The analogy between spin glasses and Yang–Mills fluids

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A dictionary of correspondence is established between the dynamical variables for spin-glass fluid and Yang–Mills plasma. The Lie-algebraic interpretation of these variables is presented for the two theories. The noncanonical Poisson bracket for the Hamiltonian dynamics of an ideal spin glass is shown to be identical to that for the dynamics of a Yang–Mills fluid plasma, although the Hamiltonians differ for the two theories. This Poisson bracket is associated to the dual space of an infinite-dimensional Lie algebra of semidirect-product type.

I. INTRODUCTION

A. Background physics

Halperin and Saslow[1] and Andreev[2] have introduced condensed-matter theories of spin glasses, i.e., disordered magnetic spin systems whose ground states are degenerate under rotations. Condensed-matter systems whose ground states are degenerate under a continuous symmetry are often described macroscopically in terms of an order parameter field taking values in the Lie group associated with that symmetry. The order parameter field describing the low-temperature configurations of a spin glass in the Halperin–Saslow–Andreev theory is a spatially varying orthogonal matrix $O(\mathbf{x})$, acting on classical spin vectors at each point $\mathbf{x}$. The matrix $O(\mathbf{x})$ is assumed to be slowly varying in space (see also Toulouse,[3] Henley *et al.*,[4] Bray and Moore,[5] Saslow,[6] and Henley[7]). The spins themselves may be eliminated in the Halperin–Saslow–Andreev theory and their dynamics replaced by that of the order parameter field, $O(\mathbf{x},t)$.

Singularities in the order parameter field are called defects. These defects can be classified topologically by conventional homotopy theory (Toulouse and Kléman,[8] Volovik and Mineev,[9] Mermin,[10] and Michel[11]). The presence of defects (singularities in the order parameter field) suggests introducing additional degrees of freedom that may be described by gauge fields associated to the symmetry group of the degenerate ground state. For spin glasses, the symmetry group is SO(3) and these additional gauge fields have been introduced heuristically (in Dzyaloshinskii and Volovik,[12,13] Hertz,[14] José and Hertz,[15] and Dzyaloshinskii[16,17]) by replacing ordinary space derivatives with covariant derivatives according to the SO(3) minimal-coupling prescription in Hamilton's principle at the level of Ginzburg–Landau mean field theory for the order-parameter dynamics. (See also Fischer[18] and Rozhkov[19].) This Ginzburg–Landau type of model could presumably be derived from a lattice model in three dimensions (by the renormalization group method, for example), but as yet no explicit connection seems to have been made between the macroscopic gauge fields and microscopic concepts such as frustration in more than two dimensions. For the two-dimensional case, the concept of local exchange invariance on a frustrated planar lattice leads naturally to an analogy between nonlinear spin-glass hydrodynamics and Yang–Mills SO(3) gauge theory (see, e.g., Refs. 12 and 13).

The Ginzburg–Landau theory with covariant derivatives describes the dynamics of isolated defects in terms of dynamics of a gauge field. Interactions among defects (Andreev[2]) and defect cores (Kawasaki and Brand[20]) may be introduced by modifying the Hamiltonian or free energy of the system. For the case of spin glasses, the phenomenological theory so defined lacks the couplings between space and spin indices that complicate the free energies of superfluid ^{3}He-B (Toulouse and Kléman[8]), and cholesteric liquid crystals (Toulouse and Kléman,[8] Bouligand *et al.*,[21] Mermin[10]), which can also be described by order parameter fields taking values in SO(3). Other generalizations also exist, such as (1) local anisotropy (Saslow[22]), (2) remanence, an external field, or a tendency toward ferromagnetism (Halperin and Saslow[1]), and (3) dissipation, e.g., spin diffusion and relaxation of the order parameter (Halperin and Saslow[1]). Recent reviews of spin glasses are given in Fischer[18] and Chowdhury and Mookerjee.[23]

B. Problem statement

As one can glean from the previous remarks, there exists at least a partial analogy between fluid dynamics with internal degrees of freedom (e.g., spin-glass dynamics, superfluids, and other quantum liquids) and Yang–Mills fluid dynamics. This analogy was introduced for spin glasses by Dzyaloshinskii and Volovik[12,13] and Volovik and Dotsenko,[24] and discussed for superfluids and other quantum liquids by Dzyaloshinskii and Volovik[13] and Khalatnikov and Lebedev.[25] Here, we propose to examine this analogy in the framework of the Hamiltonian formulation of nonlinear hydrodynamic theories. In Sec. II, we present a unification of the nondissipative theories of spin-glass dynamics, Yang–Mills plasmas, and Yang–Mills magnetohydrodynamics that combines their various Hamiltonian formulations into a single Poisson bracket, which we associate in Sec. III to the dual space of a Lie algebra endowed with two different types of nontrivial generalized two cocycles.

