Hilbert Spaces of Holomorphic Functions: Zero Sets, Invariant Subspaces, and Toeplitz Operators

by

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Abstract

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Several problems are considered in the setting of Hilbert spaces of holomorphic functions on the unit disc. In Chapter 1, the main result is a characterization of the zero sets of a large class of such spaces. The characterization is in terms of Gram matrices of reproducing kernels associated with the points of a sequence in the disc. The construction involved in the proof is then applied to a smaller class of spaces to characterize the elements of a space having a fixed inner function as a factor. Specializing to the case of the Dirichlet space, the construction gives a characterization of the boundary-zero sets.

This construction gives rise to a wandering vector of the shift operator of multiplication by z. In the case of the Dirichlet space and certain generalizations, the subspaces invariant under the shift operator are generated by wandering vectors. Furthermore, the wandering subspace of an invariant subspace is one-dimensional. Therefore, the problem of describing the invariant subspaces of Dirichlet-type spaces reduces to the problem of describing the wandering vectors. For invariant subspaces of the Dirichlet space determined by zero sets, inner divisors, and boundary-zero sets, the aforementioned construction produces a wandering vector and generator. It is not known whether other invariant subspaces of the Dirichlet space exist.

In Chapter 2, variants of the classical Toeplitz operators on H^2 are studied. A characterization is obtained for the bounded, harmonic symbols giving rise to a bounded Toeplitz operator on a Dirichlet-type space. The relationship between the characterizing condition and multipliers of the holomorphic and harmonic Dirichlet spaces is examined.

Professor Donald Sarason
Dissertation Committee Chair

Contents

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Chapter 1

Zero Sets and Invariant Subspaces

Let f be a holomorphic function on a region G. If E is the set of zeroes of f, then E does not have a limit point in G, unless $f = 0$ on all of G.

Conversely, let E be a subset of G without a limit point in G. Then there is a function f holomorphic in G having E as its set of zeroes. Furthermore, f can be chosen to have a zero of prescribed multiplicity at each $z \in E$.

Thus, the problem of characterizing the zero sets of holomorphic functions on a region is completely solved. The situation changes, however, if we consider smaller sets of functions.

Definition 1.1. Let $\mathcal{H}$ be a set of holomorphic functions on a region G; let $E \subset G$. Then E is a *zero set* of $\mathcal{H}$ if there is $f \in \mathcal{H}$, not identically zero, having E as its set of zeroes.

A sequence $\{z_n\}$ in G will be called a zero set of $\mathcal{H}$ if there is nonzero $f \in \mathcal{H}$ having a zero at each z_n of multiplicity equal to the number of times the point z_n appears in the sequence, and having no other zeroes.

Example 1.2. If $H(G)$ denotes the set of holomorphic functions on the region G, then $E \subset G$ is a zero set of $H(G)$ iff E does not have a limit point in G.

Another example is the following classical result.

Theorem 1.3. *Let $0 < p \leq \infty$. A sequence $\{z_n\}$ in $\mathbb{D}$ is a zero set of $H^p(\mathbb{D})$ iff $\sum(1 - |z_n|) < \infty$.*

The condition of the theorem is known as the *Blaschke condition*; a sequence satisfying the Blaschke condition is a *Blaschke sequence.* If $\{z_n\}$ is a (finite or infinite) Blaschke sequence, then $\{z_n\}$ is a zero set of the *Blaschke product*

$$\lambda \prod_n \frac{z_n - z}{1 - \overline{z_n} z} \frac{\overline{z_n}}{|z_n|},$$

where λ is a unimodular constant, and $\overline{z_n}/|z_n|$ is to be interpreted as 1 if $z_n = 0$. Every Blaschke product belongs to H^p for every $p \in (0, \infty]$.

For the remainder of this chapter, $\mathcal{H}$ will denote a Hilbert space of holomorphic functions on $\mathbb{D}$ satisfying the following hypotheses:

1. $\mathcal{H}$ contains the polynomials;
2. the shift operator M_z of multiplication by z is bounded on $\mathcal{H}$; and
3. if $f \in \mathcal{H}$ and $f(0) = 0$, then there is $g \in \mathcal{H}$ such that $f = zg$.

Further assumptions about $\mathcal{H}$ will be made later in the chapter.

Definition 1.4. For $w \in \mathbb{D}$ and $j = 0, 1, 2, \ldots$, the *jth-order reproducing kernel* at w is a function $k_w^j \in \mathcal{H}$ such that $\langle f, k_w^j \rangle = f^{(j)}(w)$ for all $f \in \mathcal{H}$.

The existence of $k_w^j \in \mathcal{H}$ is equivalent to the boundedness of the functional $f \mapsto f^{(j)}(w)$ on $\mathcal{H}$. If it exists, it is unique.

Now, some examples.

Example 1.5. The Hardy space H^2 can be regarded as the set of all $f \in H(\mathbb{D})$ such that $\sum |\hat{f}(n)|^2$ is finite, where $\hat{f}(n)$ is the power-series coefficient of z^n. The inner product is given by $\langle f, g \rangle_{H^2} = \sum \hat{f}(n)\overline{\hat{g}(n)}$. Using this formula for the inner product, the power series for the reproducing kernels can be calculated:

$$k_w^j(z) = \sum_{n=j}^{\infty} \frac{n!}{(n-j)!} \overline{w}^{n-j} z^n = \frac{j!\, z^j}{(1 - \overline{w} z)^{j+1}}.$$

Example 1.6. The Bergman space L_a^2 is the subspace of $L^2(\mathbb{D})$ (with respect to normalized Lebesgue measure) consisting of the holomorphic functions. Using the formula $\langle f, g \rangle_{L_a^2} = \sum (n+1)^{-1} \hat{f}(n)\overline{\hat{g}(n)}$, the reproducing kernels are found:

$$k_w^j(z) = \sum_{n=j}^{\infty} \frac{(n+1)!}{(n-j)!} \overline{w}^{n-j} z^n = \frac{(j+1)!\, z^j}{(1 - \overline{w} z)^{j+2}}.$$

Example 1.7. The Dirichlet space D consists of those $f \in H(\mathbb{D})$ having $f' \in L^2_a$; the norm is given by $\|f\|_D^2 = \|f\|_{H^2}^2 + \|f'\|_{L^2_a}^2$. The quantity $\|f'\|_{L^2_a}^2 = \int_{\mathbb{D}} |f'|^2 dA = \sum n|\hat{f}(n)|^2$ is called the *Dirichlet integral* of f, denoted $D(f)$. The formula for the Dirichlet integral in terms of the power-series coefficients of f makes it clear that $D \subset H^2$. There is also a formula, due to J. Douglas [8], in terms of integrals over $\partial\mathbb{D}$:

$$\int_{\mathbb{D}} |f'|^2 \, dA = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right|^2 \frac{dt}{2\pi} \frac{d\theta}{2\pi}. \tag{1.1}$$

The inner integral is the *local Dirichlet integral* of f at $e^{i\theta}$, denoted $D_{e^{i\theta}}(f)$, and can be regarded as a function on $\partial\mathbb{D}$.

As above, the reproducing kernels can be calculated:

$$k_w^j(z) = \sum_{n=j}^{\infty} \frac{n!}{(n+1)(n-j)!} \overline{w}^{n-j} z^n = z^j \frac{d^j}{dt^j} \left(\frac{1}{t} \log \frac{1}{1-t} \right) (\overline{w} z).$$

Example 1.8. Let μ be a finite, positive, Borel measure on $\partial\mathbb{D}$. The Dirichlet-type space $D(\mu)$ is the set of $f \in H(\mathbb{D})$ having a local Dirichlet integral that is integrable with respect to μ. Equation (1.1) says that $D = D(\frac{d\theta}{2\pi})$. The norm is given by $\|f\|_\mu^2 = \|f\|_{H^2}^2 + \int D_\lambda(f) d\mu(\lambda)$. That $D(\mu) \subset H^2$ is shown in [14] (also see Corollary 2.4).

The space $D(\mu)$ differs from the previous examples in that the powers of z are not generally orthogonal. This makes it difficult to show existence of reproducing kernels by exhibiting them explicitly. However, the fact that H^2 has reproducing kernels implies their existence for the subset $D(\mu)$:

$$|f^{(j)}(w)| \le C\|f\|_{H^2} \le C\|f\|_\mu,$$

where C is a constant not depending on f. From the boundedness of the functional on $D(\mu)$ of evaluation of the jth derivative at w follows the existence of $k_w^j \in D(\mu)$.

1.1 Zero Sets in $\mathbb{D}$

Let $\{z_n\}$ be a sequence in $\mathbb{D}$. The goal of this section is to produce a necessary and sufficient condition for $\{z_n\}$ to be a zero set of $\mathcal{H}$.

For a moment it will be convenient to let $\{\alpha_j\}$ be the subsequence of all the distinct points of the sequence $\{z_n\}$. For each j, let m_j be the number of times α_j appears in the sequence $\{z_n\}$. Since no sequence with a limit point in $\mathbb{D}$ can be a zero set of $\mathcal{H}$, we may assume that each m_j is finite. The point 0 may or may not appear in $\{z_n\}$; let $m \geq 0$ be the number of times that 0 does appear.

In addition to assumptions (1)–(3) above, for each j assume that $\mathcal{H}$ has reproducing kernels at α_j of orders $0, 1, \ldots$, and $m_j - 1$. Assume further that $\mathcal{H}$ has a reproducing kernel at 0 of order m (if 0 is some α_j, this assumes one order more than previously).

Let $\mathcal{M}$ be the subspace of $\mathcal{H}$ of functions having for all j a zero at α_j of multiplicity at least m_j. What is in question in this section is whether $\mathcal{M}$ contains elements other than the function 0. Assumption (3) about $\mathcal{H}$ guarantees that if $\mathcal{M}$ contains nonzero elements, then $\mathcal{M}$ contains an element having a zero at $z = 0$ of multiplicity exactly m (meaning no zero at all at $z = 0$ if $m = 0$). Hence, if $k_0^m \perp \mathcal{M}$, then $\mathcal{M} = \{0\}$.

