POISSON STRUCTURES OF SUPERCONDUCTORS

Darryl D. HOLM and Boris A. KUPERSHMIDT
Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Received 29 November 1982

We present Poisson structures for a hamiltonian, time-dependent , formulation of the Ginzburg–Landau theory of superconductor dynamics. The resulting equations are seen to be related to the London theory. Application of irreversible thermodynamics leads to an extension of these equations to the dissipative case.

Introduction. For superconductors, there are two phenomenological theories: the London theory and the Ginzburg–Landau theory [1], which are mutually complementary. The London theory is time dependent, but purely classical; while the Ginzburg–Landau theory is independent of time, but uses quantum mechanical arguments to predict the effect of a magnetic field on a supercurrent.

There have been various attempts to generalize the Ginzburg–Landau theory to account for time dependence. One may refer to, e.g., ref. [2] for a survey of the literature. These attempts include purely microscopic approaches [3] that start from BCS theory, in the special case of a gapless superconductor; hybrid microscopic and phenomenological approaches [4]; and purely phenomenological approaches [5]. Despite this activity, an accepted time-dependent Landau–Ginzburg equation for the general case does not yet seem to exist. The present paper addresses the problem from the phenomenological viewpoint.

First, for the case without dissipation, we construct a dynamical Ginzburg–Landau theory in the hamiltonian formulation. The key element of this construction is the method of transformation of Poisson brackets. The resulting time-dependent Ginzburg–Landau equations pass over to the London equations in a way which establishes a correspondence between Ginzburg–Landau variables and London variables. Such a correspondence lends credence to this particular theory, because of its proper London limit. Moreover, the phase of the order parameter and the number density of charge carriers turn out to be canonically conjugate, as one would expect by analogy to the quantum mechanical situation.

However, our results at this stage differ from those of ref. [5] for Ginzburg–Landau dynamics without dissipation.

To explore the physical consequences of this difference, one should include dissipation, since time dependence in superconductors generally involves dissipation. To account for dissipation, we modify our dynamical Ginzburg–Landau equations according to the rules of irreversible thermodynamics: generalized forces and fluxes are introduced formally into the equations and then identified in terms of state variables via their transformation properties as scalars, vectors, etc. In this way kinetic coefficients appear, which can be made to satisfy Onsager relations and the requirement of positive-entropy production rate.

The resulting equations for the dissipative Ginzburg–Landau theory agree, in a special case, with earlier work [3,4], which relies upon microscopic theory, as mentioned before. However, our results differ again from those in ref. [5], because of the underlying difference in the nondissipative limit.

The plan of the paper is as follows. Starting from the Ginzburg–Landau free energy (with both electric- and magnetic-field energies included), we introduce auxiliary variables which are canonically conjugate to the real and imaginary parts of the complex order parameter, $\psi = \psi^1 + i\psi^2$, of the Ginzburg–Landau theory. These conjugate variables are related algebraically to the total charge density ρ, in a manner similar to that in the canonical theory of superfluids [6]. Use of a thermodynamic formula for variations of the free energy identifies terms in the canonical equations, which reduce to the usual Ginzburg–Landau equations in the proper static limit.

Next, we eliminate the auxiliary canonical variables by transforming to a new, noncanonical hamiltonian structure, which involves only the original variables $\rho, \psi^1, \psi^2, \boldsymbol{E}, \boldsymbol{A}$; where $\boldsymbol{E}$ is the electric field and $\boldsymbol{A}$ is the magnetic vector potential. In this transformed hamiltonian structure, one sees that the modulus-squared of the order parameter, $|\psi|^2$, is preserved in time by *any* hamiltonian, in particular by the Ginzburg–Landau hamiltonian. This result differs from that of ref. [5].

A re-transformation of the hamiltonian structure leads to a *canonical* Poisson bracket in the following pairs of variables: total charge density ρ, paired with the phase of the order parameter Φ; and components of the electric field E_i, paired with corresponding components of the magnetic vector potential A_i. A final choice of new variables transforms the time-dependent, non-dissipative Ginzburg–Landau equations into the London equations, via a map in which the phase gradient of the order parameter is related to the London "velocity" of superconducting charge carriers.