During the past few years, Poisson bracket methods have been used to derive nonlinear hydrodynamic equations for various complex fluid systems. These systems include spin glasses (Dzyaloshinskii and Volovik[13]); hydrodynamics of defects in the continuum description of condensed matter, e.g., vortices in superfluid ^{4}He and disclinations in a planar magnet (Volovik and Dotsenko[24]); rotating superfluid ^{4}He and ^{3}He with spin and orbital angular momentum (Khalatnikov and Lebedev,[25] Holm and Kupershmidt[26]); as well as Yang–Mills plasmas (Gibbons, Holm, and Kupershmidt,[27] Holm and Kupershmidt[28]).

The Poisson bracket method provides a guide for determining conservation laws and a framework for studying Lyapunov stability of equilibrium solutions (see Holm *et al.*[29]), as well as a structure for pointing out similarities and differences among various theories. We emphasize the latter structural aspect in this work, by showing that the Poisson brackets for spin glasses and Yang–Mills plasmas are isomorphic. Thus, although the Hamiltonians and physical interpretations of the two theories differ, the Lie-algebraic nature of their Hamiltonian structures is the same. This Lie-algebraic nature allows us in Sec. IV to set up a dictionary of correspondence between the dynamical variables for spin-glass fluid and Yang–Mills plasma.

II. SPIN-GLASS DYNAMICS AND YANG–MILLS MAGNETOHYDRODYNAMICS

The gauge-field formulation of the nonlinear hydrodynamic equations describing the continuum dynamics of defects in condensed matter is developed in Dzyaloshinskii and Volovik.[12,13,30] In this formulation, gauge fields are introduced via the minimal-coupling hypothesis in Hamilton's principle as additional variables coupled to the defects, represented in turn as densities of gauge charges. Physical applications include crystals with continuously distributed dislocations and disclinations; superfluid HeII with vortices; liquid crystals with rotational disclinations; and two-dimensional spin glasses, regarded as the continuum limit of a planar lattice of magnets with disclinations.

The problem of formulating nonlinear dynamical equations for ideal (nondissipative) media containing continuously distributed defects is addressed here via the Hamiltonian approach. That is, the dynamics of a continuously defected medium is represented in Hamiltonian form, i.e., as

$$\partial_t \mathbf{u} = \{H,\mathbf{u}\}, \tag{2.1}$$

for Hamiltonian H and Poisson bracket $\{\ ,\ \}$ defined on the space of dynamical variables $\mathbf{u}$.

An example of such a system and the starting point for the present analysis is the theory for spin glass (continuum limit of an antiferromagnet having nonzero equilibrium disclination density) of Volovik and Dotsenko.[24] In this theory, the gauge-charge density G is the three-component magnetization density, which generates the internal symmetry group of three-dimensional rotations. The corresponding gauge potential A_i, $i = 1,2,3$, transforms under these internal symmetry rotations like a gauge field (see, e.g., Drechsler and Mayer[31]). The disclination density is identified with the gauge-field intensity

$$B_{ik} = A_{i,k} - A_{k,i} + [A_i, A_k] \tag{2.2a}$$

or, componentwise,

$$B^{\alpha}_{ik} = A^{\alpha}_{i,k} - A^{\alpha}_{k,i} + t^{\alpha}_{\beta\gamma} A^{\beta}_{i} A^{\gamma}_{k}, \tag{2.2b}$$

with notation explained below.

