank* of [3]. In the context of this paper we say that a subset S of an injective module $\mathcal{M}$ is *definable over* $A \subset M$ if S is the set of all solutions in $\mathcal{M}$ of some finite system of linear equations in one free variable and with constants from A; and we say that an element m of M is *definable over* A if it is the unique solution of such a system. (Note that such a linear equation might be of a form like $r.v = s.a + t.b$, for some r, s, t in Λ and a and b in A, so that we are really dealing with finite systems of linear equations over the submodule of $\mathcal{M}$ generated by A.)

The formal definition of positive Deissler rank follows.

Definition 1 Let $\mathcal{M}$ be an injective module, $b \in M$, $A \subset M$.

(i) $\mathrm{rk}^+(b,\ A,\ \mathcal{M}) = 0$ iff b is definable over A in $\mathcal{M}$.

(ii) For any ordinal $\xi > 0$, $\mathrm{rk}^+(b,\ A,\ \mathcal{M}) = \xi$ iff ξ is the least ordinal α such that for some finite system of linear equations $\varphi(v)$ with constants from A, φ is consistent and for every solution c of φ in $\mathcal{M}$, $\mathrm{rk}^+(b,\ A \cup \{c\},\ \mathcal{M}) < \alpha$.

(iii) If $\mathrm{rk}^+(b,\ A,\ \mathcal{M}) \neq \xi$ for all ordinals ξ, then $\mathrm{rk}^+(b,\ A,\ \mathcal{M}) = \infty$.

(iv) $\mathrm{rk}^+(\mathcal{M}/A) = \sup\{\mathrm{rk}^+(b,\ A,\ \mathcal{M}) + 1 : b \in \mathcal{M}\}$.

If Λ is a commutative Noetherian ring and $\mathcal{M}$ is an indecomposable injective Λ-module, then $\mathcal{M}$ has the form $\mathcal{M} = \mathrm{E}(\Lambda/P)$ where P is a prime ideal of Λ and $\mathrm{E}(\text{—})$ denotes the injective envelope. In [3] it was shown that if $\mathcal{M} = \mathrm{E}(\Lambda/P)$ and $A = \Lambda/P$ then $\mathrm{rk}^+(\mathcal{M}/A)$ is 1 or 2, and that $\mathrm{rk}^+(\mathcal{M}^{(\kappa)}/A^{(\kappa)}) \leq \omega$ for any infinite cardinal κ.

In this paper we calculate exact values for positive rank in the case of infinite powers. In order to understand the meaning of these results, we need the hierarchy of submodules in indecomposable injectives introduced by Matlis [6]. For $\mathcal{M}$ an indecomposable as above, $\mathrm{A}_i(\mathcal{M}) = \mathrm{A}_i = \{a \in M : P^i.a = 0\}$. $(\mathrm{A}_i)_{i\in\omega}$ is an increasing sequence of submodules of $\mathcal{M}$, $\mathcal{M} = \bigcup_{i\in\omega} \mathrm{A}_i$, A_1 is isomorphic as an $\mathcal{M}$-module to the quotient field K of Λ/P, and each $\mathrm{A}_{i+1}/\mathrm{A}_i$ is a finite dimensional vector space over K. The positive Deissler rank of an indecomposable does not "see" the levels of the hierarchy; the fact that $\mathrm{rk}^+(\mathcal{M}/A) \leq 2$ reflects the indecomposability of $\mathcal{M} = \mathrm{E}(A)$: elements of $\mathcal{M}$ are only one step away from being definable from A. The elements of A_1 are exactly the elements definable from A, and so $\mathrm{rk}^+(\mathcal{M}/A) = 1$ iff $\mathcal{M} = A_1$. (It is not known if there are any similar kinds of results in the non-commutative case.) In $\mathcal{M}^{(\kappa)}$ (κ infinite), positive Deissler rank reflects the levels of the Matlis hierarchy precisely, as we shall see below.

The definition of rank as given above is often awkward to work with in practice. It is sometimes more convenient to work with a description of the entire body of the recursive computation of rank all at once. We repeat the definition of an *analysis* of rank from [3], specialized for the purposes of this paper.

Definition 2 An analysis of $\mathrm{rk}^+(b,\ A,\ \mathcal{M})$ is a labeled rooted tree $\langle T,\ <,\ \lambda\rangle$ of the following sort:

(i) For each node $t \in T$, the label $\lambda(t)$ is a pair $\langle \varphi(v),\ C\rangle$ where $C \subset M$ and φ is a consistent finite system of linear equations in the one variable v over C, and in particular if t is the root of T, then $\lambda(t) = \langle \varphi,\ A\rangle$ for some such φ.

(ii) For each $t \in T$, if t is a leaf (terminal node) of T and $\lambda(t) = \langle \varphi,\ C\rangle$ then b is defined by φ over C in $\mathcal{M}$.