Dissipation is then added to the dynamical Ginzburg–Landau equations, according to the prescription of irreversible thermodynamics mentioned earlier, and the results are compared with those in the literature.

Ginzburg–Landau dynamics without dissipation. In the static Ginzburg–Landau theory [1] the free energy is given in terms of a complex order parameter $\psi = \psi^1 + \mathrm{i}\psi^2$, and a magnetic vector potential $\boldsymbol{A}$. The Ginzburg–Landau equations are obtained by stationarity of the free energy under variations with respect to ψ and $\boldsymbol{A}$. To make this theory dynamical, one adds the electrical energy to the free energy (which becomes also a function of the total charge density ρ) and regards the modified energy as a hamiltonian,

$$H = \int \left[\tfrac{1}{2}|(\mathrm{i}\nabla - e^*\boldsymbol{A})\psi|^2 + F(\rho, s, |\psi|^2) + \tfrac{1}{2}(E^2 + B^2)\right] \mathrm{d}^3x \,, \tag{1}$$

where $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$ and we employ natural units, with $\hbar = m^* = c = 1$, respectively for Planck's constant $\hbar$, the Cooper pair mass m^*, and the speed of light c. Dependence of the free energy F upon entropy density s is unimportant at this stage, because without dissipation the entropy density is taken to be time independent.

To construct the canonical hamiltonian equations, variables ϕ^1 and ϕ^2 are introduced, which are assumed to be canonically conjugate to ψ^1 and ψ^2, respectively. These four conjugate variables are related to the total charge density ρ by

$$\rho = (\psi^2\phi^1 - \psi^1\phi^2) \,, \tag{2}$$

in a manner similar to the canonical theory of superfluids [6].

By taking into account the thermodynamic relation $\mu = \delta F/\delta\rho$, for the chemical potential μ, one finds the following canonical equations:

$$\dot{\phi}^1 = -\delta H/\delta\psi^1 \,, \quad \dot{\phi}^2 = -\delta H/\delta\psi^2 \,,$$
$$\dot{\psi}^1 = \delta H/\delta\phi^1 = \mu\psi^2 \,, \quad \dot{\psi}^2 = \delta H/\delta\phi^2 = -\mu\psi^1 \,, \tag{3}$$

where, e.g., $\dot{\phi}^1 = \partial_t\phi^1$ is the partial derivative with respect to time. The last two equations imply

$$\partial_t|\psi|^2 = 0 \,, \quad |\psi|^2 = (\psi^1)^2 + (\psi^2)^2 \,,$$
$$\dot{\Phi} = -\mu, \ \Phi = \tan^{-1}(\psi_2/\psi_1) \,. \tag{4}$$

The equation $\dot{\Phi} = -\mu$ is the Josephson phase relation [7]. Thus, in the absence of dissipation, the Josephson phase relation controls the time dependence of the order parameter. As we shall see below, the remaining dynamics of the Ginzburg–Landau theory arises essentially as in the London theory.

The hamiltonian (1) produces electromagnetic equations,

$$\dot{\boldsymbol{E}} = \delta H/\delta \boldsymbol{A}$$
$$= \operatorname{curl} \boldsymbol{B} - e^*(\psi^1\nabla\psi^2 - \psi^2\nabla\psi^1) + e^{*2}|\psi|^2\boldsymbol{A} \,,$$
$$\dot{\boldsymbol{A}} = -\delta H/\delta \boldsymbol{E} = -\boldsymbol{E} \,. \tag{5}$$

These are in standard form, provided we identify the current density $\boldsymbol{J}$ as

$$\boldsymbol{J} = e^*(\psi^1\nabla\psi^2 - \psi^2\nabla\psi^1) - e^{*2}|\psi|^2\boldsymbol{A} \,. \tag{6}$$

The static Ginzburg–Landau equations can be recovered from (3) and (5) by setting all time derivatives to zero and ignoring the equations for $\dot{\psi}$.

