Image reconstruction from few views by non-convex optimization

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Abstract— **Image reconstruction for fan-beam computed tomography (CT) from projection data containing a small number of views is investigated. An iterative algorithm is developed that seeks to minimize the total p-variation of the reconstructed image subject to the constraint that the estimated projection data agree with the available data to within a specified data tolerance ϵ. A preliminary investigation on the dependence of image quality as a function of p and ϵ is performed.**

I. Introduction

RECENTLY , we have developed an iterative image reconstruction algorithm that can be effective in situations when projection data are collected at a small number of views [1], [2]. The algorithm seeks the image with the minimum total variation (TV) that agrees with the projection data to within a specified data tolerance, an approach first considered in Refs. [3], [4]. The image total variation is the ℓ_1 norm of the image gradient magnitude. This optimization problem tends to identify images with sparse gradients, and if such an assumption applies, the resulting reconstruction can be very accurate. In medical imaging the underlying object function can have an approximately sparse image gradient, because images are often slowly varying within particular organs, and rapid variations in the images are mainly confined to the boundaries between different tissues. In this work, we investigate a different optimization problem that may increase the effectiveness of reconstruction from sparse data.

Mathematically, image reconstruction from few views can be thought of as an underdetermined linear system. Under conditions of ideal data, there will not be a unique solution to this linear system. The solution space can, however, be narrowed down by making certain assumptions about the image. For example, requiring that the image pixels contain non-negative values can substantially reduce the set of possible images that are consistent with the data. Another powerful assumption is that the desired image is sparse in some representation, e.g., pixels. A sparse solution can be obtained by minimizing the ℓ_0 norm of the image, while constraining the image to be consistent with the available projection data. Minimization of the ℓ_0 norm, however, is a combinatorial optimization problem and not practical. It turns out that for linear constraints satisfying modest conditions, ℓ_0 minimization is equivalent to ℓ_1 minimization, which leads to a convex optimization problem for which there exist practical algorithms. Another interesting possibility is to minimize the ℓ_p norm where $0 < p < 1$, which, under ideal conditions and linear constraints, should yield the sparsest solution with fewer measurements than with $p = 1$. Such an optimization problem is non-convex and likely has many local solutions, but there may be an advantage in the case of data inconsistencies caused by noise.

In this work, we investigate image reconstruction by minimizing the ℓ_p norm of the image gradient magnitude, or total p-variation (TpV), constraining the image so that its projections are within a set tolerance of the available projection data. The results show that the TpV algorithm can yield reconstructed images substantially more accurate than our previous TV algorithm.

In Sec. II we state the optimization problem that we use for the non-convex reconstruction algorithm. In Sec. III, we heuristically argue why the non-convex TpV-norm may have some advantage over the convex TV-norm. Finally, in Sec. IV

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we show an example of image reconstruction by TpV-norm minimization in the context of few-view fan-beam CT scanning.

II. Method

For the TpV algorithm the optimization problem we seek to solve is:

$$\vec{f}^* = \text{argmin } \|\nabla f\|_{TpV} \text{ subject to } \|\mathbf{M}\vec{f} - \vec{g}\| \leq \epsilon. \tag{1}$$

The vector $\vec{f}$ denotes the pixel/voxel representation of the image; $\vec{g}$ is the discrete set of projection data; and $\mathbf{M}$ is the system matrix, which for the examples shown below is the ray-driven projection of the image matrix. The quantity $\|\nabla f\|_p$ is the p-norm of the image gradient magnitude, and for 2D images we define it by

$$\|\nabla f\|_p^p \equiv \sum_{i,j} \left(\sqrt{(f_{i,j} - f_{i-1,j})^2 + (f_{i,j} - f_{i,j-1})^2}\right)^p. \tag{2}$$