(iii) For each $t \in T$, if t is not a leaf of T and $\lambda(t) = \langle \varphi,\ C\rangle$ then the successors of t are in one-to-one correspondence with the solutions of φ in $\mathcal{M}$, the second component of the labels of these successors ranging over all sets of the form $C \cup \{m\}$, m a solution of φ in $\mathcal{M}$.

(iv) $\langle T,\ <\rangle$ is a well-founded tree, that is, the branches of T are finite.

The usual foundation rank of $\langle T,\ <\rangle$ is denoted "rank(T)". It is clear that if $\langle T,\ <,\ \lambda\rangle$ is an analysis of $\mathrm{rk}^+(b,\ A,\ \mathcal{M})$, then $\mathrm{rk}^+(b,\ A,\ \mathcal{M}) \leq \mathrm{rank}(T)$. If these two ranks are actually equal, then $\langle T,\ <,\ \lambda\rangle$ is called an *accurate* analysis of $\mathrm{rk}^+(b,\ A,\ \mathcal{M})$ (and such always exists) [3].

For the purposes of stating and proving our theorem, we fix the same notation as in [4]:

Let Λ be a commutative Noetherian ring, P a prime ideal of Λ, κ an infinite cardinal, $E = \mathrm{E}(\Lambda/P)^{(\kappa)}$, $B = (\Lambda/P)^{(\kappa)}$ (so that $E = \mathrm{E}(B)$). For each $i \in \omega$ let $B_i = \{m \in E : P^i m = 0\}$. Note that if $A_i \subset \mathrm{E}(\Lambda/P)$ is the i-th level of the Matlis hierarchy in $\mathrm{E}(\Lambda/P)$ then $B_i = A_i^{(\kappa)}$. For the basic facts of ideal theory used below, we refer the reader to any standard reference, in particular to Northcott [7].

Theorem 3

(i) If $a \in B_{n+1} \setminus B_n$ then $\mathrm{rk}^+(a,\ B,\ E) = n$.

(ii) $\mathrm{rk}^+(E/B) = \sup\{n+1 : B_{n+1} \setminus B_n \neq \varnothing\}$.

In Theorem 2.5 of [4] these were inequalities rather than equalities. Thus our result provides a positive and general solution to the conjecture of Section 2.9 of [4].

Proof: Part (ii) follows immediately from (i) and the definitions. From Theorem 2.5 of [4] it follows that every element of B_1 has rank 0, and by the same argument as in the proof of Theorem 2.4 of [4] it follows that any element of E not in B_1 has rank at least 1. Thus B_1 consists precisely of the elements of rank 0. Again from Theorem 2.5 of [4] the elements of $B_2 \setminus B_1$ have rank less than or equal to 1, so they must have rank exactly 1. Thus the theorem holds when $n = 0$ or $n = 1$.

So assume that $n \geq 2$ and that for every $c \in B_n \setminus B_{n-1}$, $\mathrm{rk}^+(c,\ B,\ E) = n-1$. Let $b \in B_{n+1} \setminus B_n$. Since $b \notin B_n$ there must be $p_0,\ \ldots,\ p_{n-1} \in P$ such that $p_{n-1} \cdot \ldots \cdot p_0.b \neq 0$, but for any $q_n,\ \ldots,\ q_1 \in P$, $q_n \cdot \ldots \cdot q_1 p_0.b = 0$. Thus $p_0.b \in B_n \setminus B_{n-1}$ and so $\mathrm{rk}^+(p_0.b,\ B,\ E) = n-1$. Suppose that $\mathrm{rk}^+(b,\ B,\ E) = m < n$. Fix an accurate analysis $\langle T,\ <,\ \lambda\rangle$ of $\mathrm{rk}^+(b,\ B,\ E)$, hence of (tree) rank m. By Theorem 2.1 part (iii) of [3] we can assume without loss of generality that the only formulas appearing in $\langle T,\ <,\ \lambda\rangle$ that define a singleton are the formulas at the leaves. (Note that there is unique division in E by elements of $\Lambda \setminus P$: the formulas defining a singleton are exactly those equivalent to a single equation $r.v = a$ with $r \notin P$.) We show how to construct from this tree an analysis of $\mathrm{rk}^+(p_0.b,\ B,\ E)$ of rank $m-1$, a contradiction.