Next, we transform the canonical bracket in the variables (ϕ^1, ψ^1), (ϕ^2, ψ^2), (E, A), into a noncanonical one, by elimination of ϕ^1, ϕ^2 in favor of the charge density ρ, given by (2). In terms of $(\rho, \psi^1, \psi^2, E, A)$, the new bracket is

$$\{H, K\} = \psi^2\left(\frac{\delta H}{\delta\rho}\frac{\delta K}{\delta\psi^1} - \frac{\delta K}{\delta\rho}\frac{\delta H}{\delta\psi^1}\right) - \psi^1\left(\frac{\delta H}{\delta\rho}\frac{\delta K}{\delta\psi^2} - \frac{\delta K}{\delta\rho}\frac{\delta H}{\delta\psi^2}\right) + \left(\frac{\delta H}{\delta A_i}\frac{\delta K}{\delta E_i} - \frac{\delta K}{\delta A_i}\frac{\delta H}{\delta E_i}\right), \quad (7)$$

where we sum over repeated indices. The first two terms here are the same as the bracket which appears in the theory of superfluid liquid ^{3}He-A [8].

The dynamical equations follow from the noncanonical bracket (7) as,

$$\dot{\psi}^1 = \{H, \psi^1\} = \psi^2\delta H/\delta\rho = \mu\psi^2 ,$$

$$\dot{\psi}^2 = \{H, \psi^2\} = -\psi^1\delta H/\delta\rho = -\mu\psi^1 ,$$

$$\dot{\rho} = \{H, \rho\} = -\psi^2\delta H/\delta\psi^1 + \psi^1\delta H/\delta\psi^2 := -\psi^2\xi^1 + \psi^1\xi^2 , \quad (8)$$

with electromagnetic equations as before, in (5). The expression in the $\dot{\rho}$ equation also appears in the statement of gauge invariance of the hamiltonian H, namely,

$$\psi^1\delta H/\delta\psi^2 - \psi^2\delta H/\delta\psi^1 + \mathrm{div}(\delta H/\delta A) = 0 . \quad (9)$$

Restated in terms of charge density ρ and current density J, the gauge invariance equation (9) becomes

$$\dot{\rho} + \mathrm{div}\, J = 0 , \quad (10)$$

which, of course, expresses local conservation of charge. Thus, the current density J and charge density ρ are properly identified by (6) and (2), respectively.

In terms of the new variables $|\psi|^2$ and Φ, defined in (4), the Poisson bracket (7) regains its canonical form; with canonical variables (ρ, Φ), (E, A). Moreover, the modulus-squared of the order parameter, $|\psi|^2$, is a "Casimir function", whose time derivative is zero for *any* hamiltonian.

In terms of $|\psi|^2$ and Φ, the hamiltonian (1) is given by

$$H = \int \{\tfrac{1}{2}[(\nabla|\psi|)^2 + (\nabla\phi - A)^2 e^*|\psi|^2] + F(\rho, s, |\psi|^2) + \tfrac{1}{2}(E^2 + B^2)\}\mathrm{d}^3x . \quad (11)$$

The new canonical hamiltonian equations are

$$\dot{\rho} = \delta H/\delta\Phi = -\mathrm{div}[(\nabla\Phi - A)|\psi|^2 e^*] = -\mathrm{div}\, J, \quad (12)$$

$$\dot{\Phi} = -\delta H/\delta\rho = -\mu , \quad (13)$$

$$\dot{E} = \delta H/\delta A = \mathrm{curl}\, B - e^*(\nabla\Phi - A)|\psi|^2 , \quad (14)$$

$$\dot{A} = -\delta H/\delta E = -E , \quad (15)$$

where $B = \mathrm{curl}\, A$. As usual, Gauss's law,

$$\nabla\cdot E = \rho , \quad (16)$$

is preserved by the dynamics.

Suppose we now transform to variables

$$\rho^s = e^*|\psi|^2 , \quad \mathfrak{v} = \nabla\Phi - A , \quad \chi = \Phi . \quad (17)$$

We then obtain the following equations, by taking the gradient of (13) and adding to (15):

$$\dot{\rho} = -\mathrm{div}\, \rho^s\mathfrak{v} , \quad \dot{\mathfrak{v}} = -\nabla\mu + E ,$$

$$\dot{E} = -\rho^s\mathfrak{v} + \mathrm{curl}\, B , \quad \dot{A} = E . \quad (18)$$

These equations are just the London equations, originally introduced by London in 1935 to explain the Meissner effect. Thus, the dynamical Ginzburg–Landau equations transform into the "classical" London equations under the mapping (17), which relates the phase gradient $\nabla\Phi$ of the order parameter in the Ginzburg–Landau theory to the superconducting "velocity" $\mathfrak{v}$ in the London theory.

