HAMILTONIAN STRUCTURE FOR ALFVÉN WAVE TURBULENCE EQUATIONS

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A hamiltonian formulation using a noncanonical Poisson bracket is presented for a nonlinear fluid system that includes reduced magnetohydrodynamics and the Hasegawa–Mima equation as limiting cases. Nonlinear integral invariants for the system are found to be in the kernel of the noncanonical Poisson bracket. This Poisson bracket is given a Lie algebraic interpretation.

Introduction. Planar Alfvén wave turbulence in magnetized plasmas has been modelled by differential equations of the form [1,2]

$$\partial_t \nabla^2\phi + [\phi, \nabla^2\phi] - [\psi, \nabla^2\psi] = 0\,, \tag{1}$$

$$\partial_t \psi + [\phi, \psi] + \alpha[\psi, \chi] = 0\,, \tag{2}$$

$$\partial_t \chi + [\phi, \chi] - [\psi, \nabla^2\psi] = 0\,, \tag{3}$$

for functions ϕ, ψ, and χ of (x, y), with constant parameter α. Here, $[a, b]$ is the jacobian,

$$[a, b] = a_x b_y - a_y b_x\,, \tag{4}$$

that is, $[a, b]$ is the canonical Poisson bracket for functions a, b in the plane. The function ϕ represents the electrostatic potential for drift waves, with drift velocity

$$\boldsymbol{v} = \hat{\boldsymbol{z}} \times \nabla\phi\,. \tag{5}$$

The function ψ is the normalized magnetic flux, so that the magnetic field in the plane is given by

$$\boldsymbol{B} = \hat{\boldsymbol{z}} \times \nabla\psi\,. \tag{6}$$

Finally, the function χ is the normalized deviation of particle density from a constant equilibrium value.

The details of the physical approximations which lead to eqs. (1)–(3) are discussed in refs. [1–4]. These equations include the physics of both ideal reduced magnetohydrodynamics (RMHD) [5–8] in the low-beta limit and the Hasegawa–Mima (HM) equation [9,10].

Low-beta RMHD is recovered from (1)–(3) by neglecting the constant parameter α in (2). This limit decouples the field χ from the other fields, ϕ and ψ. The evolution of ϕ and ψ then constitutes the RMHD description. The RMHD system (i.e., (1) and (2) with $\alpha = 0$) is commonly used to simulate nonlinear shear-Alfvén dynamics in tokamaks, see ref. [4].

The Hasegawa–Mima equation is recovered from the system (1)–(3) by assuming

$$\alpha\chi = \phi\,. \tag{7}$$

This amounts physically to linearizing the adiabatic Maxwell–Boltzman limit for the electrons, see refs. [1–4]. Then, via (7) eq. (3) implies that $[\psi, \nabla^2\psi] = \partial_t\chi$, eq. (2) becomes $\partial_t\psi = 0$ and thus decouples from the system, and eq. (1) becomes

$$\partial_t(\nabla^2\phi - \alpha^{-1}\phi) + [\phi, \nabla^2\phi] = 0\,, \tag{8}$$

which is the HM equation for the electrostatic field. The HM equation (8) describes ideal drift wave turbulence in a low-beta plasma. It also has a hydrodynamic interpretation for geostrophic fluid flow [11–13].

A general evolution equation

$$\partial_t u = A(u)\,, \tag{9}$$

with an operator A is said to be *in hamiltonian form*,

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if there is a Poisson bracket operation $\{\ ,\ \}$ on the space of functionals of u and a hamiltonian functional $H(u)$ such that (9) is equivalent to

$$\partial_t F(u) = \{H, F\} , \tag{10}$$

for all functionals F. Recall that a Poisson bracket operation should be bilinear, skew-symmetric, and satisfy the Jacobi identity. The bracket $[\ ,\ ]$ in (4) is such an operation on functions of (x, y).