In our notation, Latin indices $i,j,k,\ldots$, run from 1 to n ($n = 3$ for three-dimensional space), script Latin indices $\mathscr{a},\mathscr{b},\mathscr{c},\ldots$, run from 0 to n, and the charge G belongs to the dual $\mathfrak{g}^*$ of the gauge-symmetry Lie algebra $\mathfrak{g}$, with $A_i \in \mathfrak{g}$. The adjoint representation map ad: $\mathfrak{g} \to \operatorname{End} \mathfrak{g}$ denotes multiplication in $\mathfrak{g}$: $\operatorname{ad}(y)z = [y,z]$. Another map $\mathfrak{g} \to \mathfrak{g}^*$, $\mathfrak{g} \ni y \to {}^*y \in \mathfrak{g}^*$ is defined by the rule

$$\langle {}^*y, z\rangle = (y,z), \tag{2.3}$$

where $(\ ,\ )$ is an invariant symmetric nondegenerate form on $\mathfrak{g}$ (e.g., the Killing form, for $\mathfrak{g}$ semisimple). The structure constants of $\mathfrak{g}$ are denoted $t^{\gamma}_{\alpha\beta}$ [see (2.2)] in a basis with elements e_α, where Greek indices run from 1 to $M = \dim \mathfrak{g}$. In this basis, we have the commutator relation

$$[e_\alpha, e_\beta] = t^{\gamma}_{\alpha\beta} e_\gamma. \tag{2.4}$$

We denote $A_i = A^{\alpha}_{i} e_\alpha$ and $G = G_\alpha e^\alpha$, where e^α, $\alpha = 1,\ldots,M$, are elements of the dual basis, satisfying $\langle e^\alpha, e_\beta\rangle = \delta^{\alpha}_{\beta}$. The rule (2.3) associates to each element $y \in \mathfrak{g}$ a corresponding dual element ${}^*y \in \mathfrak{g}^*$, via

$${}^*y_\beta z^\beta := \langle {}^*y, z\rangle = (y,z) =: y^\alpha g_{\alpha\beta} z^\beta, \tag{2.5}$$

where $g_{\alpha\beta} = (e_\alpha, e_\beta)$ is the matrix of the invariant form in the basis $\{e_\alpha\}$.

To the linear operation ad on $\mathfrak{g}$, there corresponds another linear operation ad* (essentially minus the transpose of ad, in a matrix representation), which acts on $\mathfrak{g}^*$ as defined by

$$\langle \operatorname{ad}^*(y)\,{}^*z, x\rangle := \langle {}^*z, [x,y]\rangle \tag{2.6}$$

for $x,y \in \mathfrak{g}$ and ${}^*z \in \mathfrak{g}^*$. In components, then,

$$\begin{aligned}(\operatorname{ad}^*(y)\,{}^*z)_\alpha x^\alpha &= \langle \operatorname{ad}^*(y)\,{}^*z, x\rangle = -\langle {}^*z, \operatorname{ad}(y)x\rangle \\ &= -\langle {}^*z, [y,x]\rangle = -z_\gamma t^{\gamma}_{\beta\alpha} y^\beta x^\alpha,\end{aligned}$$

so that

$$(\operatorname{ad}^*(y)\,{}^*z)_\alpha = -y^\beta t^{\gamma}_{\beta\alpha} z_\gamma. \tag{2.7}$$

We may now define $(n+1)$ covariant derivative operators acting on $\mathfrak{g}$-valued functions of space and time. Namely,

$$\mathbf{D} = \nabla - \operatorname{ad}(\mathbf{A}), \tag{2.8a}$$

with n spatial components

$$D_i = \partial_i - \operatorname{ad}(A_i), \tag{2.8b}$$

and

$$D_t = \partial_t - \operatorname{ad}(A_0), \tag{2.9}$$

for the time component. Similarly, one defines $(n+1)$ covariant derivative operators acting on $\mathfrak{g}^*$-valued functions

$$\mathbf{D}^* = \nabla - \operatorname{ad}^*(\mathbf{A}), \tag{2.10a}$$

with components

$$D^*_i = \partial_i - \operatorname{ad}^*(A_i), \tag{2.10b}$$

and

$$D^*_t = \partial_t - \operatorname{ad}^*(A_0). \tag{2.11}$$

If ϕ and $\bar{\phi}$ are functions of space and time with values in $\mathfrak{g}$ and

$\mathfrak{g}^*$, respectively, then, for example, we have the partial-derivative relations

$$\langle\bar{\phi},\phi\rangle_{,i} = \langle (D_i^*\bar{\phi}),\phi\rangle + \langle\bar{\phi},(D_i\phi)\rangle, \tag{2.12}$$

since

$$D_i^* = -(D_i)^\dagger, \tag{2.13}$$

where † stands for the "adjoint."

