Topics in Geometric Mechanics: Geometrical Monodromy

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Geometric phases are ubiquitous in science and technology: from the coriolis-induced rotation of Foucault's pendulum in classical mechanics to the precession of polarization of an optical pulse along a birefringent fiber in modern telecommunications to quantum mechanics where the Aharanov-Bohm effect for electrons interacting with a spatially confined magnetic field introduces interference among the phases of the wave-function depending on the virtual paths of the electrons around the region containing the magnetic field. Mathematically, the geometrical phase is associated with the integral of a natural mechanical connection for these problems (whose components are, for example, the magnetic vector potential in the Aharanov-Bohm effect) around a closed loop in phase space. In particular, the geometrical phase is nonzero when this connection has curvature. See Marsden and Ratiu [1999] for a more detailed exposition.

Recently, classical mechanical systems have been discovered for which the geometrical phase is multi-valued on the global phase space (Duistermaat [1980]). Mathematically, this means the fiber bundle associated with the natural mechanial connection for these problems (again, for example, the magnetic vector potential in the Aharanov-Bohm effect) is non-trivial. This multi-valuedness is known as ***geometric monodromy***, in analogy with the same term arising in analytic function theory. In classical mechanics geometric monodromy prevents a global parametrization based on action-angle coordinates. In quantum mechanics geometric monodromy prevents quantization in integer quantum numbers. In retrospect, it is shocking that such a fundamental property went unrecognized for so long. This happened in part because the geometric phase itself went largely unrecognized until the 1970's.

Many familiar systems exhibit geometrical monodromy. One example of such a system is the spherical pendulum! Recall the setup for the spherical pendulum: imagine a bob of mass m at the end of an inextensible string of length ℓ. Let $\mathbf{e}_z = (0,0,1)$ correspond to the direction of gravity. The Lagrangian is given by

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}} - \mathbf{x}^T\mathbf{e}_z$$

where time has been rescaled by gravitational time scale $\sqrt{\ell/g}$ and position by length scale ℓ.

Configuration space is $Q = S^2 \subset \mathbb{R}^3$ and tangent bundle is $TQ = TS^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$. Lagrangian is invariant under rotations about $\mathbf{e}_z$, i.e.,

$$L(\mathbf{x},\dot{\mathbf{x}}) = L(\mathbf{R}_z\mathbf{x},\mathbf{R}_z\dot{\mathbf{x}})$$

The infinitesimal generator for this S^1 symmetry on Q is,

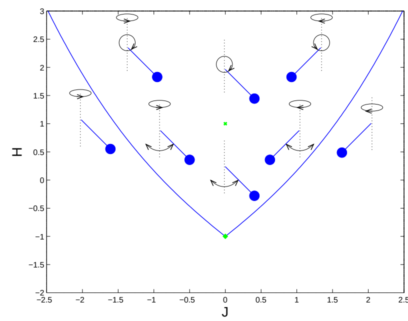
$$\psi(\mathbf{x}) = \frac{d}{ds}(\mathbf{R}_z(s)\mathbf{x})|_{s=0} = (\mathbf{e}_z \times \mathbf{x})$$

Noether's theorem implies $\partial L/\partial\dot{\mathbf{x}}^T\psi(\mathbf{x})$ is invariant, i.e.,

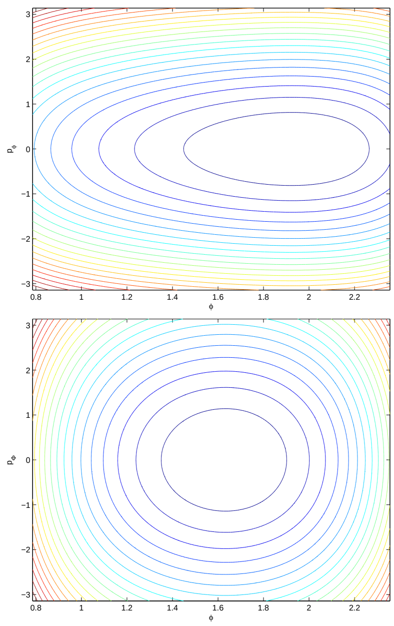
$$J(\mathbf{x},\dot{\mathbf{x}}) = (\mathbf{e}_z \times \mathbf{x})^T\dot{\mathbf{x}}, \quad J_t = 0$$

$J(\mathbf{x},\dot{\mathbf{x}})$ is the momentum map associated with the above symmetry.

Define the energy momentum map: $\mathcal{EM} = (J,H)$. The global phase space of the spherical pendulum can be nicely visualized on the energy momentum diagram as shown in the figure.



Fiber structure superposed on the $\mathcal{EM}$ diagram.



L-R: Reduced phase space for $J = 1.5, 4$.

Changing to spherical coordinates reveals the geometrical phase in the spherical pendulum:

$$\mathbf{x}(t) = \mathbf{R}_z(-\theta(t))\mathbf{R}_y(-\phi(t) + \pi/2)\mathbf{e}_x$$

where $\mathbf{R}_z$ and $\mathbf{R}_y$ are the usual S^1 rotation matrices about the y and z axes. From the reduced phase space portraits, it is clear that $\phi(t)$ is periodic with some period T. However, $\mathbf{x}(t)$ is not periodic. Rather, after every period of ϕ, $\mathbf{x}(t)$ rotates about the vertical by an amount given by the ***spherical pendulum phase formula***:

$$\Delta\theta = \theta(t+T) - \theta(t) = 2J \int_{x^-}^{x^+} \frac{dx}{(1-x^2)\sqrt{P(x)}} \tag{1}$$

where $P(x) = 2(H-x)(1-x^2) - J^2$ and $P(x^{\pm}) = 0$.

To see this note that

$$\begin{aligned}\mathbf{e}_x &= \mathbf{R}_y(\phi(t) - \pi/2)\mathbf{R}_z(\theta(t))\mathbf{x}(t)\\ &= \mathbf{R}_y(\phi(t) - \pi/2)\mathbf{R}_z(\theta(t+T))\mathbf{x}(t+T)\end{aligned}$$

Thus,

$$\mathbf{x}(t+T) = \mathbf{R}_z(\theta(t) - \theta(t+T))\mathbf{x}(t) = \mathbf{R}_z(-\Delta\theta)\mathbf{x}(t)$$

for $t \in [0, T)$ and $\Delta\theta$ is the spherical pendulum phase (1).

If Γ is the range of the energy-momentum diagram of physical relevance, then geometrical monodromy in the spherical pendulum manifests itself in the non-triviality of the bundle $\mathcal{EM}^{-1}(\Gamma)$, i.e., it is not diffeomorphic to $T^2 \times S^1$.

An excellent example of a mechanical system that may benefit from such an analysis is the falling cat. A recent paper in Nature muddles geometric phase and monodromy in the falling cat, but in doing so, suggests the very interesting possibility that the falling cat can exhibit geometric monodromy (I. Stewart [2004]). Geometric phase represents the phase shift that occurs in dynamical systems on surfaces with nonzero curvature and is a property of a trajectory in phase space; whereas geometric monodromy determines how these trajectories fit together.

Understanding the global dynamical structure of the falling cat would undoubtedly advance the theoretical treatment of classical mechanical systems. Moreover, as a concrete classical mechanics problem, the falling cat could also popularize tools for the global analysis of mechanical systems within the broader scientific community. These tools are of particular importance to researchers in structural, biomolecular, and celestial mechanics where dynamic problems with a similar geometric structure, e.g., rigid rotation, geometric phases, and holonomic constraints naturally arise.

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