For notational simplicity, let $\{\varphi_n\}$ enumerate the set of functions $\bigcup_j \{k^0_{\alpha_j}, \ldots, k^{m_j-1}_{\alpha_j}\}$, and let $\psi = k_0^m$. For each n, let G_n be the $n \times n$ Gram matrix $\{\langle \varphi_j, \varphi_k \rangle\}_{k,j=1}^n$.

Claim 1.9. *The set $\{\varphi_1, \ldots, \varphi_n\}$ is linearly independent in $\mathcal{H}$.*

Proof. Suppose $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are such that $\sum \lambda_j \varphi_j = 0$. Then for all $f \in \mathcal{H}$, the relation

$$\sum \overline{\lambda_j} \langle f, \varphi_j \rangle = \langle f, \sum \lambda_j \varphi_j \rangle = 0 \tag{1.2}$$

holds. However, each $\langle f, \varphi_j \rangle$ is the value of f or a derivative of f at some fixed point. Unless all $\lambda_j = 0$, one can find a polynomial f not satisfying (1.2). Since $\mathcal{H}$ contains all the polynomials, it must be that all $\lambda_j = 0$. Thus, $\{\varphi_1, \ldots, \varphi_n\}$ is linearly independent. □

Since $\{\varphi_1, \ldots, \varphi_n\}$ is linearly independent, G_n is invertible (for this and other properties of Gram matrices, see [11]). For each n, let $\mathbf{b}_n$ be the vector $(\langle \varphi_1, \psi \rangle, \ldots, \langle \varphi_n, \psi \rangle)^t$. Let S_n be the number $\mathbf{b}_n^t G_n^{-1} \overline{\mathbf{b}_n}$. Since G_n and G_n^{-1} are, in fact, positive definite matrices, as a result S_n is real and positive.

It will be shown shortly that the sequence $\{S_n\}$ is nondecreasing and has $\|\psi\|^2$ as an upper bound.

Theorem 1.10. *The subspace* $\mathcal{M}$ *is nontrivial iff* $\lim S_n < \|\psi\|^2$.

This theorem provides a characterization of those sequences that are contained in a zero set of $\mathcal{H}$. There are many spaces, such as the examples given above, in which every subset of a zero set is a zero set. In such spaces, the theorem characterizes the zero sets.

In the case where the points of $\{z_n\}$ are all distinct and nonzero, in many spaces (including H^2, L^2_a, and D) the components of each $\mathbf{b}_n$ are all 1, and $\psi = k^0_0$ is the constant 1 and has norm 1. In this case, the condition of the theorem is that the sum of all the elements of G_n^{-1} does not increase to 1 as $n \to \infty$.

Proof. The reproducing properties of the kernels $\{\varphi_1, \varphi_2, \ldots\}$ are such that a function $f \in \mathcal{H}$ belongs to $\mathcal{M}$ iff $\langle f, \varphi_n \rangle = 0$ for all n. Such an f will be sought among differences of elements of $\mathcal{M}^\perp$ and constant multiples of ψ. Therefore, given n, consider the system of equations in n complex variables $c_1, c_2, \ldots, c_n$ given by

$$\langle \sum_{j=1}^{n} c_j \varphi_j - \lambda_n \psi, \varphi_k \rangle = 0 \qquad \text{for } k = 1, 2, \ldots, n, \tag{1.3}$$

where λ_n is a constant to be specified later. By expanding out the inner product, the system (1.3) can be expressed as the matrix equation $G_n \mathbf{c}_n = \lambda_n \overline{\mathbf{b}}_n$, where $\mathbf{c}_n$ is the $n \times 1$ vector $(c_1, \ldots, c_n)^t$, and G_n and $\mathbf{b}_n = (b_1, \ldots, b_n)^t$ are as defined above.

Since $\{\varphi_1, \ldots, \varphi_n\}$ is linearly independent in $\mathcal{H}$ by the claim, the system (1.3) has a unique solution $\mathbf{c}_n = \lambda_n G_n^{-1} \overline{\mathbf{b}}_n$. Let $\tilde{f}_n = \sum_{j=1}^n c_j \varphi_j$; let $f_n = \tilde{f}_n - \lambda_n \psi$. By construction, $\tilde{f}_n \in \mathcal{M}^\perp$ and $f_n \perp \{\varphi_1, \ldots, \varphi_n\}$.

The norm of $\tilde{f}_n$ can be calculated:

$$\begin{aligned}\|\tilde{f}_n\|^2 &= \langle\sum_j c_j\varphi_j, \sum_k c_k\varphi_k\rangle \\ &= \sum_k \overline{c_k}\langle\sum_j c_j\varphi_j, \varphi_k\rangle \\ &= \sum_k \overline{c_k}\lambda_n\langle\psi, \varphi_k\rangle \\ &= \lambda_n\sum_k \overline{c_k}\overline{b_k}.\end{aligned}$$

In order to simplify this and subsequent calculations, choose λ_n so that $\sum c_k b_k = 1$. This is accomplished by letting $\lambda_n = 1/\mathbf{b}_n^t G_n^{-1}\overline{\mathbf{b}_n} = 1/S_n$, for then $\sum c_k b_k = \mathbf{b}_n^t\mathbf{c}_n = \mathbf{b}_n^t\lambda_n G_n^{-1}\overline{\mathbf{b}_n} = 1$. Now we have that $\|\tilde{f}_n\|^2 = \lambda_n$.

Claim 1.11. *$\tilde{f}_n$ is the solution to the extremal problem*

$$\inf\left\{\left\|\sum_{j=1}^n \gamma_j\varphi_j\right\| : \sum \gamma_j b_j = 1\right\}.$$

Proof of Claim. If $\mathcal{S}$ is the span of $\{\varphi_1, \ldots, \varphi_n\}$, then the problem becomes

$$\inf\{\|\varphi\| : \varphi \in \mathcal{S},\ \langle\varphi, \psi\rangle = 1\}.$$

If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$, then a solution is given by $\varphi = P\psi/\|P\psi\|^2$. It is geometrically clear that this solution is unique. Since $\tilde{f}_n \in \mathcal{S}$ and $\tilde{f}_n - \lambda_n\psi \perp \mathcal{S}$ by construction, $\tilde{f}_n = \lambda_n P\psi$. Then $\|P\psi\|^2 = 1/\lambda_n$; hence $\tilde{f}_n = P\psi/\|P\psi\|^2 = \varphi$. □

Continuing the proof of the theorem, the claim implies that $\{\lambda_n\}$ is a nonincreasing sequence (hence $\{S_n\}$ is nondecreasing). Let $\lambda = \lim \lambda_n$. Also, $\{\tilde{f}_n\}$ is norm-bounded, so a subsequence converges weakly to a limit $\tilde{f} \in \mathcal{H}$. In fact, $\|\tilde{f}_n - \tilde{f}\| \to 0$: if $\mathbf{c}_n = (c_1^n, \ldots, c_n^n)^t$, then for $l \geq n$,

$$\begin{aligned}\langle\tilde{f}_l, \tilde{f}_n\rangle &= \langle\sum_{j=0}^l c_j^l\varphi_j, \sum_{k=0}^n c_k^n\varphi_k\rangle \\ &= \sum_{k=0}^n \overline{c_k^n}\langle\sum_{j=0}^l c_j^l\varphi_j, \varphi_k\rangle \\ &= \sum_k \overline{c_k^n}\lambda_l\overline{b_k} = \lambda_l.\end{aligned}$$

Therefore

$$\|\tilde{f}_n - \tilde{f}\|^2 = \lambda_n - 2\operatorname{Re}\langle \tilde{f}, \tilde{f}_n\rangle + \langle \tilde{f}, \tilde{f}\rangle = \lambda_n - 2\lambda + \lambda \to 0.$$

Let $f = \tilde{f} - \lambda\psi$. Then $f_n \to f$ in norm. For each k, since $f_n \perp \varphi_k$ for every $n \geq k$, $f \perp \varphi_k$. Therefore $f \in \mathcal{M}$.

Recall that ψ was chosen so that $\psi \perp \mathcal{M} \Rightarrow \mathcal{M} = \{0\}$ (with the converse being trivial). Since $f = \tilde{f} - \lambda\psi$ and $\tilde{f} \in \mathcal{M}^\perp$, $\mathcal{M} \neq \{0\}$ iff $f \neq 0$. Since

$$\begin{aligned}
\|f\|^2 = \langle f, f\rangle &= \|\tilde{f}\|^2 - 2\operatorname{Re}\lambda\langle \tilde{f}, \psi\rangle + \lambda^2\|\psi\|^2 \\
&= \lambda - 2\lambda\operatorname{Re}\lim_{n\to\infty}\langle \sum_{j=1}^{n} c_j\varphi_j, \psi\rangle + \lambda^2\|\psi\|^2 \\
&= \lambda - 2\lambda\operatorname{Re}\lim_{n\to\infty}\sum_{j=1}^{n} c_j b_j + \lambda^2\|\psi\|^2 \\
&= \lambda - 2\lambda\cdot 1 + \lambda^2\|\psi\|^2 = \lambda(\lambda\|\psi\|^2 - 1),
\end{aligned}$$

$\lambda \geq 1/\|\psi\|^2$ and $f \neq 0$ iff $\lambda > 1/\|\psi\|^2$. Since $\lambda = 1/\lim S_n$, we have $S_n < \|\psi\|^2$ and $\mathcal{M} \neq \{0\}$ iff $\lim S_n < \|\psi\|^2$. $\square$

Remark. Since both the Blaschke condition (Theorem 1.3) and Theorem 1.10 apply to the Hardy space H^2, the condition of Theorem 1.10 must be equivalent to the Blaschke condition. The author has not been able to prove this directly. If a direct proof were known, perhaps the methods could be applied in other spaces, transforming the rather algebraic condition of Theorem 1.10 into a simpler, geometric condition. In the Bergman space and the Dirichlet space, among other spaces, no geometric characterization of the zero sets is known.

Remark. The result of Theorem 1.10 overlaps with a result of P. Malliavin [13]. He gives a characterization of the subsets of $\overline{\mathbb{D}}$ on which a nonzero element of certain spaces of holomorphic functions can vanish. The characterization is in terms of capacities defined by extremal problems involving complex measures on $\overline{\mathbb{D}}$. By applying Malliavin's capacity condition to finite sets, his result can be shown to be equivalent to Theorem 1.10 in the case of sequences of distinct points in $\mathbb{D}$. His result pertains only to spaces in which the powers of z are orthogonal and have norm at least one, among other assumptions. Spaces excluded include the Dirichlet-type spaces $D(\mu)$, where the powers of z are not generally orthogonal, and the Bergman space, where the norms of the powers of z approach zero. Malliavin's result also does not consider zero sets with multiplicities. On the other hand, Theorem 1.10 only applies to subsets of $\mathbb{D}$; however, see Section 1.4.