Remark. At this point, we have shown that time dependence of the Ginzburg–Landau theory arises, from the phenomenological viewpoint, as a consequence of the Josephson phase relation (5) coupled to essentially the London equations, with the added proviso that the modulus-squared of the order parameter is independent of time. These results appear obvious and natural when the problem is approached in terms of hamiltonian structure.

Recently this problem has also been addressed in terms of constrained variational principles by Geurst [5] who comes tantalizingly close to our answer, but finds $\delta H/\delta|\psi|^2 = 0$, instead of $\partial_t|\psi|^2 = 0$. Now, in terms of hamiltonian structure, $|\psi|^2$ is a "Casimir function" for superconductivity, i.e., $|\psi|^2$ is preserved by any hamiltonian expressible in the physical variables. Consequently, $|\psi|^2$ must be preserved in time by the Ginzburg–Landau hamiltonian (1). Hence, $\partial_t|\psi|^2 = 0$ is the proper result.

Ginzburg–Landau dynamics with dissipation. The results obtained in the previous section can be extended to the dissipative case by following standard techniques, see e.g. ref. [6]. When dissipation occurs, the dynamical equations for a superconductor become, formally,

$$\dot{\rho} + \operatorname{div}(\boldsymbol{J} + \boldsymbol{K}) = 0\,, \quad \dot{\psi}^1 - \mu\psi^2 = Z^1\,,$$
$$\dot{\psi}^2 + \mu\psi^1 = Z^2\,, \quad \dot{s} + \operatorname{div}\boldsymbol{\sigma} = R/T\,,$$
$$\dot{\boldsymbol{E}} = \operatorname{curl}\boldsymbol{B} - (\boldsymbol{J} + \boldsymbol{K})\,, \quad \dot{\boldsymbol{A}} = -\boldsymbol{E}\,, \tag{19}$$

where s is the entropy density, which is now time dependent; $\boldsymbol{J}$ is the supercurrent density given in (6), which continues to satisfy

$$\operatorname{div}\boldsymbol{J} = \xi^1\psi^2 - \xi^2\psi^1\,, \tag{20}$$

with ξ^1, ξ^2 given, as in (8), by $\xi^1 = \delta H/\delta\psi^1$, $\xi^2 = \delta H/\delta\psi^2$; H as before; $\mu = \delta H/\delta\rho$ and $T = \delta H/\delta s$. These state variables, plus the field $\boldsymbol{E}$ and $\boldsymbol{A}$, must determine the other functions: the normal current density $\boldsymbol{K}$, the order dissipation parameters Z^1 and Z^2, the entropy flux $\boldsymbol{\sigma}$ and the dissipation function R.

By requiring that the energy conservation law be of the form

$$\dot{E} + \operatorname{div}\boldsymbol{Q} = 0\,, \tag{21}$$

we find the following expressions:

$$\boldsymbol{Q} = T\boldsymbol{\sigma} + \mu K\,,$$
$$R = -\boldsymbol{\sigma}\cdot\nabla T + \boldsymbol{K}\cdot(E - \nabla\mu) - (\xi^1 Z^1 + \xi^2 Z^2)\,. \tag{22}$$

Then, separation of scalar and vector components of the generalized forces and fluxes results in

$$Z^1 = -(\beta_1 + \gamma_1|\psi|^2)\xi^1 + (\beta_2 - \gamma_2|\psi|^2)\xi^2\,,$$
$$Z^2 = -(\beta_2 + \gamma_2|\psi|^2)\xi^1 - (\beta_1 - \gamma_1|\psi|^2)\xi^2\,, \tag{23}$$

for the scalar components and

$$\begin{pmatrix} -\boldsymbol{\sigma} \\ K \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \nabla T \\ E - \nabla\mu \end{pmatrix} \tag{24}$$

for the vector components. In ref. [5], the thermoelectric contribution to $\boldsymbol{K}$ is attributed to Ginzburg [9] (see also ref. [10]).