We also have

$$^*[D_a(w)] = D_a^*(^*w), \quad \forall w\in\mathfrak{g}, \quad \forall a\in(0,...,n). \tag{2.14}$$

Indeed, for any $y\in\mathfrak{g}$, we have, denoting $x = A_a$,

$$\begin{aligned}\langle ^*[D_a(w)],y\rangle &= (D_a(w),y) = (w_{,a} - \mathrm{ad}_x(w),y)\\ &= (w_{,a},y) - ([x,w],y)\\ &= (w_{,a},y) + (w,[x,y]),\end{aligned} \tag{2.15a}$$

and

$$\begin{aligned}\langle D_a^*(^*w),y\rangle &= \langle ^*w_{,a},y\rangle - \langle \mathrm{ad}_x^*(^*w),y\rangle\\ &= (w_{,a},y) + \langle ^*w,[x,y]\rangle\\ &= (w_{,a},y) + (w,[x,y]).\end{aligned} \tag{2.15b}$$

Comparison of (2.15a) and (2.15b) proves (2.14).

From the covariant derivative operators, one defines the fields

$$[D_i,D_t] = \mathrm{ad}(E_i), \tag{2.16a}$$

$$[D_i,D_j] = \mathrm{ad}(B_{ij}), \tag{2.16b}$$

with spatial components

$$E_i = A_{i,t} - A_{0,i} + [A_i,A_0] = F_{0i} = -F_{i0}, \tag{2.17a}$$

$$B_{ij} = A_{i,j} - A_{j,i} + [A_i,A_j] = -F_{ij}, \tag{2.17b}$$

where subscript-comma notation is used for partial derivatives, e.g., $A_{0,i} = (\partial A_0/\partial x^i)$. In n spatial dimensions, the one-form $\mathbf{E}$ has n spatial components E_i, and the two-form B has $n(n-1)/2$ independent spatial components B_{ij}, with $B_{ij} = -B_{ji}$ (skew symmetric).

In Yang–Mills plasma theory (Gibbons, Holm, and Kupershmidt,[27] Holm and Kupershmidt[28]), the Yang–Mills fields $F_{a\ell}$ appearing in (2.17) satisfy

$$^*(D_a F^{a\ell}) = J^\ell, \quad a,\ell, = 0,1,...,n, \tag{2.18}$$

where J^ℓ with components $J^0 = G$, $J^i = Gv^i$, is the gauge current density, with v^i, $i = 1,...,n$, denoting velocity components of the moving medium. Script indices are raised and lowered by the Lorentz metric, with signature $(n-1)$. The gauge charge is conserved, since

$$\begin{aligned}D_\ell^* J^\ell &= D_\ell^{*}{}^*(D_a F^{a\ell}) \quad [\text{by } (2.18)]\\ &= ^*(D_\ell D_a F^{a\ell}) \quad [\text{by } (2.14)]\\ &= -\tfrac{1}{2}\mathrm{ad}(F_{a\ell})F^{a\ell} = 0 \quad [\text{by (2.16) and (2.17)}].\end{aligned} \tag{2.19}$$

In the Volovik–Dotsenko spin-glass theory, the structure constants $t^\alpha_{\beta\gamma}$ in (2.2) for the gauge symmetry algebra are those of so(3): $t^\alpha_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}$, the totally antisymmetric tensor in dim $\mathfrak{g} = 3$ dimensions, with $\epsilon_{123} = -1$. Let K_i be the defect momentum density and ρ the inertial mass density of the defects. The Poisson bracket for spin glass proposed by Volovik and Dotsenko[24] is then expressible as