1.2 Inner Divisors

Recall that H^∞ denotes the space of bounded functions $f \in H(\mathbb{D})$, with norm given by $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. As shown by P. Fatou [9], if $f \in H^\infty$ then f has radial limits at almost every point of $\partial\mathbb{D}$. If the radial limit of $u \in H^\infty$ satisfies $|u(e^{i\theta})| = 1$ for almost all θ, then u is said to be an *inner function*. Every Blaschke product is an inner function. If ν is a positive, finite measure on $\partial\mathbb{D}$ that is singular with respect to Lebesgue measure, define S on $\mathbb{D}$ by

$$S(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\nu(e^{i\theta})\right).$$

Then S is an inner function without zeroes in $\mathbb{D}$, known as a *singular inner function.* Every inner function u is the product of a Blaschke product and a singular inner function, where either factor may be constant.

Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\mathbb{D}$ satisfying hypotheses (1)–(3) defined earlier. Now suppose also that $\mathcal{H} \subset H^2$, and that $\|\cdot\|_{\mathcal{H}} \geq \|\cdot\|_{H^2}$. Let u be an inner function, and let $\mathcal{M} = uH^2 \cap \mathcal{H}$.

Example 1.12. Suppose $\mathcal{H} = D(\mu)$, the Dirichlet-type space associated with the measure μ. Suppose $F \in H^2$ and $f = uF \in \mathcal{M}$. It follows from the formula in [16] for the local Dirichlet integral that in fact, $F \in D(\mu)$. Thus, $\mathcal{M}$ consists of those $f \in D(\mu)$ having u as a divisor. The general problem of determining which inner functions u are divisors of a non-zero function in $D(\mu)$, or even in D, is unsolved; in fact, the problem of characterizing the zero sets of $D(\mu)$ is the special case where u is a Blaschke product.

The problem of determining whether $\mathcal{M} \neq \{0\}$ will now be considered. Assume that u is not a Blaschke product, that case being covered by the results of Theorem 1.10.

For $n \geq 1$, define $\varphi_n \in \mathcal{H}$ by $\langle g, \varphi_n \rangle_{\mathcal{H}} = \langle z^n g, u \rangle_{H^2}$ for all $g \in \mathcal{H}$. Such a φ_n exists because it represents a bounded functional on $\mathcal{H}$:

$$|\langle z^n g, u \rangle_{H^2}| \leq \|z^n g\|_{H^2} \|u\|_{H^2} = \|g\|_{H^2} \cdot 1 \leq \|g\|_{\mathcal{H}},$$

by the assumptions on $\mathcal{H}$. Since u is inner, $\overline{u(e^{i\theta})} = 1/u(e^{i\theta})$ for almost all θ. Hence,

$$\langle g, \varphi_n \rangle_{\mathcal{H}} = \langle z^n g, u \rangle_{H^2} = \int g(e^{i\theta}) \overline{u(e^{i\theta})} e^{in\theta} \frac{d\theta}{2\pi} = \widehat{\left(\tfrac{g}{u}\right)}(-n).$$

If $g \in \mathcal{M}$, say $g = uF$ for $F \in H^2$, then $\langle g, \varphi_n \rangle_{\mathcal{H}} = \hat{F}(-n) = 0$. Thus $\varphi_n \perp \mathcal{M}$. Now suppose $g \in \mathcal{H}$ and $g \perp \{\varphi_1, \varphi_2, \ldots\}$. Since $\widehat{\left(\tfrac{g}{u}\right)}(-n) = 0$ for all $n \in \mathbb{N}$, it follows that $\frac{g}{u} \in H^2$; hence $g \in \mathcal{M}$. Therefore $\mathcal{M} = \{\varphi_1, \varphi_2, \ldots\}^{\perp}$.

Claim 1.13. *The set $\{\varphi_1, \varphi_2, \ldots\}$ is linearly independent in $\mathcal{H}$.*

Proof. Suppose $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\sum \lambda_j \varphi_j = 0$. Let $p(z) = \sum \overline{\lambda_j} z^j$. Then for all $g \in \mathcal{H}$,

$$\langle g, \sum \lambda_j \varphi_j \rangle_{\mathcal{H}} = \sum \overline{\lambda_j} \langle z^j g, u \rangle_{H^2} = \langle pg, u \rangle_{H^2} = 0.$$

By choosing $g = z^n$ for each $n \in \mathbb{N}$, we get that $\widehat{\left(\tfrac{p}{u}\right)}(-n) = 0$ for all n; hence $\frac{p}{u} \in H^\infty$ (as a polynomial, p is bounded). Since u is assumed not to be a Blaschke product, u has a nontrivial singular factor S. Then S is also a factor of p. Since the only polynomial with a nontrivial singular factor is 0, $p = 0$. Therefore $\{\varphi_1, \varphi_2 \ldots\}$ is linearly independent. □

Let m be the largest power of z that divides u. Let $\psi = k_0^m$. Since $\mathcal{H} \subset H^2$, the existence of the reproducing kernel $k_0^m \in \mathcal{H}$ follows by the argument in Example 1.8 that showed the existence of reproducing kernels in $D(\mu)$. Suppose $\psi \perp \mathcal{M}$. If $f = uF \in \mathcal{M}$ with $F \in H^2$, from $f \perp \psi$ follows that $F(0) = 0$. However, by assumption (3) about $\mathcal{H}$, the function g defined by $g(z) = f(z)/z = u(z)F(z)/z$ belongs to $\mathcal{M}$. By replacing f with g and repeating the argument indefinitely, it must be that $F = 0$. Thus if $\psi \perp \mathcal{M}$, then $\mathcal{M} = \{0\}$.

Having defined $\{\varphi_1, \varphi_2, \ldots\}$ and ψ, define G_n, $\mathbf{b}_n$, and S_n as in Section 1.1. Observe that the only properties of $\{\varphi_1, \varphi_2, \ldots\}$ and ψ used in the proof of Theorem 1.10 are that $\mathcal{M} = \{\varphi_1, \varphi_2, \ldots\}^\perp$, that $\{\varphi_1, \varphi_2, \ldots\}$ is linearly independent, and that $\psi \perp \mathcal{M}$ iff $\mathcal{M} = \{0\}$. Therefore, the proof of Theorem 1.10 proves the following:

Theorem 1.14. *The subspace* $\mathcal{M}$ *is nontrivial iff* $\lim S_n < \|\psi\|^2$.

This theorem characterizes the inner functions that are H^2-divisors of nonzero elements of $\mathcal{H}$.

1.3 Invariant Subspaces of Dirichlet-type Spaces

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ will be called *invariant* if it is invariant under the operator M_z of multiplication by z; that is, if $z\mathcal{M} \subset \mathcal{M}$. An important problem is to characterize the invariant subspaces of $\mathcal{H}$.

Example 1.15. Let $\mathcal{H} = H^2$. The invariant subspaces of H^2 were characterized by A. Beurling [4]. He showed that they are precisely the subspaces generated by inner functions; that is, those of the form uH^2, where u is an inner function.

Definition 1.16. A *wandering vector* of $\mathcal{H}$ is an element of $\mathcal{H}$ that is orthogonal to its orbit under M_z; that is, an $f \in \mathcal{H}$ satisfying $\langle f, z^k f\rangle = 0$ for all $k \geq 1$. The *wandering subspace* of $\mathcal{H}$ is $\mathcal{H} \ominus z\mathcal{H}$.

Example 1.17. Again, let $\mathcal{H} = H^2$, and let u be an inner function. Then for $k \geq 1$, $\langle u, z^k u\rangle = \langle 1, z^k\rangle = 0$. Thus u is a wandering vector of H^2. If $\mathcal{M} = uH^2$ is the invariant subspace generated by u, then the wandering subspace of $\mathcal{M}$ is the one-dimensional space $\mathbb{C}u$.

Example 1.18. Let $\mathcal{H}$ be the Dirichlet-type space $D(\mu)$. The invariant subspaces of $D(\mu)$ have not been characterized for general μ, or even for $D = D(d\theta/2\pi)$ (see D. Sarason [19] for results when μ is finitely atomic). However, S. Richter and C. Sundberg [17] have shown that if $\mathcal{M}$ is any invariant subspace of $D(\mu)$, then $\mathcal{M}$ is generated by its wandering subspace, which is one-dimensional.

In this respect, the situation of $\mathcal{H} = D(\mu)$ mirrors that of $\mathcal{H} = H^2$: an invariant subspace $\mathcal{M}$ is generated by the one-dimensional subspace $\mathcal{M} \ominus z\mathcal{M}$. Hence, a characterization of the invariant subspaces is equivalent to a characterization of the wandering vectors. In H^2, the wandering vectors are the inner functions. In $D(\mu)$, the wandering vectors are not completely determined for general μ. An important result is that wandering vectors of $D(\mu)$ are multipliers of $D(\mu)$. This fact, trivial in the H^2 case, was proved by Richter and Sundberg [17]. The final part of their argument is to show that wandering vectors of $D(\mu)$ are bounded functions. By refining the argument, it can be shown that norm-one wandering vectors of $D(\mu)$ are contractive multipliers of $D(\mu)$:

Theorem 1.19. *Let μ be a positive, finite, Borel measure on $\partial\mathbb{D}$. Let f be a wandering vector of $D(\mu)$ of norm one. Then f is a contractive multiplier of $D(\mu)$; that is, $\|fg\| \le \|g\|$ for all $g \in D(\mu)$.*

See Richter and Sundberg [18] for the case of $D(\mu) = D$, Sarason [19] for the case of μ finitely atomic. The proof below is a modification of that of Theorem 3.1 of [17]. Theorem 1.19 was proved independently by A. Aleman [1].

Proof. As explained in [17], the multiplier norm of f is the maximum of its norm in $D(\mu)$ and its sup-norm. Hence it suffices to show that $\|f\|_\infty \le 1$.

