Effects of disorder on solitary waves of the cubic-quintic nonlinear Schrödinger equation

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The cubic-quintic nonlinear Schrödinger equation (CQNLSE)

$$i\psi_z + \psi_{tt} + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = 0 \qquad (1)$$

(with $z \geq 0, t \in \mathbb{R}, \varepsilon_q > 0$) is one of the simplest non-integrable extensions to the cubic nonlinear Schrödinger equation (CNLSE) for which solitary wave solutions are readily obtained. The CQNLSE is of great interest since it appears in a wide variety of physical systems, such as, optical fibers with nonlinear saturation, fiber lasers [1], Langmuir plasmas [2], and in many pattern forming systems. Therefore, it is an ideal test bed for studying effects of perturbations, interactions, and collisions on solitary waves. Because of the non-integrability of the CQNLSE, the dynamics can be far richer than the dynamics of solitons of the CNLSE. For example, small perturbations can make the solitary waves unstable, lead to front formation or to pulse splitting. Effects of deterministic perturbations on the solitary wave dynamics have been studied in detail [3], [4], but effects of stochastic perturbations due to disorder in the system, and effects of interactions and collisions between the pulses are not yet understood.

The family of solitary wave solutions $\psi = \eta e^{i\chi} g_0(x)$ with

$$\chi = \alpha_0 + \beta(t - y_0) + (\eta^2 - \beta^2)z,\ x = \eta(t - y_0 - 2\beta z),$$

$$g_0(x) = \sqrt{2}\left[(1 - (4/3)\varepsilon_q\eta^2)^{1/2}\cosh(2x) + 1\right]^{-1/2}$$

is parametrized by $\eta, \beta, \alpha_0, y_0$, determining the amplitude/width, frequency/velocity, phase shift and time shift correspondingly.

We study the effects of perturbations by adding a small perturbation ($\varepsilon << 1$) to (1)

$$i\psi_z + \psi_{tt} + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = \varepsilon P(\psi, z, t). \quad (2)$$

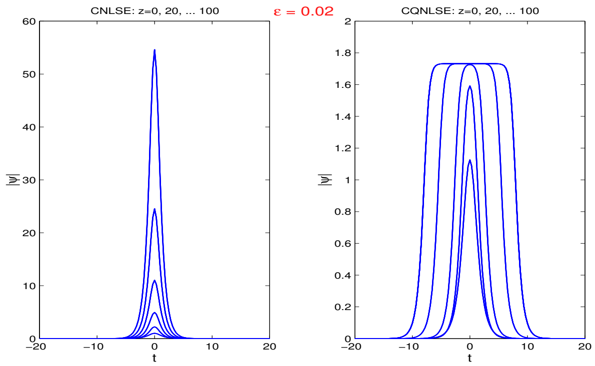
The effects of P are in general of two major types. The first is distortions in the parameters of the solitary waves and the second is emmission of radiation. The former can be expressed by making the parameters of the solitary wave $(z,t)-$dependent functions. The general perturbation method for studying these effects is based on direct scattering and requires completely analyzing the spectrum of a certain linearized operator, see [4]. One of our subprojects has been to numerically find the eigenfunctions of the continuous spectrum. This has not been finished.

During this summer we focused on the special case of a **linear gain perturbation**, i.e. $P(\psi, z, t) = i\psi$ in the deterministic case and $P(\psi, z, t) = i\xi(z)\psi$, where $\xi(z)$ satisfies $\langle\xi(z)\rangle = 0$, $\langle\xi(z)\xi(z')\rangle = a^2\delta(z - z')/3$, and δ is the Dirac delta function. For this special kind of P the general method is not needed when studying first order effects (in dz) because the effects of radiation are of second order and one can use a simple conservation law to study the $z-$evolution of the parameters of the solitary wave. In order to verify our predictions we have also developed a 6th order (in dz) numerical solver using the split-step method for $z-$stepping and Fourier transform for differentiation in t. Using the code we have been able to verify all the formulas given in this report.