Consider first any leaf t of T which is not at the maximum level m of T. The formula part of $\lambda(t)$ is some formula (system of equations) $\varphi(v, c_1, \ldots, c_k)$ defining b; replace it by the formula $\varphi(v, p_0.c_1, \ldots, p_0.c_k)$ which clearly defines $p_0.b$. Now consider those leaves at the maximum level m of T; we eliminate them from T as follows. Let t be the predecessor of such a leaf (so all the successors of t are leaves). There are consistent $\varphi_0(v)$ over B and for each $i \in \{1, \ldots, m-1\}$, a solution e_{i-1} of $\varphi_{i-1}(v)$ in E and consistent $\varphi_i(v)$ over $B \cup \{e_j : j < i\}$, with $\lambda(t) = \langle \varphi_{m-1}, B \cup \{e_j : j < m-1\}\rangle$. Since φ_{m-1} is a finite system of linear equations over $B \cup \{e_0, \ldots, e_{m-2}\}$, that is, over the submodule of E generated by this set, it can be written in the form

$$\varphi_{m-1}(v) = \bigwedge_{i=1}^{k} (r_i.v = \sum_{j=0}^{m-2} r_{ij}.e_j + c_i)$$

where each c_i is in B. Since t is not a leaf, φ_{m-1} does not define a singleton and thus $r_i \in P$ for each i. Let J be the ideal generated by $\{r_1, \ldots, r_k\}$. Without loss of generality, by Theorem 2.6 of [4], we may assume that $\varphi_{m-1}(v)$ is a complete formula and so determines the annihilator I of its solutions. Note that I is necessarily a P-primary ideal, in fact by Theorem 0.3 of [4], since κ is infinite, an ideal is P-primary iff it is the annihilator of some element of E. Since J contains I, for some l, $P \supset J \supset P^l$. Thus P must be a minimal prime of J, so J has a P-primary component J_1, and $J_1 = J_{\Lambda \setminus P} = \{s \in \Lambda : rs \in J \text{ for some } r \in \Lambda \setminus P\}$ (since any other prime ideal belonging to J must meet $\Lambda \setminus P$). Let

$$Z = \{\alpha \in \kappa : e_0(\alpha) = \ldots = e_{m-2}(\alpha) = c_0(\alpha) = \ldots = c_k(\alpha) = b_\alpha = 0\}.$$

Z is cofinite and J_1 is P-primary, so there is $d \in E$ with $\mathrm{ann}(d) = J_1$ and $\mathrm{supp}(d) \subset Z$ (where $\mathrm{supp}(d) = \{\alpha \in \kappa : d(\alpha) \neq 0\}$). Choose e_{m-1} a solution of φ_{m-1} in E such that $e_{m-1}|Z = d|Z$. This is clearly possible since $J_1 \supset J$. Since each successor of t is a leaf, b is definable over $B \cup \{e_j : j < m\}$, by some formula $\varphi_m(v)$ of the form "$s.v = \sum_{j=0}^{m-1} s_j.e_j + c$" with $s \notin P$ and $c \in B$.

Let $\varphi^*(v)$ be $s.v = \sum_{j=0}^{m-1} p_0 s_j.e_j$. Then $p_0.b$ is the unique solution of φ^* in E. Now for every $i \in Z$, $p_0 s_{m-1}.e_{m-1}(i) = 0$, so $p_0 s_{m-1} \in \mathrm{ann}(d) = J_1$. Thus there is $r \in \Lambda \setminus P$ such that $rp_0 s_{m-1} \in J$. Let $\varphi^{**}(v)$ be $rs.v = \sum_{j=0}^{m-1} rp_0 s_j.e_j$. Since $r \notin P$, $p_0.b$ is also the unique solution of φ^{**} in E. Now $rp_0 s_{m-1} \in J$, so there are $t_1, \ldots, t_k \in \Lambda$ such that $rp_0 s_{m-1} = \sum_{i=1}^{k} t_i r_i$. Since for $1 \leq i \leq k$, $r_i.e_{m-1} = \sum_{j=0}^{m-2} r_{ij}.e_j + c_i$,

$$\begin{aligned} rp_0 s_{m-1}.e_{m-1} &= (\sum_{i=1}^{k} t_i r_i).e_{m-1} \\ &= \sum_{i=1}^{k} t_i.(\sum_{j=0}^{m-2} r_{ij}.e_j + c_i) \\ &= \sum_{j=0}^{m-2} (\sum_{i=1}^{k} t_i r_{ij}).e_j + \sum_{i=1}^{k} t_i.c_i \end{aligned}$$

Thus φ^{**} can be rewritten in terms of $e_0, \ldots, e_{m-2}$ alone, as

$$\varphi^*_{m-1}(v) = \text{“}rs.v = \sum_{j=0}^{m-2}(rp_0s_j + (\sum_{i=1}^{k} t_i r_{ij})).e_j + \sum_{i=1}^{k} t_i.c_i\text{”}.$$

Replace the formula part of $\lambda(t)$ by $\varphi^*_{m-1}(v)$, and delete all the successors of t.

Once we have carried out the above process for all the leaves of T, the result will clearly be an analysis of $\mathrm{rk}^+(p_0.b,\ B,\ E)$, necessarily of rank $m-1$ since we have truncated all the maximal leaves of T. This is the promised contradiction.

The authors have considered the problem of the Deissler rank complexity of arbitrary injective modules (over a commutative Noetherian ring), that is, the case of arbitrary direct sums of indecomposables, but Conjecture 2.15 of [4] remains open.

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