Denote by $\mathcal{M}$ the invariant subspace of $D(\mu)$ generated by f. Let k be the largest power of z dividing f. Then each function in $\mathcal{M}$ is of the form $z^k h$ for some $h \in D(\mu)$, and if $f = z^k g$, then

$$\frac{\|f\|_\mu}{|g(0)|} = \inf\left\{\frac{\|z^k h\|_\mu}{|h(0)|} : z^k h \in \mathcal{M}\right\}. \tag{1.4}$$

For $N \in \mathbb{R}$ and $h \in D(\mu)$ with inner-outer factorization $h = uF$, define the cutoff function $h_N = u(F \wedge e^N)$, where $F \wedge e^N$ is the outer function determined by $|(F \wedge e^N)(e^{i\theta})| = \min\{|F(e^{i\theta})|, e^N\}$. By Corollary 2.3 of [17], for all $\lambda \in \partial\mathbb{D}$, $D_\lambda(h_N) \le D_\lambda(h)$; hence $\|h_N\|_\mu \le \|h\|_\mu$. By Lemma 2.4 of [17], $f_N = z^k g_N \in \mathcal{M}$. Then by (1.4),

$$\frac{\|f\|_\mu}{|g(0)|} \le \frac{\|f_N\|_\mu}{|g_N(0)|}$$

for all $N \in \mathbb{R}$. Then for all $N \in \mathbb{R}$, $\|f\|_\mu^2 \le \|f_N\|_\mu^2 \frac{|g(0)|^2}{|g_N(0)|^2}$,

$$\|f\|_\mu^2 - \|f_N\|_\mu^2 \le \|f_N\|_\mu^2 \left(\frac{|g(0)|^2}{|g_N(0)|^2} - 1 \right) \le \frac{|g(0)|^2}{|g_N(0)|^2} - 1 \tag{1.5}$$

since $\|f_N\|_\mu \le \|f\|_\mu = 1$.

Let F be the outer factor of g; let $v = \log|F|$. Since $f \in H^2$, v is integrable on $\partial\mathbb{D}$. For $N \in \mathbb{R}$ let $E_N = \{\theta \in (-\pi, \pi) : v(e^{i\theta}) > N\}$. Since g and g_N have the same inner factor,

$$\begin{aligned} \log \frac{|g(0)|^2}{|g_N(0)|^2} &= \log \frac{|F(0)|^2}{|F_N(0)|^2} = 2(\log|F(0)| - \log|F_N(0)|) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\log|F(e^{i\theta})| - \log|F_N(e^{i\theta})|)\, d\theta \\ &= \frac{1}{\pi} \int_{E_N} (v(e^{i\theta}) - N)\, d\theta. \end{aligned} \tag{1.6}$$

Since v is integrable, $\int_{E_N} (v(e^{i\theta}) - N)\, d\theta$ decreases to 0 as $N \to \infty$; hence $\frac{|g(0)|^2}{|g_N(0)|^2}$ decreases to 1.

Fix $N > 0$, and suppose that $\frac{|g(0)|^2}{|g_N(0)|^2} > 1$. Choose $M \ge N$ such that $\frac{|g(0)|^2}{|g_M(0)|^2} < 1 + 2(1 - e^{-2N})$. Then since $D_\lambda(f_M) \le D_\lambda(f)$ for all $\lambda \in \partial\mathbb{D}$,

$$\begin{aligned} \|f\|_\mu^2 - \|f_M\|_\mu^2 &= \|f\|_{H^2}^2 - \|f_M\|_{H^2}^2 + \int_{\partial\mathbb{D}} (D_\lambda(f) - D_\lambda(f_M))\, d\mu(\lambda) \\ &\ge \|f\|_{H^2}^2 - \|f_M\|_{H^2}^2 \\ &= \int_{\partial\mathbb{D}} (|g|^2 - |g_M|^2) \\ &= \frac{1}{2\pi} \int_{E_M} (e^{2v(e^{i\theta})} - e^{2M})\, d\theta \\ &= e^{2M} \frac{1}{2\pi} \int_{E_M} (e^{2v(e^{i\theta}) - 2M} - 1)\, d\theta. \end{aligned}$$

By equation (1.6) and the estimates $e^x - 1 \ge x$ for all x and $\log x \ge (x-1) -$

$(x-1)^2/2$ for $x \geq 1$,

$$\begin{aligned}
\|f\|_\mu^2 - \|f_M\|_\mu^2 &\geq e^{2M}\frac{1}{2\pi}\int_{E_M} 2(v(e^{i\theta}) - M)\, d\theta \\
&= e^{2M} \log \frac{|g(0)|^2}{|g_M(0)|^2} \\
&\geq e^{2M}\left[\left(\frac{|g(0)|^2}{|g_M(0)|^2} - 1\right) - \frac{\left(\frac{|g(0)|^2}{|g_M(0)|^2} - 1\right)^2}{2}\right] \\
&= e^{2M}\left(\frac{|g(0)|^2}{|g_M(0)|^2} - 1\right)\left[1 - \frac{\left(\frac{|g(0)|^2}{|g_M(0)|^2} - 1\right)}{2}\right] \\
&> e^{2M}\left(\frac{|g(0)|^2}{|g_M(0)|^2} - 1\right) e^{-2N} \\
&\geq \frac{|g(0)|^2}{|g_M(0)|^2} - 1,
\end{aligned}$$

contradicting (1.5). Therefore $|g(0)| = |g_N(0)|$. By the definition of the cutoff function, $|F(0)| = |F_N(0)|$. Since F and F_N are outer functions,

$$\int \log |F(e^{i\theta})|\, d\theta = \int \log |F_N(e^{i\theta})|\, d\theta;$$

hence $|F(e^{i\theta})| = |F_N(e^{i\theta})|$ almost everywhere. Thus $|F(e^{i\theta})| \leq e^N$ almost everywhere for all $N > 0$; therefore $\|f\|_\infty = \|F\|_\infty \leq 1$. □

Now let $\mathcal{H}$ and $\mathcal{M}$ be as in Theorem 1.10 or as in Theorem 1.14. Note that in either case, $\mathcal{M}$ is an invariant subspace of $\mathcal{H}$. The construction in the proof of Theorem 1.10 produces $f \in \mathcal{M}$ of the form $\tilde{f} - \lambda\psi$, where $\tilde{f} \in \mathcal{M}^\perp$, $\lambda \in \mathbb{C}$, and $\psi = k_0^m$ is a reproducing kernel at 0. If $k \geq 1$, then $z^k f \in \mathcal{M}$ since $\mathcal{M}$ is invariant, and

$$\langle f, z^k f\rangle = \langle \tilde{f} - \lambda\psi, z^k f\rangle = 0 - \lambda\langle \psi, z^k f\rangle = 0,$$

since $z^k f$ has a zero at $z = 0$ of multiplicity at least $m + k$. Thus, f is a wandering vector of $\mathcal{M}$. Hence if $\mathcal{M} = D(\mu)$ (or H^2), f generates $\mathcal{M}$.

1.4 Zero Sets in $\partial\mathbb{D}$

In this section, it will be necessary to assume that $\mathcal{H} = D$, the Dirichlet space. The problem to be considered is that of characterizing the boundary-zero sets of D, and to produce a generator of each associated invariant subspace of D. A complication arises from the fact that radial limits of a function in D need not exist at every point of $\partial\mathbb{D}$. The following results, respectively of Beurling [3] and L. Carleson [6], help quantify the extent of the problem:

Theorem 1.20. *If $f \in D$, then the radial limit function $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists for $e^{i\theta}$ outside a set of (outer logarithmic) capacity zero.*

However:

Theorem 1.21. *If $E \subset \partial\mathbb{D}$ is a closed set of capacity zero, then there is $f \in D$ having radial limit zero precisely on E.*

This suggests regarding sets of capacity zero as negligible sets.

Definition 1.22. A property which holds off a set of capacity zero will be said to hold *quasi-everywhere*. A set $E \subset \partial\mathbb{D}$ is *quasi-closed* if there are open subsets of $\partial\mathbb{D}$ of arbitrarily small capacity whose complements in E are closed. The set E *quasi-contains* the set F if $F \setminus E$ has capacity zero. A function f on $\partial\mathbb{D}$ is *quasi-continuous* if the restriction of f to $\partial\mathbb{D} \setminus U$ is continuous for open sets $U \subset \partial\mathbb{D}$ of arbitrarily small capacity. A *quasi-support* of a positive measure μ on $\partial\mathbb{D}$ is a quasi-closed set E such that $\mu(\partial\mathbb{D} \setminus E) = 0$ and E is "quasi-minimal" with respect to this property; that is, if F is quasi-closed and $\mu(\partial\mathbb{D} \setminus F) = 0$, then F quasi-contains E.

It will be necessary to use a different capacity, one used by Richter, W. Ross, and Sundberg [15, page 19]. The definition makes use of the *harmonic Dirichlet space*: Recall equation (1.1) for the Dirichlet integral:

$$D(f) = \int_{\mathbb{D}} |f'|^2 \, dA = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right|^2 \frac{dt}{2\pi} \frac{d\theta}{2\pi}.$$

The Dirichlet integral can be defined for harmonic functions, becoming

$$D(f) = \int_{\mathbb{D}} \left(\left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}} \right|^2 \right) dA = \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right|^2 \frac{dt}{2\pi} \frac{d\theta}{2\pi}. \tag{1.7}$$

A harmonic function f on $\mathbb{D}$ for which the first integral of (1.7) is finite will have an almost-everywhere-defined boundary function satisfying (1.7). Conversely, a function f on $\partial\mathbb{D}$ for which the right side of (1.7) is finite can be extended using the Poisson integral to a harmonic function on $\mathbb{D}$ satisfying (1.7).

Define the *harmonic Dirichlet space* $\mathcal{D}$ to be the set of functions on $\partial\mathbb{D}$ for which the Dirichlet integral is finite. Define the norm by

$$\|f\|_{\mathcal{D}}^2 = D(f) + \frac{1}{2}|f(0)|^2.$$

Define a capacity of a set $E \subset \partial\mathbb{D}$ by:

$$\operatorname{cap}(E) = \inf\{\|f\|_{\mathcal{D}}^2 : f \in \mathcal{D},\ 0 \le f \le 1, \text{ and } f = 1 \text{ quasi-everywhere on } E\}. \tag{1.8}$$

This capacity is comparable to the square of the outer logarithmic capacity (see [15]), so the properties of Definition 1.22 are the same for the two capacities.