1) In the deterministic case the perturbation $P = i\psi$ has a first order effect of an increase in amplitude and/or width of the solitary wave. This is modeled by making η $z-$dependent: $\eta = \eta(z)$. Using the conservation law for the L_2 norm of ψ for (1) one obtains an ODE for $\|\psi\|_2$ in (2), and using this, an explicit formula (c =const.)

$$\eta(z) = (3/(4\varepsilon_q))^{1/2}\tanh\left(c(\eta_0)e^{2\varepsilon z}\right).$$

Therefore, the amplitude converges to a constant value and because the $\|\psi\|_2$ has to keep increasing exponentially, **forward and backward propagating fronts develop**. An example of the evolution of $|\psi|$ is in the first figure, where on the left, for comparison, we show the same plots for



CNLSE. The position of the front (at half maximum of $|\psi|$) in (2) can now also be easily found:

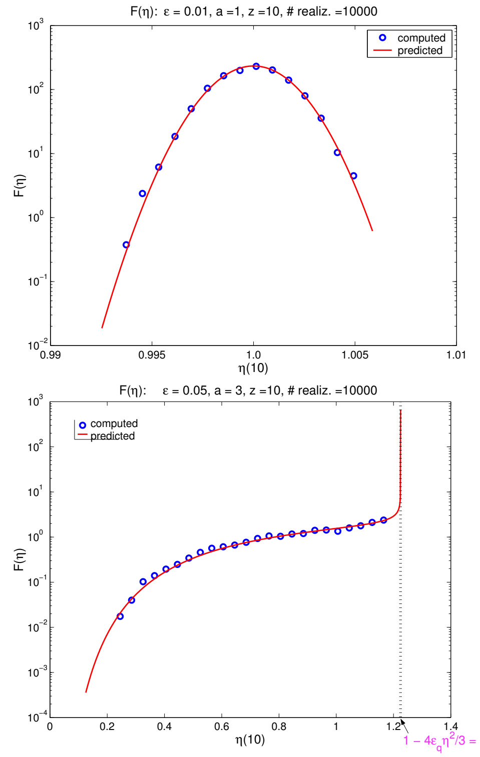
$$t_{\rm fr} = \frac{1}{2\eta}\operatorname{arccosh}\left(4+3\left(1-(4/3)\varepsilon_q\eta^2\right)^{-1/2}\right).$$

2) In the stochastic case, $P = i\xi(z)\psi$, we are interested in knowing the probability of $\eta(z)$ (and $t_{\rm fr}$) falling within a given interval. For this end we find (using the central limit theorem) the probability distribution functions (pdf). For η we get

$$F(\eta) = \left(\frac{\varepsilon_q}{2\pi a^2\varepsilon^2 z}\right)^{1/2}\times$$
$$\frac{\exp\left(\frac{-3}{8a^2\varepsilon^2 z}\ln^2\left[\frac{1}{c(\eta_0)}\operatorname{arctanh}\left(\left(\frac{4\varepsilon_q}{3}\right)^{1/2}\eta\right)\right]\right)}{\left(1-\frac{4\varepsilon_q}{3}\eta^2\right)\operatorname{arctanh}\left(\left(\frac{4\varepsilon_q}{3}\right)^{1/2}\eta\right)}.$$

This distribution function is defined for $0 \le \eta < (3/(4\varepsilon_q))^{1/2}$. The shape of F is determined mainly by $a^2\varepsilon^2 z$. The two basic shapes are shown in the second figure . A numerical verification using 10000 direct Monte Carlo simulations is also shown. The upper plot is for $a^2\varepsilon^2 z$ below a certain threshhold - the pdf is concave down (similarly to $F(\eta)$ for CNLSE). In the lower plot $a^2\varepsilon^2 z$ is above the threshhold and clearly events corresponding to values of η close to the supremum $\eta = (3/(4\varepsilon_q))^{1/2}$ become more frequent. These events are front formations.

For a range of values of η close to the supremum, i.e. for front solutions, we can also study the pdf of the front position $G(t_{\rm fr})$. We have shown that this pdf is truly log-normal with a tail



extending to $t_{\rm fr} = \infty$. We have been able to numerically verify the shape of $G(t_{\rm fr})$ too.

To conclude, we have analyzed the effect of a linear gain perturbation with and without disorder on the propagation of solitary waves in the CQNLSE. In the disordered case we have found the pdf's of both the amplitude parameter and the front position and shown that for either large distance z and/or strong disorder a large number of solitary waves evolve into fronts.

Acknowledgements

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References

[1] J.M. Soto-Crespo, N.N. Akhmediev, V.V. Afanasjev, and S. Wabnitz, Phys. Rev. E 55, 4783 (1997).

[2] C. Zhou, X.T. He, and S. Chen, Phys. Rev. A 46, 2277 (1992).

[3] W. van Saarloos and P.C. Hohenberg, Phys. D 56, 303 (1992).

[4] J. Yang and D.J. Kaup, SIAM J. Appl. Math. 60, 967 (2000).