$$\begin{aligned}\{H,F\} = -\int d^n x \Bigg\{ &\frac{\delta F}{\delta K_j}\left[(K_i\partial_j + \partial_i K_j + B^\alpha_{ji}G_\alpha)\frac{\delta H}{\delta K_i} + B^\alpha_{ji}\frac{\delta H}{\delta A^\alpha_i} + \rho\partial_j\frac{\delta H}{\delta\rho}\right]\\ &+ \frac{\delta F}{\delta G_\alpha}\left[t^\gamma_{\alpha\beta}G_\gamma\frac{\delta H}{\delta G_\beta} + (\partial_i\delta^\beta_\alpha - t^\beta_{\alpha\gamma}A^\gamma_i)\frac{\delta H}{\delta A^\beta_i}\right] + \frac{\delta F}{\delta\rho}\partial_i\rho\frac{\delta H}{\delta K_i}\\ &+ \frac{\delta F}{\delta A^\alpha_j}\left[B^\alpha_{ji}\frac{\delta H}{\delta K_i} + (\delta^\alpha_\beta\partial_j + t^\alpha_{\beta\gamma}A^\gamma_j)\frac{\delta H}{\delta G_\beta}\right]\Bigg\},\end{aligned} \tag{2.20}$$

in three dimensions ($n=3$) and for functionals H and F of the dynamical variables $(K_j,G_\alpha,\rho,A^\alpha_j)$. In Hamiltonian matrix form, the spin-glass equations corresponding to the Poisson bracket (2.20) are

$$\partial_t \begin{vmatrix} K_j \\ \rho \\ A^\alpha_j \\ G_\alpha \end{vmatrix} = -\begin{vmatrix} (K_i\partial_j + \partial_i K_j + B^\alpha_{ji}G_\alpha) & \rho\partial_j & B^\beta_{ji} & 0 \\ \partial_i\rho & 0 & 0 & 0 \\ B^\alpha_{ji} & 0 & 0 & (\delta^\alpha_\beta\partial_j + t^\alpha_{\beta\gamma}A^\gamma_j) \\ 0 & 0 & (\delta^\beta_\alpha\partial_i - t^\beta_{\alpha\gamma}A^\gamma_i) & t^\gamma_{\alpha\beta}G_\gamma \end{vmatrix} \begin{vmatrix} \delta H/\delta K_i \\ \delta H/\delta\rho \\ \delta H/\delta A^\beta_i \\ \delta H/\delta G_\beta \end{vmatrix} \tag{2.21a}$$

for Hamiltonian [Volovik and Dotsenko,[24] Eq. (6.11)]

$$H = \int d^n x\left[\frac{1}{2\rho}|\mathbf{K}|^2 + \frac{1}{2}\rho^*\mathbf{A}_\alpha\cdot\mathbf{A}^\alpha + \frac{1}{2\chi}G^*_\alpha G^\alpha\right], \tag{2.21b}$$

with constant susceptibility χ.

In general, Hamiltonian equations are expressible as

$$\partial_t\mathbf{u} = \mathsf{b}\cdot\frac{\delta H}{\delta\mathbf{u}} = \{H,\mathbf{u}\}, \tag{2.22}$$

where the Hamiltonian matrix b defines the Poisson bracket $\{H,F\}$ in terms of the dynamical variables $\mathbf{u}$ according to the standard form

$$\{H,F\} = \int d^n x \frac{\delta F}{\delta \mathbf{u}} \cdot \mathsf{b} \cdot \frac{\delta H}{\delta \mathbf{u}}. \tag{2.23}$$

The spin-glass Poisson bracket in (2.20) defined by the Hamiltonian matrix b given in (2.21a) is bilinear, skew symmetric, and satisfies the Jacobi identity. To demonstrate the last property (which is neither self-evident nor trivial), we map the Hamiltonian matrix b in (2.21a) into *an affine* form, by using the invertible transformation

$$\mathbf{P} = \mathbf{K} + G_\alpha \mathbf{A}^\alpha, \tag{2.24}$$

and leaving ρ, $\mathbf{A}^\alpha$, and G_α unchanged. Under such a map, the Hamiltonian matrix b changes according to the chain rule, i.e.,

$$\mathsf{b}_1 = \mathsf{J}\cdot\mathsf{b}\cdot\mathsf{J}^\dagger, \tag{2.25}$$

where J is the Fréchet derivative of the map (2.24) and $\mathsf{J}^\dagger$ is its adjoint. The resulting Hamiltonian formulation of the spin-glass equations in the new variables $(P_i,\rho,A_i^\alpha,G_\alpha)$ is found [after matrix multiplication as in (2.25) and elimination of old variables **K** in favor of new ones **P**], to be