Note that because $D \subset H^2$, a function in D having radial limit zero on a set of positive Lebesgue measure must be the zero function. Hence we let $E \subset \partial\mathbb{D}$ be a set of positive capacity and Lebesgue measure zero. Define

$$\mathcal{N} = \mathcal{N}_E = \{f \in D : f|_E = 0 \text{ quasi-everywhere}\}. \tag{1.9}$$

Then $\mathcal{N}$ is an invariant subspace of D (see [5, p. 295]). The goal of this section is to determine those E for which $\mathcal{N}_E \neq \{0\}$, and to produce a generator of $\mathcal{N}$.

As shown by B. Fuglede [10], there is a quasi-closed set E^* that quasi-contains E, and that is quasi-contained in all quasi-closed sets that quasi-contain E. Such a set E^* is known as a *quasi-closure* of E; note that every set whose symmetric difference with such an E^* has capacity zero is also a quasi-closure of E.

Lemma 1.23. *If E^* is a quasi-closure of E, then $\mathcal{N}_E = \mathcal{N}_{E^*}$.*

Proof. For $f \in D$, let $Z(f)$ be the set of points in $\partial\mathbb{D}$ where the radial limit of f is zero. Suppose $f \in \mathcal{N}_{E^*}$; then $Z(f)$ quasi-contains E^*. Since $\operatorname{cap}(E \setminus E^*) = 0$, $Z(f)$ quasi-contains E. Hence $f \in \mathcal{N}_E$.

Conversely, suppose $f \in \mathcal{N}_E$. It is shown by Richter, Ross, and Sundberg [15] that functions in the harmonic Dirichlet space $\mathcal{D}$ are quasi-continuous; hence the boundary function of $f \in D$ is quasi-continuous. Since $\{0\}$ is closed, it follows that $Z(f)$ is a quasi-closed set, which quasi-contains E by the assumption on f. Then by the definition of quasi-closure, $Z(f)$ quasi-contains E^*. Thus $f \in \mathcal{N}_{E^*}$. $\square$

Now fix a quasi-closure E^* of E. Let μ be a positive measure on $\partial\mathbb{D}$ such that E^* is a quasi-support of μ, the functional $g \mapsto \int g\, d\mu$ is bounded on $\mathcal{D}$, and $\operatorname{cap}(E^*) = \mu(\partial\mathbb{D})$. The existence of such a measure is shown in [15].

Let $\hat{\nu}(n)$ denote the Fourier-Stieltjes coefficient $\int e^{-in\theta}\, d\nu(e^{i\theta})$ of a measure ν. Let $g\mu$ be the measure ν such that $d\nu = g\, d\mu$. Then for each $j \in \mathbb{N}$, the functional $g \mapsto \widehat{(g\mu)}(-j)$ is bounded on D:

$$|\widehat{(g\mu)}(-j)| = \left| \int g(e^{i\theta}) e^{ij\theta}\, d\mu(e^{i\theta}) \right| \leq C\|z^j g\|_{\mathcal{D}} \leq C\|z^j g\|_D \leq C\|M_z\|^j \|g\|_D,$$

since the multiplication operator M_z is bounded on D. Let $\varphi_j \in D$ represent this functional; explicitly, $\varphi_j(z) = \sum_{n=0}^{\infty} \frac{\hat{\mu}(n+j)}{n+1} z^n$. In order to use the construction of Theorem 1.10, it must be shown that $\Phi = \{\varphi_j : j \in \mathbb{N}\}$ is linearly independent and that $\Phi^\perp = \mathcal{N}$.

Lemma 1.24. *Φ is linearly independent in D.*

Proof. Suppose $\sum_{j=1}^{n} \lambda_j \varphi_j = 0$. Let $p(z) = \sum \overline{\lambda_j} z^j$. Then for $g \in D$,

$$0 = \langle g, \sum \lambda_j \varphi_j \rangle = \sum \overline{\lambda_j} \int g(e^{i\theta}) e^{ij\theta}\, d\mu(e^{i\theta}) = \int gp\, d\mu. \tag{1.10}$$

Choosing $g(z) = z^n$ for each $n \in \mathbb{N}$, from (1.10) we conclude that the measure $p\mu$ is absolutely continuous with respect to Lebesgue measure, by the F. and M. Riesz Theorem. However, since μ is supported on a set of Lebesgue measure zero, $p\mu$ is the zero measure. Therefore $p = 0$ $[\mu]$-almost everywhere. Since p has at most n zeroes and finite sets have capacity zero, p must be the zero polynomial. Therefore Φ is linearly independent. $\square$

Now suppose $g \perp \Phi$. Then $g\mu$ is the zero measure, so $g = 0$ $[\mu]$-almost everywhere. As above, $Z(g)$ is quasi-closed. Since $\mu(\partial\mathbb{D} \setminus Z(g)) = 0$ and E^* is a quasi-support of μ, $Z(g)$ quasi-contains E^*. Thus $g \in \mathcal{N}_{E^*} = \mathcal{N}$.

Let $\psi = 1 = k_0^0$; as before, if $\psi \perp \mathcal{N}$ then $\mathcal{N} = \{0\}$. The proof of Theorem 1.10 may now be applied to prove the following:

Theorem 1.25. *$\mathcal{N} \neq \{0\}$ iff $\lim S_n < 1$.*

The construction also produces a wandering vector and generator of $\mathcal{N}$.

In the case of the Dirichlet space D, the construction of Theorem 1.10 has produced a generator of all invariant subspaces determined by a zero set in $\mathbb{D}$, or more generally by an inner function, and those determined by a zero set in $\partial\mathbb{D}$. Another type of invariant subspace is an intersection of two invariant subspaces, one of each type. Suppose $\mathcal{M} = uH^2 \cap D$ for some inner function u and $\mathcal{N} = \mathcal{N}_E$ for some $E \subset \partial\mathbb{D}$ of positive capacity and Lebesgue measure zero. Let Φ_1, Φ_2 be linearly independent subsets of $\mathcal{H}$ such that $\Phi_1^\perp = \mathcal{M}$ and $\Phi_2^\perp = \mathcal{N}$, as defined above. Let $\Phi = \Phi_1 \cup \Phi_2$. Then $\Phi^\perp = \mathcal{M} \cap \mathcal{N}$. That Φ is linearly independent follows from the fact that an inner product with an element of Φ_1 is an integral with respect to an absolutely continuous measure, while an inner product with an element of Φ_2 is an integral with respect to a singular measure, and no nontrivial linear combination of an absolutely continuous measure and a singular measure can be zero. As before, $\psi = 1$ gives $\psi \perp (\mathcal{M} \cap \mathcal{N}) \Rightarrow \mathcal{M} \cap \mathcal{N} = \{0\}$. Then the proof of Theorem 1.10 produces a generator of $\mathcal{M} \cap \mathcal{N}$, as well as a necessary and sufficient condition for $\mathcal{M} \cap \mathcal{N}$ to be nontrivial.

Richter and Sundberg [17] have shown that every invariant subspace of D is of the form $uH^2 \cap \mathcal{N}$, where u is an inner function and $\mathcal{N}$ contains an outer function. It has been conjectured that any such $\mathcal{N}$ is of the form $\mathcal{N}_E$ as in (1.9), for some E. In light of Richter and Sundberg's results, this conjecture can be reformulated to say that the above constructions produce all wandering vectors of D.

Chapter 2

Toeplitz Operators on Dirichlet-type Spaces

In this chapter, we study operators of the type $f \mapsto P(\varphi f)$ on Dirichlet-type spaces $D(\mu)$ (see Example 1.8), where φ is a function on $\mathbb{D}$ or $\partial\mathbb{D}$ and P is a projection. These operators are variants of the classical Toeplitz operators on H^2, and will be referred to as Toeplitz operators. The function φ is called the *symbol* of the operator $f \mapsto P(\varphi f)$, which will be denoted T_φ.

The properties of a Toeplitz operator can depend both on its symbol and on the projection P used in the definition of the operator. There are several possible projections that can be used to define Toeplitz operators on $D(\mu)$.

Example 2.1. Let L^2_a be the Bergman space, a subspace of $L^2(\mathbb{D})$. Let P_B be the orthogonal projection of $L^2(\mathbb{D})$ onto L^2_a, known as the *Bergman projection*. It can be expressed as an integral operator, or in terms of reproducing kernels:

$$(P_B f)(z) = \int f(w) \frac{1}{(1 - z\overline{w})^2}\, dA(w) = \langle f, k_z^B \rangle_{L^2(\mathbb{D})}, \tag{2.1}$$

where dA denotes normalized Lebesgue measure on $\mathbb{D}$.

If φ is a function on $\mathbb{D}$ such that $\varphi D(\mu) \subset L^2(\mathbb{D})$, then a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_B(\varphi f)$.

Example 2.2. The Hardy space H^2 can be identified with a subspace of $L^2(\partial\mathbb{D})$, with radial limits transforming an H^2 function on $\mathbb{D}$ to its boundary function, and the Poisson integral doing the reverse. The orthogonal projection of $L^2(\partial\mathbb{D})$ onto $H^2(\mathbb{D})$ is known as the *Szegö projection*, and will be denoted P_H. Like P_B, the Szegö projection can be expressed as an integral operator or in terms of reproducing kernels:

$$(P_H f)(z) = \int f(e^{i\theta}) \frac{1}{1 - ze^{-i\theta}} \frac{d\theta}{2\pi} = \langle f, k_z^{H^2} \rangle_{L^2(\partial\mathbb{D})}.$$

Since $D(\mu) \subset H^2$, every element of $D(\mu)$ has a boundary function defined almost everywhere on $\partial\mathbb{D}$. Then if φ is a function on $\partial\mathbb{D}$ such that $\varphi D(\mu) \subset L^2(\partial\mathbb{D})$, a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_H(\varphi f)$.

Before proceeding with the next example, a harmonic analogue of $D(\mu)$ will be defined.