The optimization problem specified by Eqs. (1) and (2) depends on two parameters ϵ and p. When $p = 1$, TpV optimization reduces to TV optimization. The data constraint in Eq. (1) allows for the fact that the data $\vec{g}$ may be inconsistent, i.e. there may be no image $\vec{f}$ that satisfies $\mathbf{M}\vec{f} = \vec{g}$. Such inconsistency can occur when the projector model does not coincide with the actual projection or when the data contain noise. In general the minimum data tolerance ϵ_{min}, the minimum ϵ such that there exists an $\vec{f}$ satisfying $\|\mathbf{M}\vec{f} - \vec{g}\| \leq \epsilon$, is larger than zero. As ϵ increases, images with smaller TpV are obtained. In general, the standard TV norm will not identify the image with the sparsest image gradient, here, because the image constraint is ellipsoidal, not linear. Reducing p, however, brings Eq. (1) closer to the ideal $p = 0$ case that does yield the sparse image gradient solution. When the underlying image function satisfies the assumption of image gradient sparseness, the reduction of p to values below $p = 1$ can improve the accuracy of the image reconstruction.

III. Motivation for TpV-norm minimization

To motivate the use of the TpV-norm, we briefly describe the theory of sparse linear system inversion. If it is known that the underlying solution to a linear system is sparse in some representation, it may only be necessary to obtain a small set of measurements to find the solution. To find the sparsest solution that satisfies the available data, one needs to solve:

$$\vec{f}^* = \text{argmin } \|\vec{x}\|_0 \text{ subject to } \mathbf{A}\vec{x} = \vec{b}, \tag{3}$$

where $\mathbf{A}$ is a rectangular $M \times N$ matrix with $M < N$; $\vec{x}$ (length N) is the solution we seek, and $\vec{b}$ (length M) is the available data. The norm $\|\cdot\|_0$ is defined:

$$\|\vec{x}\|_0 \equiv \lim_{p \to 0} \sum_i |x_i|^p,$$

where p approaches zero from the positive real numbers. This norm essentially counts the number of non-zero elements in $\vec{x}$, so minimizing it gives the sparsest solution satisfying the available data. There is no known algorithm that solve Eq. (3) in a way that scales well with problem size. Instead, one can solve the following optimization problem

$$\vec{f}^* = \text{argmin } \|\vec{x}\|_p \text{ subject to } \mathbf{A}\vec{x} = \vec{b}, \tag{4}$$

where

$$\|\vec{x}\|_p \equiv \sum_i |x_i|^p.$$

When $p \leq 1$ this optimization problem can in many cases identify a solution to $\mathbf{A}\vec{x} = \vec{b}$ this is either the sparsest or "close" to the sparsest possible solution. Most research efforts on solving Eq. (4) have focussed on $p = 1$, because the optimization problem is convex in this case. But one of us [5] has considered the case when $p < 1$. A mathematical analysis of this problem is beyond the scope of this article, but we give an example with a small linear system that illustrates the possible advantage of considering the non-convex $p < 1$ problem.

Consider the following underdetermined linear system:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1.5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}. \tag{5}$$

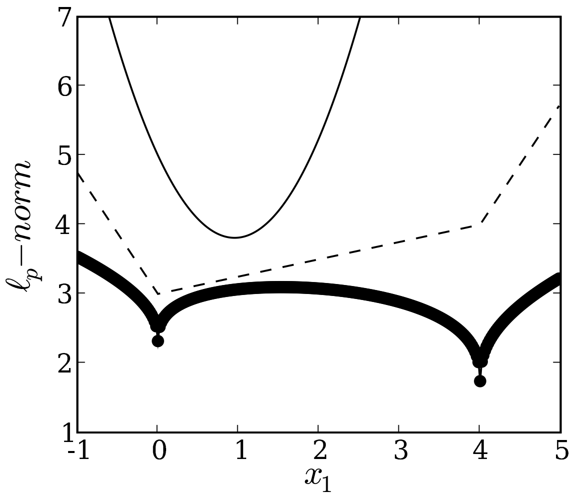

Fig. 1. Plots of the ℓ_p-norm from Eq. (7) for the p values: $p = 2.0$ (solid curve), $p = 1.0$ (dashed curve), and $p = 0.4$ (solid curve with dots).