According to the thermodynamics of irreversible processes, the kinetic coefficients must (a) satisfy Onsager relations and (b) keep R positive. To satisfy the Onsager relations requires $\alpha_{12} = \alpha_{21}$ and $\gamma_2 = 0$. The requirement that R be positive leads to the conditions $\beta_1 \pm \gamma_1|\psi|^2 > 0$; $\alpha_{11}, \alpha_{22} > 0$; sign α_{12} = sign$[(E - \nabla\mu)\cdot(\nabla T)]$, or else $\alpha_{12} = 0$.

A special case of eqs. (19) can be compared with results in the literature [3–5]. When

$$0 = \alpha_{11} = \alpha_{12} = \alpha_{21} = \gamma_1\,, \quad \alpha_{22} = \alpha > 0\,, \tag{25}$$

we have the following expressions to subsitute into eqs. (19):

$$Z^1 = -\beta_1\xi^1 + \beta_2\xi^2\,, \quad Z^2 = -\beta_2\xi^1 - \beta_1\xi^2\,, \quad \boldsymbol{\sigma} = 0\,,$$
$$K = \alpha(E - \nabla\mu)\,, \quad R = \beta_1[(\xi^1)^2 + (\xi^2)^2] + \alpha(E - \nabla\mu)^2 \tag{26}$$

In particular, the time-dependent Ginzburg–Landau equation for the complex order parameter, $\psi = \psi^1 + i\psi^2$, becomes

$$\dot{\psi} + i\mu\psi = -\beta\xi := Z\,, \tag{27}$$

where $\xi = \xi^1 + i\xi^2$, $\beta = \beta^1 + i\beta^2$, $Z = Z^1 + iZ^2$. In this limiting case, aside from differences in notation, we recover the results of Schmid [4], which are obtained by a hybrid microscopic and phenomenological approach, and of Gor'kov and Eliashberg [3], obtained from the microscopic theory in the special case of a gapless superconductor.

The result (27), however, differs from the result of Geurst [5], who finds, for example, that in this limit the Josephson relation (4) continues to hold. In contrast, we find that the modulus and phase of the order parameter obey

$$\partial_t|\psi|^2 = 2\,\mathrm{Re}(Z^*\psi)\,, \quad \dot{\Phi} + \mu = -\mathrm{Im}(Z^*\psi)/|\psi|^2\,. \tag{28}$$

And in particular, the Josephson relation (4) does not continue to hold. Evidently this discrepancy is due to the difference in the underlying non-dissipative equations noted earlier.

The results obtained here could also be extended to (a) include order parameters which take values in a non-abelian Lie algebra, or (b) let the "charges" lie in the dual space to a Lie algebra and interact, say, via a

Yang–Mills field. Such extensions would, thus, combine earlier results obtained for ^{3}He-A [8] and chromohydrodynamics [11]. A possible application of extension (a) is to superconductors with spin paramagnetism.

We would like to thank Fredrick Tappert for stimulating discussions. We are also grateful to the Center for Nonlinear Studies at Los Alamos, whose facilities made this work possible.

References

[1] D. Saint-James, G. Sarma and E.J. Thomas, Type II superconductivity (Pergamon, New York, 1969).
[2] M. Cyrot, Rep. Prog. Phys. 36 (1973) 103.
[3] L.P. Gor'kov and G.M. Eliashberg, Sov. Phys. JETP 27 (1968) 328.
[4] A. Schmid, Phys. Kondens. Mater. 5 (1966) 302.
[5] J.A. Geurst, Physica 101B (1980) 82.
[6] I.M. Khalatnikov and V.V. Lebedev, J. Low Temp. Phys. 32 (1978) 789.
[7] B.D. Josephson, Phys. Lett. 1 (1962) 251.
[8] D.D. Holm and B.A. Kupershmidt, Phys. Lett. 91A (1982) 425.
[9] V.L. Ginzburg, Zh. Eksp. Teor. Fiz. 14 (1944) 177.
[10] V.L. Ginzburg, Sov. Phys. Usp. 21 (1978) 381.
[11] J. Gibbons, D.D. Holm and B.A. Kupershmidt, Phys. Lett. 90A (1982) 281.