In the sequel, if ν is a measure on $\partial\mathbb{D}$, then $P\nu$ denotes the *Poisson integral* of ν, the integral with respect to ν of the Poisson kernel: $(P\nu)(z) = \int \frac{1-|z|^2}{|z-\lambda|^2}\, d\nu(\lambda)$. Note that the Poisson kernel itself is the Poisson integral of the point mass δ_λ. If g is a function on $\partial\mathbb{D}$, then Pg denotes the Poisson integral of the measure $g\frac{d\theta}{2\pi}$.

If $\lambda \in \partial\mathbb{D}$ and δ_λ denotes the point mass at λ, then from the definitions in Examples 1.7 and 1.8 it follows that $f \in D(\delta_\lambda)$ iff f has a finite local Dirichlet integral at λ. The following criterion of Richter and Sundberg [16] for $D_\lambda(f)$ to be finite will be useful:

Proposition 2.3. *Let $\lambda \in \partial\mathbb{D}$. Then a function $f \in D(\delta_\lambda)$ iff $f = \alpha + (z - \lambda)f_\lambda$ for some constant α and function $f_\lambda \in H^2$. If this is the case, then α is the radial limit $f(\lambda)$ of f at λ, and $D_\lambda(f) = \|f_\lambda\|_2^2$.*

Remark. It is shown in [16] that in fact, if $f \in D(\delta_\lambda)$ then $f(z) \to f(\lambda)$ as z approaches λ within any disc tangent to $\partial\mathbb{D}$ at λ. Also, P. Chernoff [7] has shown that if $D_\lambda(f) < \infty$, then the Fourier series of f at λ converges to $f(\lambda)$.

Corollary 2.4. $D(\mu) \subset H^2$.

Proof. Let $f \in D(\mu)$. Since $\int D_\lambda(f)\, d\mu(\lambda)$ is finite, there is at least one λ such that $f \in D(\delta_\lambda)$. For any such λ, by the proposition there are $\alpha \in \mathbb{C}$ and $f_\lambda \in H^2$ such that $f = \alpha + (z - \lambda)f_\lambda$. Therefore $f \in H^2$. $\square$

The following analogue of the Douglas formula (1.1) for the Dirichlet integral will be used, and is proved by Richter and Sundberg in [16].

Proposition 2.5. *If $f \in H^2$, then*

$$\int_{\partial\mathbb{D}} D_\lambda(f)\, d\mu(\lambda) = \int_{\mathbb{D}} |f'|^2 P\mu\, dA. \tag{2.2}$$

Like the Douglas formula, equation (2.2) can be extended to harmonic functions.

Proposition 2.6. *Let f be a harmonic function on $\mathbb{D}$ of the form $f = f_+ + f_-$, where f_+, $\overline{f_-} \in D(\mu)$ and $f_-(0) = 0$. Then*

$$\int_{\partial\mathbb{D}} D_\lambda(f)\, d\mu(\lambda) = \int_{\mathbb{D}} \left(\left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \overline{z}} \right|^2 \right) P\mu\, dA = \int_{\mathbb{D}} (|f_+'|^2 + |\overline{f_-}'|^2) P\mu\, dA. \tag{2.3}$$

Proof. For $\lambda \in \partial\mathbb{D}$ and functions $g, h \in D(\delta_\lambda)$, define

$$D_\lambda(g,h) = \int_{\partial\mathbb{D}} \frac{g(\lambda) - g(e^{it})}{\lambda - e^{it}} \overline{\left(\frac{h(\lambda) - h(e^{it})}{\lambda - e^{it}} \right)} \frac{dt}{2\pi}.$$

Then $D_\lambda(\cdot,\cdot)$ is a sesquilinear form, and $D_\lambda(g) = D_\lambda(g,g)$. Hence

$$D_\lambda(f) = D_\lambda(f_+ + f_-, f_+ + f_-) = D_\lambda(f_+) + 2\,\mathrm{Re}\, D_\lambda(f_+, f_-) + D_\lambda(f_-). \tag{2.4}$$

Since $D_\lambda(f_-) = D_\lambda(\overline{f_-})$, the proposition will follow by integrating (2.4) with respect to μ and applying Proposition 2.5, once it is shown that $D_\lambda(f_+, f_-) = 0$ for $[\mu]$-almost every λ.

Since f_+, $\overline{f_-} \in D(\mu)$, both belong to $D(\delta_\lambda)$ for $[\mu]$-almost every λ; fix such a λ. By Proposition 2.3, choose g_+, $g_- \in H^2$ such that $f_+ = f_+(\lambda) + (z-\lambda)g_+$ and $\overline{f_-} = \overline{f_-}(\lambda) + (z-\lambda)g_-$. Then

$$\begin{aligned}
D_\lambda(f_+, f_-) &= \int_{\partial\mathbb{D}} \frac{f_+(\lambda) - f_+(e^{it})}{\lambda - e^{it}} \overline{\left(\frac{f_-(\lambda) - f_-(e^{it})}{\lambda - e^{it}} \right)} \frac{dt}{2\pi} \\
&= \int_{\partial\mathbb{D}} \frac{f_+(\lambda) - f_+(e^{it})}{\lambda - e^{it}} \, \frac{\overline{f_-}(\lambda) - \overline{f_-}(e^{it})}{\lambda - e^{it}} \, \frac{\lambda - e^{it}}{\overline{\lambda} - e^{-it}} \, \frac{dt}{2\pi} \\
&= \int_{\partial\mathbb{D}} g_+(e^{it}) g_-(e^{it}) (-\lambda e^{it}) \frac{dt}{2\pi} \\
&= 0.
\end{aligned}$$

$\square$

Define the harmonic Dirichlet-type space $\mathcal{D}(\mu)$ to be the set of functions $f \in L^2(\partial\mathbb{D})$ such that $D_\lambda(f)$ is integrable with respect to μ. For such an f, the harmonic extension $f(z) = (Pf)(z)$ to $\mathbb{D}$ satisfies (2.3); in the usual way, elements of $\mathcal{D}(\mu)$ can be regarded as functions on $\partial\mathbb{D}$ or as functions on $\mathbb{D}$. Define the norm by $\|f\|^2 = \int D_\lambda(f)\, d\mu(\lambda) + \|f\|^2_{L^2(\partial\mathbb{D})}$.

Proposition 2.7. *$\mathcal{D}(\mu)$ is a reproducing-kernel Hilbert space containing $D(\mu)$ as a closed subspace.*

Proof. Suppose $f \in \mathcal{D}(\mu)$; write $f = f_+ + f_-$, with $f_+, \overline{f_-} \in D(\mu)$ and $f_-(0) = 0$. If $w \in \mathbb{D}$, then by the existence of H^2 reproducing kernels,

$$\begin{aligned} |f(w)| &= |f_+(w) + f_-(w)| \le |f_+(w)| + |\overline{f_-}(w)| \\ &\le \|k_w^{H^2}\|_{H^2}(\|f_+\|_{H^2} + \|\overline{f_-}\|_{H^2}) \le C\|f\|_{L^2(\partial\mathbb{D})} \\ &\le C\|f\|_{\mathcal{D}(\mu)}. \end{aligned}$$

Thus, the functional of evaluation at w is bounded on $\mathcal{D}(\mu)$, as was to be proved.

Similarly, for $k \in \mathbb{N}$

$$|\hat{f}(-k)| = |\hat{f}_-(-k)| = |\widehat{\overline{f_-}}(k)| = |\overline{f_-}^{(k)}(0)|/k! \le C\|\overline{f_-}\|_{H^2} \le C\|f\|_{\mathcal{D}(\mu)}.$$

Therefore if $\{f_n\}$ is a sequence in $D(\mu)$ converging in $\mathcal{D}(\mu)$ to f, then $\hat{f}(-k) = 0$ for all $k \in \mathbb{N}$. Thus $D(\mu)$ is closed in $\mathcal{D}(\mu)$. □

Example 2.8. If φ is a function on $\partial\mathbb{D}$ such that $\varphi D(\mu) \subset \mathcal{D}(\mu)$, then a Toeplitz operator T_φ can be defined on $D(\mu)$ by $T_\varphi f = P_\mu(\varphi f)$, where P_μ is the orthogonal projection of $\mathcal{D}(\mu)$ onto $D(\mu)$.

There are advantages to using each of the projections in Examples 2.1, 2.2, and 2.8. The Bergman projection can be used for the largest collection of symbols, as the requirement that $\varphi D(\mu) \subset L^2(\partial\mathbb{D})$ is the weakest requirement among the three. Using the Szegö projection has the advantage of giving rise to the best-understood sort of Toeplitz operator. The theory of Toeplitz operators is most often studied in settings where the range of the projection is the domain of the operator; such is the case if $P = P_\mu$.

2.1 Bergman Toeplitz Operators on D

The problem to be studied in this section is to determine the symbols φ for which the Toeplitz operator $T_\varphi f = P_B(\varphi f)$ is bounded on D. It will be assumed that φ is a bounded, harmonic function on $\mathbb{D}$.

The Bergman projection is one of a family of projections of $L^2(\mathbb{D})$ onto L^2_a. For $\alpha > -1$, define the operator P_α by:

$$(P_\alpha f)(z) = (\alpha+1)\int \frac{(1-|w|^2)^\alpha}{(1-z\overline{w})^{\alpha+2}} f(w)\,dA(w).$$

Clearly $P_0 = P_B$. If $1 \le p < \infty$ and $p(\alpha+1) > 1$, then P_α is bounded on $L^p(\mathbb{D})$ and fixes the holomorphic functions in $L^p(\mathbb{D})$, as shown in [22, section 4.2].

The main result of this section hinges on the following lemma.

Lemma 2.9. *For $f \in D$, $(T_\varphi f)' = \frac{\partial\varphi}{\partial z} f + P_1(\varphi f')$.*

Note that if $\frac{\partial\varphi}{\partial z} f \in L^2(\mathbb{D})$, then the right side of the equation is $P_1\big(\frac{\partial}{\partial z}(\varphi f)\big)$. Thus the lemma says that in a restricted sense, differentiation intertwines P_B and P_1.