As there are two data points and three unknowns, the solution space is described by a line that can be parametrized as:

$$\vec{x} = (x_1, 2 - x_1/2, 1 - x_1/4)^T. \tag{6}$$

The linear system in Eq. (5) is designed so that the line corresponding to its solution space intersects the x_1-axis at $x_1 = 4$. This point is the sparsest solution of Eq. (5). Now we illustrate how the non-convex norm minimization allows us to identify this solution. With the parametrization of Eq. (6) it is simple to write down the expression for the norm $\|\cdot\|_p$ as a function of x_1:

$$\|\vec{x}\|_p = |x_1|^p + |2 - x_1/2|^p + |1 - x_1/4|^p. \tag{7}$$

And in Fig. 1, we plot the values of the various norms for different values of p. It is clear from this plot that minimizing $\|\cdot\|_p$ for different values of p lead to different solutions. The minimum energy solution is obtained with the $p = 2$ case, yielding the solution $\vec{x}^*_{\ell 2} = (0.952, 1.524, 0.762)^T$, which is not sparse since all components are non-zero. The $p = 1$ case, the lowest value of p maintaining convexity, gives $\vec{x}^*_{\ell 1} = (0, 2, 1)^T$, which is sparser than the minimum energy solution but not the sparsest possible solution. For $p = 0.4$ we see that the sparsest solution is identified by the minimum $\ell_{p=0.4}$-norm.

Returning to the image reconstruction problem at hand, we seek to minimize the TpV-norm which is the ℓp-norm of the image gradient magnitude. We do this because for many underlying image functions the image gradient magnitude is approximately sparse. We take the same algorithm for the convex $p = 1$ case to attempt to solve the non-convex $p < 1$ problem in Eq. (1). We demonstrate the possible advantage of TpV-norm minimization in the next section.

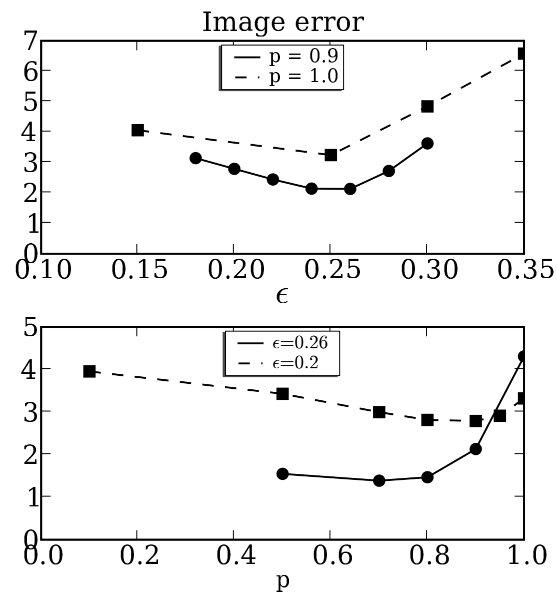

Fig. 2. Image error $\|\vec{f} - \vec{f}_{\text{true}}\|$ of reconstructed images as a function ϵ and p.

IV. Results

We apply the TpV algorithm to 2D fan-beam CT image reconstruction from few view projection data. The parameters of the simulated scan are: radius is 40 cm; source to detector distance is 80 cm; the detector has 512 bins covering 41.3 cm. The projection data contain only 25 views. The image array is a 20×20 cm^2 square composed of 256×256 pixels. In this study we used the Shepp-Logan phantom, and added Gaussian noise to the projection data at a level of 0.1% of the actual bin values. Using the same set of projection data, images are reconstructed by solving Eq. (1) for various values of ϵ and p. First, plots of the image error, $\|\vec{f} - \vec{f}_{\text{true}}\|$, are shown fixing ϵ and varying p