$$\partial_t \begin{vmatrix} P_i \\ \rho \\ A_i^\alpha \\ G_\alpha \end{vmatrix} = - \begin{vmatrix} P_k\partial_i + \partial_k P_i & \rho\partial_i & \partial_k A_i^\beta - A_{k,i}^\beta & G_\beta \partial_i \\ \partial_k \rho & 0 & 0 & 0 \\ A_k^\alpha \partial_i + A_{i,k}^\alpha & 0 & 0 & \delta_\beta^\alpha \partial_i + t_{\beta\gamma}^\alpha A_i^\gamma \\ \partial_k G_\alpha & 0 & \delta_\alpha^\beta \partial_k - t_{\alpha\gamma}^\beta A_k^\gamma & t_{\alpha\beta}^\gamma G_\gamma \end{vmatrix} \begin{vmatrix} \delta H/\delta P_k \\ \delta H/\delta\rho \\ \delta H/\delta A_k^\beta \\ \delta H/\delta G_\beta \end{vmatrix}, \tag{2.26a}$$

with Hamiltonian

$$H = \int d^n x \left[\frac{1}{2\rho} |\mathbf{P} + G_\alpha \mathbf{A}^\alpha|^2 + \frac{1}{2}\rho^* \mathbf{A}_\alpha \cdot \mathbf{A}^\alpha + \frac{1}{2\chi} G_\alpha^* G^\alpha \right]. \tag{2.26b}$$

By being affine (linear plus constant) in the dynamical variables, the Hamiltonian matrix b_1 in (2.26a) yields a Poisson bracket [given by (2.23) with b replaced by b_1] that may be associated to the dual space of a certain Lie algebra with a generalized two-cocycle on it (Kupershmidt[32]). In this case, the Lie algebra is of semidirect-product type,

$$\mathfrak{g}_1 = D \circledS \{\Lambda^0 \oplus [(\Lambda^0 \otimes \mathfrak{g}) \circledS (\Lambda^{n-1} \otimes \mathfrak{g}^*)]\}, \tag{2.27}$$

where D is the Lie algebra of vector fields on $\mathbf{R}^n$ and Λ^i is the space of differential i forms on $\mathbf{R}^n$. The dual coordinates are P_i dual to $\partial_i \in D$; ρ, to $1 \in \Lambda^0$; G_α, to $1 \otimes e^\alpha \in (\Lambda^0 \otimes \mathfrak{g})$, i.e., functions taking values in Lie algebra $\mathfrak{g}$, the symmetry algebra; and A_i^α dual to $(\partial_i \lrcorner\, d^n x) \otimes e^\alpha \in (\Lambda^{n-1} \otimes \mathfrak{g}^*)$, i.e., $(n-1)$ forms taking values in the dual symmetry algebra $\mathfrak{g}^*$. In (2.27), $\circledS$ denotes semidirect product; $\otimes$, tensor product; and $\oplus$, direct sum. Mathematical discussion of this Lie algebra is deferred until Sec. III. At this point, we only remark that association of the b_1 Poisson bracket to the dual of the Lie algebra $\mathfrak{g}_1$ assures that the Jacobi identity for the b_1 Poisson bracket is satisfied. Since b_1 is related to b in (2.21a) by the invertible transformation (2.24), the Jacobi identity is also satisfied for the Poisson bracket (2.20) defined by the Hamiltonian matrix b in (2.21a).

Remarkably enough, a gauge-covariant Poisson bracket for spin glasses exists and is canonically related via (2.17b) to the Poisson bracket corresponding to the Hamiltonian matrix b_1 expressed in (2.26a) in terms of gauge potential A_i^α. The new Hamiltonian matrix b_2 is expressed in terms of the gauge field (disclination density) B_{ij}^α, using definition (2.2) in a chain-rule matrix multiplication as in (2.25). The resulting matrix-Hamiltonian equations for spin glass are now, in terms of B_{ij}^α [cf. (2.26a)],