For instance, the vorticity equation,

$$\partial_t \omega + [(\nabla^2)^{-1}\omega, \omega] = 0 , \tag{11}$$

for vorticity ω in two-dimensional incompressible flow is in hamiltonian form with the Poisson bracket [14]

$$\{F, G\} = -\int \omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right] \mathrm{d}x\,\mathrm{d}y , \tag{12}$$

and hamiltonian functional

$$H(\omega) = -\frac{1}{2}\int \omega (\nabla^2)^{-1} \omega \,\mathrm{d}x\,\mathrm{d}y . \tag{13}$$

The operator $(\nabla^2)^{-1}$ is the inverse of the laplacian ∇^2 on the domain of flow. For (12) and (13) to yield the vorticity equation (11) in hamiltonian form (10), appropriate boundary conditions are needed, so that $\delta H/\delta\omega = -(\nabla^2)^{-1}\omega =: -\phi$ is well defined. Then, using (12) and (13), eq. (11) can be written as

$$\partial_t \omega = \{H, \omega\} = [\delta H/\delta\omega, \omega] = -[\phi, \omega] , \tag{14}$$

in hamiltonian form.

Purpose. In this paper, we will cast the system (1)–(3) into hamiltonian form, by extending the Poisson bracket (12) to include the additional plasma variables of density, χ, and magnetic flux, ψ. The extended Poisson bracket formalism we present provides a mechanism for identifying various integral invariants of the system (1)–(3). Elsewhere [3], these integral invariants will be used to derive classes of equilibrium solutions for (1)–(3) which describe solitary *electromagnetic* drift waves.

Hamiltonian formulation. Eqs. (1)–(3) can be expressed in hamiltonian form, letting $\hat{z}\cdot\mathrm{curl}\,\mathbf{v} = \nabla^2\phi =: U$, as

$$\partial_t U = \{H, U\} , \tag{15}$$

$$\partial_t \psi = \{H, \psi\} , \tag{16}$$

$$\partial_t \chi = \{H, \chi\} , \tag{17}$$

where the hamiltonian $H(U, \psi, \chi)$ is given by

$$H(U, \psi, \chi) = \frac{1}{2}\int [-U(\nabla^2)^{-1}U + |\nabla\psi|^2 + \alpha\chi^2]\,\mathrm{d}x\,\mathrm{d}y , \tag{18}$$

and the Poisson bracket $\{\ ,\ \}$ is defined to be

$$\{F, G\} = -\int \mathrm{d}x\,\mathrm{d}y \left\{ U\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right] + \psi\left(\left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta U}\right] - \left[\frac{\delta G}{\delta \psi}, \frac{\delta F}{\delta U}\right]\right) + \chi\left(\left[\frac{\delta F}{\delta \chi}, \frac{\delta G}{\delta U}\right] - \left[\frac{\delta G}{\delta \chi}, \frac{\delta F}{\delta U}\right]\right) + \chi\left[\frac{\delta F}{\delta \chi}, \frac{\delta G}{\delta \chi}\right] + \psi\left(\left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \chi}\right] - \left[\frac{\delta G}{\delta \psi}, \frac{\delta F}{\delta \chi}\right]\right)\right\} . \tag{19}$$

Using (18) and (19), eqs. (15)–(17) become

$$\partial_t U = \{H, U\} = -\left[U, \frac{\delta H}{\delta U}\right] - \left[\psi, \frac{\delta H}{\delta \psi}\right] - \left[\chi, \frac{\delta H}{\delta \chi}\right] , \tag{20}$$

$$\partial_t \psi = \{H, \psi\} = -\left[\psi, \frac{\delta H}{\delta U}\right] - \left[\psi, \frac{\delta H}{\delta \chi}\right] , \tag{21}$$

$$\partial_t \chi = \{H, \chi\} = -\left[\chi, \frac{\delta H}{\delta U}\right] - \left[\psi, \frac{\delta H}{\delta \psi}\right] - \left[\chi, \frac{\delta H}{\delta \chi}\right] . \tag{22}$$

Formulae (20)–(22) are easily verified using the following identities for arbitrary functions f, g and h,

$$\int f[g, h]\,\mathrm{d}x\,\mathrm{d}y = \int g[h, f]\,\mathrm{d}x\,\mathrm{d}y = \int h[f, g]\,\mathrm{d}x\,\mathrm{d}y . \tag{23}$$