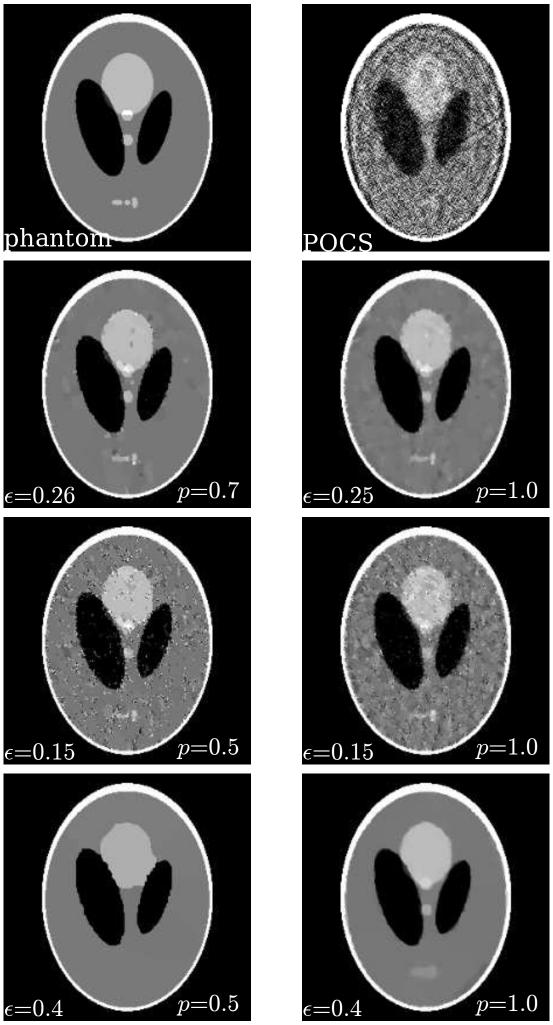

Fig. 3. Reconstructed images for 25 view noisy projection data for various values of ϵ and p.

and vice versa. Second, the reconstructed images are shown for particular values of ϵ and p.

In the top panel of Fig. 2 two curves are shown that show the dependence of the image error as a function of ϵ for $p = 1.0$ (standard TV) and $p = 0.9$. In both cases the image error does not decrease monotonically with ϵ. The reason for this is that the given projection data is inconsistent due to noise, and as a result tighter data constraints can lead to larger image errors. Looser data constraints allow for the selection of images with lower TpV, which can reduce the image error if the underlying image is sparse in its gradient magnitude. The curve corresponding to $p = 0.9$ does yield lower image errors. Thus the non-convex optimization problem may have some advantage over standard TV minimization. Fixing ϵ, we examine the image error dependence on p in the lower panel of Fig. 2. These curves clearly show a marked drop in image error as p is decreased from 1.0. A minimum image error of 3.2 is obtained for the case where p is confined to 1.0, while allowing p to be less than 1.0 yields an image error as low as 1.39.

In Fig. 3, we show reconstructed images for a few values of different ϵ and p. In the top row are shown the phantom and a reconstruction by projection onto convex sets (POCS) for reference. The POCS implementation is basic with no regularization. In the second row are shown the "best" images in terms of lowest image error, when p is allowed to vary (left) and when p is fixed at 1.0 (right). The difference in the image quality is quite apparent, and the image from the non-convex $p = 0.7$ case has fewer artifacts. In the remaining four panels, images are shown for various combinations of high and low ϵ and p. The images with $p < 1.0$ are sparser in the image gradient as there is a clear quantization in the reconstructed image gray values.

V. Conclusion

We have developed a new iterative algorithm for solving the non-convex TpV optimization problem that can reconstruct images with a sparse gradient magnitude from projection data containing few views. The non-convex extension of the TV algorithm to values of p less than one appears to lead to more accurate reconstructed images. We have shown preliminary optimization of p and ϵ values in terms of image error, but clearly other image quality metrics can be used to select optimal values of ϵ and p.

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