Proof. First, the lemma will be verified in the case of $\varphi(z) = \overline{z}^m$ and $f(z) = z^n$:

$$\begin{aligned}(T_\varphi f)(z) = P_B(\varphi f)(z) &= \int \frac{\overline{w}^m w^n}{(1-z\overline{w})^2}\,dA(w)\\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{r^m e^{-im\theta} r^n e^{in\theta}}{(1-zre^{-i\theta})^2} r\,dr\,d\theta\\ &= \frac{1}{\pi}\int_0^1 r^{m+n+1}\int_0^{2\pi}\frac{e^{i(n-m+2)\theta}}{(e^{i\theta}-rz)^2}\,d\theta\,dr\\ &= \frac{1}{\pi i}\int_0^1 r^{m+n+1}\int_{\partial\mathbb{D}}\frac{\zeta^{n-m+1}}{(\zeta-rz)^2}\,d\zeta\,dr.\end{aligned}$$

A residue calculation shows that the contour integral $\int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta-rz)^2}\,d\zeta$ is zero if $n-m+1 \le 0$, and is otherwise $2\pi i(n-m+1)(rz)^{n-m}$. Hence

$$(T_\varphi f)(z) = 2(n-m+1)z^{n-m}\int r^{2n+1}\,dr = \frac{n-m+1}{n+1} z^{n-m} \tag{2.5}$$

if $n - m \ge 0$ and zero otherwise.

The derivative of $T_\varphi f(z)$ is to be compared with:

$$\begin{aligned} P_1(\varphi f')(z) &= 2\int \frac{1-|w|^2}{(1-z\overline{w})^3} n\overline{w}^m w^{n-1}\, dA(w) \\ &= \frac{2n}{\pi i}\int_0^1 (1-r^2) r^{m+n} \int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta - rz)^3}\, d\zeta\, dr. \end{aligned}$$

Since $\int_{\partial\mathbb{D}} \frac{\zeta^{n-m+1}}{(\zeta-rz)^3}\, d\zeta = \pi i(n-m+1)(n-m)(rz)^{n-m-1}$ if $n-m \geq 1$ and is zero otherwise,

$$\begin{aligned} P_1(\varphi f')(z) &= 2n(n-m+1)(n-m)z^{n-m-1}\int (r^{2n-1} - r^{2n+1})\, dr \\ &= \frac{(n-m+1)(n-m)}{n+1} z^{n-m-1} \end{aligned}$$

for $n-m \geq 1$ and is otherwise zero. Comparing this with the derivative of the right side of (2.5), we see that the lemma holds in this case.

Now let φ be any bounded, harmonic function on $\mathbb{D}$, and f any element of D. Define φ_+ by $\varphi_+(z) = \sum_{n=0}^\infty \hat{\varphi}(n)z^n$; let $\varphi_- = \varphi - \varphi_+$. Both φ_+ and φ_- belong to $L^2(\mathbb{D})$, but they need not be bounded functions.

Since φ is bounded and $f \in D \subset L^2$, the sum $\varphi \sum \hat{f}(n)z^n$ converges in L^2 norm. Then since P_B is bounded on L^2,

$$\begin{aligned} P_B(\varphi f) &= \sum \hat{f}(n)P_B(\varphi z^n) = \sum \hat{f}(n)(\varphi_+ z^n + P_B(\varphi_- z^n)) \\ &= \varphi_+ f + \sum_{n=0}^\infty \hat{f}(n) \sum_{m=1}^\infty \hat{\varphi}(-m)P_B(\overline{z}^m z^n). \end{aligned}$$

Since φ is harmonic, $\varphi_+' = \frac{\partial\varphi}{\partial z}$; hence

$$P_B(\varphi f)' = \frac{\partial\varphi}{\partial z} f + \varphi_+ f' + \sum_{n=0}^\infty \hat{f}(n)\sum_{m=1}^\infty \hat{\varphi}(-m)P_B(\overline{z}^m z^n)'. \tag{2.6}$$

Similarly,

$$\begin{aligned} P_1(\varphi f') &= \sum n\hat{f}(n)P_1(z^{n-1}\varphi) \\ &= \sum_{n=1}^\infty n\hat{f}(n)(z^{n-1}\varphi_+ + \sum_{m=1}^\infty \hat{\varphi}(-m)P_1(\overline{z}^m z^{n-1})) \\ &= \varphi_+ f' + \sum_{n=1}^\infty \hat{f}(n)\sum_{m=1}^\infty \hat{\varphi}(-m)P_1(n\overline{z}^m z^{n-1}). \end{aligned} \tag{2.7}$$

Since $P_B(\overline{z}^m z^n)' = P_1(n\overline{z}^m z^{n-1})$ for each m and n, the lemma follows by comparing (2.7) with (2.6), and observing that the $n = 0$ term of the sum in (2.6) is zero, since $P_B(\overline{z}^m) = 0$ for all $m \geq 1$. $\square$

Theorem 2.10. *The Toeplitz operator T_φ is bounded on D iff*

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2 \, dA \leq C\|f\|_D^2 \tag{2.8}$$

for all $f \in D$, for some constant C not depending on f.

Proof. Suppose that (2.8) holds. Then since P_1 is bounded on L^2 and φ is a bounded function,

$$\begin{aligned} D(T_\varphi f) = \int |(T_\varphi f)'|^2 \, dA &= \int \left|\frac{\partial\varphi}{\partial z} f + P_1(\varphi f')\right|^2 \, dA \\ &\leq 2\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2 \, dA + 2\|P_1\|^2\|\varphi\|_\infty^2\|f'\|_{L^2}^2 \\ &\leq 2(C + \|P_1\|^2\|\varphi\|_\infty^2)\|f\|_D^2. \end{aligned}$$

Also,

$$|(T_\varphi f)(0)| = \left|\int \varphi f \, dA\right| \leq \|\varphi\|_\infty\|f\|_{L^2} \leq \|\varphi\|_\infty\|f\|_D.$$

Since

$$\|T_\varphi f\|_{H^2}^2 \leq |(T_\varphi f)(0)|^2 + D(T_\varphi f),$$

it follows that T_φ is bounded on D.

Conversely, suppose that T_φ is bounded. Then by the lemma,

$$\begin{aligned} \int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2 \, dA &\leq 2\|(T_\varphi f)'\|_2^2 + 2\|P_1(\varphi f')\|_2^2 \\ &\leq 2\|T_\varphi f\|_D^2 + 2\|P_1\|^2\|\varphi\|_\infty^2\|f'\|_2^2 \\ &\leq 2\|T_\varphi\|^2\|f\|_D^2 + 2\|P_1\|^2\|\varphi\|_\infty^2\|f\|_D^2 \\ &= C\|f\|_D^2. \end{aligned}$$

$\square$

The condition of Theorem 2.10 is equivalent to that of $\frac{\partial\varphi}{\partial z}$ being a multiplier of D into L^2_a; that is, $\frac{\partial\varphi}{\partial z}D \subset L^2_a$. The condition also says that $\left|\frac{\partial\varphi}{\partial z}\right|^2 dA$ is a *D-Carleson measure* on $\mathbb{D}$. Compare with the following theorem of D. Stegenga [21]:

Theorem 2.11. *A function g is a multiplier of D (into itself) iff $g \in H^\infty$ and $|g'|^2\,dA$ is a D-Carleson measure.*

Stegenga also gives a geometric characterization of D-Carleson measures in [21].

Since $\frac{\partial\varphi}{\partial z} = \varphi'_+$, from Theorems 2.10 and 2.11 it follows that if T_φ is bounded on D and φ_+ is a bounded function, then φ_+ is a multiplier of D. However, it is possible for T_φ to be bounded without φ_+ being bounded:

Example 2.12. Define the function g on $\mathbb{D}$ by $g(z) = \sum \frac{z^n}{n\log n \log\log n}$. Since $\sum n|\hat{g}(n)|^2$ is finite, $g \in D$. Since $D \subset BMOA$, the space of analytic functions having bounded mean oscillation on $\partial\mathbb{D}$ (see [20]), it follows from Fefferman's Theorem that we can choose a bounded, harmonic function φ such that $g = P_B\varphi = \varphi_+$. Since g is unbounded, φ_+ is not a multiplier of D. However, by a result of S. Axler and A. Shields [2], g' is a multiplier of D into L^2_a. Therefore T_φ is bounded on D.

2.2 Hardy Toeplitz Operators on $D(\mu)$

Let μ be a positive, finite Borel measure on $\partial\mathbb{D}$. In this section, the symbols $\varphi \in L^\infty(\partial\mathbb{D})$ for which the Toeplitz operator $T_\varphi f = P_H(\varphi f)$ is bounded on $D(\mu)$ will be determined.

Remark. Recall that $\|f\|^2_\mu = \|f\|^2_2 + \int D_\lambda(f)\,d\mu(\lambda)$. Since the projection P_H has norm one as an operator on $L^2(\partial\mathbb{D})$,

$$\|T_\varphi f\|_2 = \|P_H(\varphi f)\|_2 \le \|\varphi f\|_2 \le \|\varphi\|_{L^\infty(\partial\mathbb{D})}\|f\|_2 \le \|\varphi\|_\infty \|f\|_\mu.$$

Therefore T_φ is bounded on $D(\mu)$ iff $\int D_\lambda(T_\varphi f)\,d\mu(\lambda) \le C\|f\|^2_\mu$ for $f \in D(\mu)$ and some C not depending on f.

Fix $f \in D(\mu)$. Then $D_\lambda(f) < \infty$ for $[\mu]$-almost every $\lambda \in \partial\mathbb{D}$. For each such λ define $f_\lambda \in H^2$ as in Proposition 2.3.

Lemma 2.13. $\int D_\lambda(T_\varphi((z-\lambda)f_\lambda))\,d\mu(\lambda) \le \|\varphi\|_\infty^2\|f\|_\mu^2$.