Identities (23) hold, provided boundary terms may be neglected when integrating by parts. The variational derivatives of the hamiltonian (18) are

$$\delta H/\delta U = -(\nabla^2)^{-1}U =: -\phi , \tag{24}$$

$$\delta H/\delta \psi = -\nabla^2 \psi , \tag{25}$$

$$\delta H/\delta \chi = \alpha\chi . \tag{26}$$

Upon substituting the variational derivatives (24)–(26) into (20)–(22), the original equations (1)–(3) are recovered, now in hamiltonian form.

The Poisson bracket (19) is clearly bilinear and skew-symmetric in F and G. Its Jacobi identity can be verified either by direct computation using the properties of the canonical bracket $[\ ,\ ]$ in (4), or by observing that the Poisson bracket (19) can be associated to the dual of the following Lie algebra,

$$[f_1 \circledS (f_2 \oplus f_3)] \oplus [f_2 \circledS f_3] , \tag{27}$$

which is the direct sum of two semidirect products. Here, the semidirect product actions Ⓢ in (27) are defined as follows. Let an element of $f_2 Ⓢ f_3$ be written as a pair $(f_2;f_3)$. For another element $(\bar{f}_2;\bar{f}_3)$] e.g., the second semidirect product action in (27) is defined by

$$[(f_2;f_3),(\bar{f}_2;\bar{f}_3)] := ([f_2,\bar{f}_2];[f_2,\bar{f}_3]-[\bar{f}_2,f_3]). \tag{28}$$

A similar definition applies for the first semidirect product action in (27). With these definitions of the semidirect product actions in (27), dual coordinates are identified in (19), as follows: U is dual to f_1; χ is dual to f_2; and ψ is dual to f_3. For further discussion of Poisson brackets in continuum physics which are associated to the duals of semidirect product Lie algebras, see, e.g., ref. [15].

The conservation of energy H in (18) under the dynamics of (1)–(3) is now an immediate consequence of the hamiltonian formulation and skew-symmetry of the Poisson bracket (19).

Conserved quantities. Analogously to the well known conservation of integrals of the vorticity in (11), the system (1)–(3) conserves any functionals of the form

$$C=\int [F(\psi)+\chi G(\psi)+K(\omega-\chi)]\,\mathrm{d}x\,\mathrm{d}y\,, \tag{29}$$

for arbitrary (smooth) functions F, G, and K. For example, taking $F=0$, $G=0$, and $K(\xi)=\xi^2$ gives the conserved quantity

$$\int (\omega^2-2\omega\chi-\chi^2)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{30}$$

which is analogous to the enstrophy invariant for the vorticity equation (11).

Furthermore, the functionals C in (29) are "Casimirs" of the Poisson bracket (19), in the sense that

$$\{C,J\}=0\,,\quad \forall J(U,\psi,\chi)\,. \tag{31}$$

That is, the functionals C are conserved for *any* hamiltonian $J(U,\psi,\chi)$, not just for the hamiltonian H in (18).

Remarks. Although the noncanonical Poisson bracket (19) is presented here in an *ad hoc* fashion, there are a number of ways of deriving it systematically. One way is via a constrained variational principle obtained by using the Clebsch procedure, as explained in ref. [16]. In this way, a canonical (symplectic) Poisson bracket is first constructed in the space of Clebsch variables, i.e., in the combined space of the physical variables and the Lagrange multipliers for the constraint equations, (2) and (3). This canonical bracket can then be mapped into the smaller space of physical variables alone as in ref. [15], resulting in the noncanonical Poisson bracket (19).

The integral invariants (Casimirs) displayed in (29) are used in ref. [3] within the present hamiltonian context to derive classes of equilibrium and traveling-wave solutions for the system (1)–(3). These solutions include solitary electromagnetic drift waves, which generalize the well-known electrostatic drift waves [17,18] to the case of a magnetized plasma.

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