Basis pursuit denoising using a primal-dual interior point method

Guillermo A. Vera, alonzo@ece.unm.edu

Signals with a sparse representation are more easily analyzed and processed, thus the search for this representation has become the focus of intensive research. We aim to analyze an optimization principle for signal decomposition known as Basis pursuit; write a parametrizable library for its fast and efficient implementation, and test its suitability for practical applications such as signal and image denoising. As a result, a C library was created and successfully tested for 1D signal denoising and a basic extension to 2D signals is presented as proof of concept.

Given a dictionary $\Phi = [\phi_1, \phi_2, \ldots, \phi_d]$, where $\phi_k, 1 \geq k \leq d$ are vectors, a signal $\overline{s} = [s_1, s_2 \ldots s_N]$ is represented as a linear combination $\overline{s} = \sum_{k=1}^{N} \phi_k \alpha_k$ with scalar coefficients $\overline{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_k]$. The dictionary Φ can be seen as a matrix NxL with prototype signals as columns such that $\overline{s} = \Phi\overline{\alpha}$. For overcomplete dictionaries, $L \ggg N$ and $\overline{\alpha}$ is non-unique. In general, a signal's representation whose coefficients have the smallest ℓ^1 norm, is a good approximation of the sparsest representation [1].

Basis pursuit (BP) is an optimization principle for signal decomposition whose objective is to find such a representation. The basis pursuit problem is expressed as:

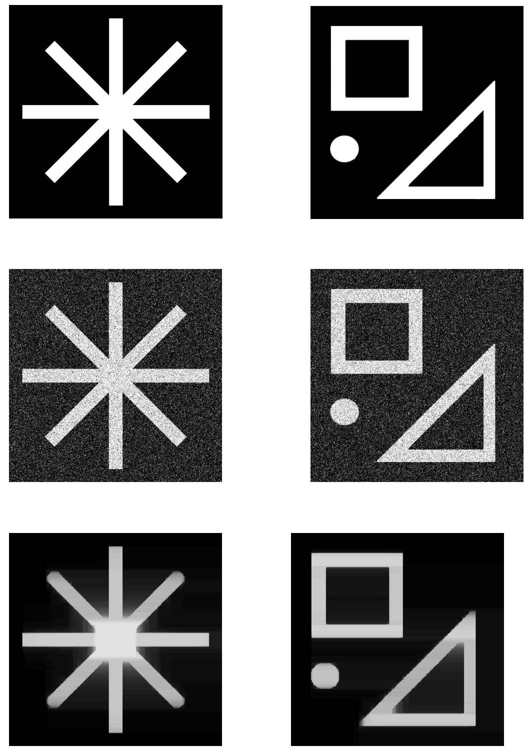
$$\min \|\overline{\alpha}\|_1 \quad \text{s.t:} \quad \overline{s} = \Phi\overline{\alpha} \tag{1}$$

For overcomplete dictionaries, the search for a solution becomes a large scale optimization problem. Primal-dual interior-point methods offer a polynomial-time solution for this problem, with impact on a wide number of signal processing applications, for example, denoising.

In denoising, one searches to minimize the noise level $\sigma^2 = \|y - \overline{s}\|_2^2$; where $\overline{s}$ represents the original signal, n the noise and $y = \overline{s} + n$, the observed signal. The basis pursuit model for signal denoising is given by the equation:

$$\min_{\alpha} \frac{1}{2}\|y - \Phi\overline{\alpha}\|_2^2 + \lambda\|\overline{\alpha}\|_1 \tag{2}$$

The solution $\overline{\alpha}$ is dependent on the parameter λ. As λ approaches 0, the equation 2 behaves as



Two set of images (left and right columns) are corrupted with white noise and processed using a Heavyside (HS) dictionary as a proof of concept. A Heavyside dictionary Φ is a matrix whose elements $\phi_k, 1 \geq k \leq d$ are step functions shifted with respect to each other. Since HS is not a separable linear transformation (the 2D transform is not equivalent to a 1D transform, performed along a single dimension followed by the same 1D transform performed in the other dimension), some distortion is noticeable in the diagonal features of the images. Other dictionaries, or compositions of several dictionaries are expected to have better results.

a simple basis pursuit problem, where $\overline{\alpha}$ is the set of coefficients to represent the noisy signal y, and the residual goes to zero. As λ grows, the residual grows to equal y. Thus, λ controls the size of the residual. Using similar arguments than for the case of wavelets coefficient thresholding [2, 1], one sets $\lambda = \sigma\sqrt{2\log(n)}$ where n is the cardinality of the dictionary and σ the noise power.

By using $A = [\phi, -\phi]$, $b = y$, $C = [\lambda; -\lambda]$, $x = [u; v]$ and $\overline{\alpha} = u - v$ equation 2 is equivalent to a least squares problem with positivity constraints:

$$\min \|Ax - b\|_2^2 + C^T x \quad \text{s.t:} \quad x \geq 0 \tag{3}$$

which is equivalent [3, 4] to the perturbed system

$$\begin{aligned} &\min C^T x + \frac{1}{2}\|\gamma x\|^2 + \frac{1}{2}\|p\|^2 \\ &\qquad \text{s.t:} \quad Ax + \delta p = b, \quad \text{and} \quad x \geq 0 \end{aligned} \tag{4}$$

where p accounts for noise or residual and γ is used to regularize the problem, ensuring that $\|x\|$ is bounded [5]. Our resulting equation represents a constrained optimization problem, with equalities and inequalities as constraints. To solve the system, the inequalities and equalities constraints are transformed into the cost function using, barrier methods [5] and Lagrange multipliers [6], respectively. The resulting system is solved by using Newton's direction and conjugate gradient iteratively, while decreasing the value of the barrier parameter μ.

A matlab, 1-D implementation of these methods was reported in 2001 [3], under the principle of reproducible research. With this implementation as an starting point, we developed and successfully tested a C library that implements basis-pursuit denoising for 1-D signals, extensible to 2-D signals. This library is to be included in an open source project called NUMIPAD [7] started at DDMA, LANL. The NUMIPAD library implements several methods/algorithms to solve inverse problems and adaptive decomposition.

Future work

The set of functions implemented will be extended to more overcomplete dictionaries and compositions of them. This will improve the ability to denoise a wide range of signals more efficiently.

Quantitative comparison with other methods shall also be carried on, in order to conclude on the suitability of the present methods for practical applications. Also, extensive performance tests will be performed to optimize the code and provide performance comparisons.

Acknowledgements

Los Alamos Report LA-UR-06-5746. Funded by the Department of Energy under contracts W-7405-ENG-36 and the DOE Office of Science's Advanced Scientific Computing Research (ASCR) Applied Mathematical Research program.

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