Proof. The result hinges on a commutation relation obtained by following $T_\varphi((z-\lambda)f_\lambda)$ by $T_{\overline{z}}$, and using composition properties of H^2 Toeplitz operators:

$$\begin{aligned} T_{\overline{z}}T_\varphi((z-\lambda)f_\lambda) &= T_{\overline{z}}T_\varphi T_{z-\lambda}f_\lambda \\ &= T_{\overline{z}\varphi(z-\lambda)}f_\lambda \\ &= T_{(1-\overline{z}\lambda)\varphi}f_\lambda \\ &= T_{\overline{z}}(z-\lambda)T_\varphi f_\lambda. \end{aligned}$$

Subtracting the end from the beginning, we see that if $g = T_\varphi((z-\lambda)f_\lambda)-(z-\lambda)T_\varphi f_\lambda$, then $T_{\overline{z}}g = 0$. Since

$$T_{\overline{z}}g = P_H\left(e^{-i\theta}\sum_{n=0}^{\infty}\hat{g}(n)e^{in\theta}\right) = P_H\left(\sum_{n=-1}^{\infty}\hat{g}(n+1)e^{in\theta}\right) = \sum_{n=0}^{\infty}\hat{g}(n+1)e^{in\theta} = 0,$$

it follows that $\hat{g}(n+1) = 0$ for all $n \ge 0$. Thus g is constant, say with constant value α. Hence

$$T_\varphi((z-\lambda)f_\lambda) = \alpha + (z-\lambda)T_\varphi f_\lambda.$$

Then by Proposition 2.3, $T_\varphi((z-\lambda)f_\lambda) \in D(\delta_\lambda)$, and

$$D_\lambda(T_\varphi((z-\lambda)f_\lambda)) = \|T_\varphi f_\lambda\|_2^2 \le \|\varphi\|_\infty^2\|f_\lambda\|_2^2 = \|\varphi\|_\infty^2 D_\lambda(f).$$

Therefore

$$\int D_\lambda(T_\varphi((z-\lambda)f_\lambda))\,d\mu(\lambda) \le \|\varphi\|_\infty^2\int D_\lambda(f)\,d\mu(\lambda) \le \|\varphi\|_\infty^2\|f\|_\mu^2.$$

□

Remark. For each $\lambda \in \partial\mathbb{D}$, $D_\lambda(\cdot)^{1/2}$ is a seminorm on $D(\delta_\lambda)$, and hence satisfies the triangle inequality. Thus it follows from the lemma and the previous remark that T_φ is bounded on $D(\mu)$ iff $\int D_\lambda(T_\varphi(f(\lambda)))\,d\mu(\lambda) \le C\|f\|_\mu^2$, for some C not depending on f.

Theorem 2.14. *T_φ is bounded on $D(\mu)$ iff*

$$\int\left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2\mu)\,dA \le C\|f\|_\mu^2,$$

for $f \in D(\mu)$ and some constant C not depending on f.

If φ satisfies the condition of the theorem, the measure $\left|\frac{\partial\varphi}{\partial z}\right|^2 dA$ will be called a *μ-Carleson measure.*

Proof. Following the previous remark, we fix $\lambda \in \partial\mathbb{D}$ and calculate the local Dirichlet integral at λ of $T_\varphi(f(\lambda))$:

$$\begin{aligned} D_\lambda(T_\varphi(f(\lambda))) &= \int D_\zeta(T_\varphi(f(\lambda)))\, d\delta_\lambda(\zeta) \\ &= \int |(P_H(\varphi f(\lambda)))'|^2 P\delta_\lambda\, dA \\ &= |f(\lambda)|^2 \int \left|\frac{\partial\varphi}{\partial z}\right|^2 P\delta_\lambda\, dA, \end{aligned}$$

where the last occurrence of φ denotes the harmonic extension of φ to $\mathbb{D}$.

Integrating with respect to μ gives

$$\begin{aligned} \int D_\lambda(T_\varphi(f(\lambda)))\, d\mu(\lambda) &= \int |f(\lambda)|^2 \int \left|\frac{\partial\varphi}{\partial z}\right|^2 \frac{1-|z|^2}{|z-\lambda|^2}\, dA(z)\, d\mu(\lambda) \\ &= \int \left|\frac{\partial\varphi}{\partial z}\right|^2 \int |f(\lambda)|^2 \frac{1-|z|^2}{|z-\lambda|^2}\, d\mu(\lambda)\, dA(z) \\ &= \int \left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2\mu)\, dA. \end{aligned}$$

The theorem now follows from the previous remark. $\square$

In the case of the Dirichlet space $D = D(\frac{d\theta}{2\pi})$, the theorem says that the Hardy Toeplitz operator T_φ is bounded on D iff

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 P(|f|^2)\, dA \le C\|f\|_D^2. \tag{2.9}$$

Compare this with Theorem 2.10, which says that the Bergman Toeplitz operator T_φ is bounded on D iff

$$\int \left|\frac{\partial\varphi}{\partial z}\right|^2 |f|^2\, dA \le C\|f\|_D^2. \tag{2.10}$$

However,

$$\||f|^2 - P(|f|^2)\|_\infty \le C_1\|f\|_{BMO}^2 \le C_2\|f\|_D^2,$$

the first inequality being due to A. Garsia (see [12, p. 221]), the second to Stegenga [20]. Therefore the two conditions (2.9) and (2.10) are equivalent.

Stegenga's Theorem 2.11 characterizing the multipliers of D can be generalized to the harmonic Dirichlet-type space $\mathcal{D}(\mu)$:

Theorem 2.15. *A bounded function φ on $\partial\mathbb{D}$ is a multiplier of $\mathcal{D}(\mu)$ iff $\left|\frac{\partial\varphi}{\partial z}\right|^2 dA$ and $\left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2 dA$ are μ-Carleson measures.*

Proof. Suppose that φ is a multiplier of $\mathcal{D}(\mu)$. By the existence of reproducing kernels in $\mathcal{D}(\mu)$, norm convergence of sequences in $\mathcal{D}(\mu)$ implies pointwise convergence. By the Cauchy-Schwarz inequality, if $f \in \mathcal{D}(\mu)$ then $|P(\varphi f)|^2 \leq P|\varphi|^2 \cdot P|f|^2$. It then follows from the closed-graph theorem that the operator M_φ of multiplication by φ is bounded on $\mathcal{D}(\mu)$.

Let $f \in \mathcal{D}(\mu)$. Then

$$\frac{\varphi(\lambda)f(\lambda) - \varphi(e^{it})f(e^{it})}{\lambda - e^{it}} = f(\lambda)\frac{\varphi(\lambda) - \varphi(e^{it})}{\lambda - e^{it}} + \varphi(e^{it})\frac{f(\lambda) - f(e^{it})}{\lambda - e^{it}}. \tag{2.11}$$

Hence

$$\begin{aligned}\int |f(\lambda)|^2 D_\lambda(\varphi)\, d\mu(\lambda) &\leq 2\|\varphi\|_\infty^2 \int D_\lambda(f)\, d\mu(\lambda) + 2\int D_\lambda(\varphi f)\, d\mu(\lambda)\\ &\leq 2(\|\varphi\|_\infty^2 + \|M_\varphi\|^2)\|f\|_{\mathcal{D}(\mu)}^2.\end{aligned}$$

Since $\varphi = \varphi \cdot 1 \in \mathcal{D}(\mu)$, by Proposition 2.6

$$\begin{aligned}\int |f(\lambda)|^2 D_\lambda(\varphi)\, d\mu(\lambda) &= \int |f(\lambda)|^2 \int_{\mathbb{D}} \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2\right) P\delta_\lambda\, dA(z)\, d\mu(\lambda)\\ &= \int \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2\right) P(|f|^2\mu)\, dA.\end{aligned}$$

Therefore $\left|\frac{\partial\varphi}{\partial z}\right|^2 dA$ and $\left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2 dA$ are μ-Carleson measures.

Conversely, suppose that $\left|\frac{\partial\varphi}{\partial z}\right|^2 dA$ and $\left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2 dA$ are μ-Carleson measures. Since $\frac{\partial\varphi}{\partial z} = \varphi_+'$ and $\overline{\frac{\partial\varphi}{\partial \bar{z}}} = \overline{\varphi_-}'$, applying the μ-Carleson condition with $f = 1$ gives that $\varphi_+, \overline{\varphi_-} \in D(\mu)$. Thus $\varphi \in \mathcal{D}(\mu)$. Then by (2.11) and Proposition 2.6,

$$\begin{aligned}\int D_\lambda(\varphi f)\, d\mu(\lambda) &\leq 2\int |f(\lambda)|^2 D_\lambda(\varphi)\, d\mu(\lambda) + 2\|\varphi\|_\infty^2 \int D_\lambda(f)\, d\mu(\lambda)\\ &= \int \left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial \bar{z}}\right|^2\right) P(|f|^2\mu)\, dA + 2\|\varphi\|_\infty^2 \int D_\lambda(f)\, d\mu(\lambda)\\ &\leq C\|f\|_{\mathcal{D}(\mu)}^2.\end{aligned}$$

Therefore φ is a multiplier of $\mathcal{D}(\mu)$. $\square$

Corollary 2.16. *A holomorphic function φ on $\mathbb{D}$ is a multiplier of $D(\mu)$ iff φ is bounded and $|\varphi'|^2\,dA$ is a μ-Carleson measure.*

Proof. Suppose φ is a multiplier of $D(\mu)$. That $|\varphi'|^2\,dA$ is a μ-Carleson measure follows as in the proof of the theorem, with $\mathcal{D}(\mu)$ replaced with $D(\mu)$, and noting that $\frac{\partial\varphi}{\partial\bar{z}} = 0$. That φ is bounded follows from the existence of reproducing kernels in $D(\mu)$: as above, M_φ is bounded on $D(\mu)$. Then

$$|\varphi(w)|\|k_w\|^2 = |\varphi(w)k_w(w)| = |\langle\varphi k_w, k_w\rangle| \leq \|\varphi k_w\|\|k_w\| \leq \|M_\varphi\|\|k_w\|^2;$$

thus $|\varphi|$ is bounded by $\|M_\varphi\|$ on $\mathbb{D}$.

If φ is bounded and $|\varphi'|^2\,dA$ is a μ-Carleson measure, then since $\frac{\partial\varphi}{\partial\bar{z}} = 0$ and $\varphi' = \frac{\partial\varphi}{\partial z}$, the theorem gives that φ is a multiplier of $\mathcal{D}(\mu)$. Since φ is holomorphic, φ is a multiplier of $D(\mu)$. $\square$

The following connection between bounded Toeplitz operators and multipliers of $\mathcal{D}(\mu)$ is an immediate consequence of Theorems 2.14 and 2.15.

Corollary 2.17. *A function $\varphi \in L^\infty(\partial\mathbb{D})$ is a multiplier of $\mathcal{D}(\mu)$ iff T_φ and $T_{\overline{\varphi}}$ are bounded on $D(\mu)$.*

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