$$\partial_t \begin{vmatrix} P_i \\ \rho \\ B_{ij}^\alpha \\ G_\alpha \end{vmatrix} = - \begin{vmatrix} P_k\partial_i + \partial_k P_i & \rho\partial_i & -B_{lm,i}^\beta + \partial_m B_{li}^\beta - \partial_l B_{mi}^\beta & G_\beta \partial_i \\ \partial_k \rho & 0 & 0 & 0 \\ B_{ij,k}^\alpha + B_{ik}^\alpha \partial_j - B_{jk}^\alpha \partial_i & 0 & 0 & t_{\beta\gamma}^\alpha B_{ij}^\gamma \\ \partial_k G_\alpha & 0 & -t_{\alpha\gamma}^\beta B_{lm}^\gamma & t_{\alpha\beta}^\gamma G_\gamma \end{vmatrix} \begin{vmatrix} \delta H/\delta P_k \\ \delta H/\delta\rho \\ \delta H/\delta B_{lm}^\beta \\ \delta H/\delta G_\beta \end{vmatrix}. \tag{2.28}$$

The Hamiltonian matrix b_2 in (2.28) is now *linear* in the dynamical variables and thus (Kupershmidt[32]) may be associated to the dual of a Lie algebra. In this case, the Lie alebra is again a semidirect product,

$$\mathfrak{g}_3 = D \circledS \{\Lambda^0 \oplus [(\Lambda^0 \otimes \mathfrak{g}) \circledS (\Lambda^{n-2} \otimes \mathfrak{g}^*)]\}, \tag{2.29}$$

with the *same* dual coordinates as in the case of b_1 associated to $\mathfrak{g}_1$in (2.27) *except* that instead of A_i^α dual to $(\partial_i \lrcorner\, d^n x) \otimes e^\alpha \in (\Lambda^{n-1} \otimes \mathfrak{g}^*)$, we now have B_{ij}^α dual to $(\partial_i \lrcorner\, \partial_j \lrcorner\, d^n x) \otimes e^\alpha \in (\Lambda^{n-2} \otimes \mathfrak{g}^*)$, i.e., $\{B_{ij}^\alpha\}$ dual to $(n-2)$ forms taking values in the dual gauge algebra, $\mathfrak{g}^*$.

Yang–Mills MHD: The Poisson matrices b_1 and b_2 for spin glasses in (2.26a) and (2.28) extend the corresponding matrices for Yang–Mills magnetohydrodynamics (YM–MHD) (Holm and Kupershmidt[28]), by allowing nonzero entries for Poisson brackets between the gauge charges and the gauge fields. The Hamiltonian for YM–MHD is (Holm and Kupershmidt[28])

$$H = \int d^n x \left[\frac{1}{2\rho} |\mathbf{P}|^2 + U(\rho) + \frac{1}{4}({}^*B_{\alpha ij} B^{\alpha}_{ij}) \right]. \tag{2.30}$$

Remarkably, when the YM–MHD Hamiltonian (2.30) is used with the *spin-glass* Hamiltonian matrices b_1 and b_2 in (2.26a) and (2.28), respectively, the same dynamical equations reemerge for YM–MHD as in Holm and Kupershmidt.[28] That is, correct YM–MHD equations reappear using the spin-glass Poisson bracket (2.28) with the YM–MHD Hamiltonian (2.30).

The spin-glass Hamiltonian matrices b_1 and b_2 extend their YM–MHD counterparts found in Holm and Kupershmidt[28] by allowing semidirect-product actions instead of simple direct sums among quantities dual to gauge charges and gauge fields, in the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_3$. Since this extension of Hamiltonian matrices is available for YM–MHD, it is natural to expect the Hamiltonian matrix for chromohydrodynamics (CHD: the non-Abelian Yang–Mills plasma theory from which YM–MHD is derived) also to have an extended counterpart. This extended counterpart may, in turn, find application in the theory of condensed matter with internal symmetry variables.

To determine this extension of the CHD Hamiltonian matrix, we propose to argue heuristically: we start from the extended YM–MHD/spin-glass Hamiltonian matrix in (2.26a) and enlarge it, by comparing its structure to that for Abelian charged fluids (Holm[33]).

There is a standard derivation (see, e.g., Friedberg[34]) of *Abelian* MHD from the ideal two-fluid Abelian plasma equations. Abelian MHD emerges in the course of this derivation in the limit that the dielectric constant vanishes (i.e., displacement current is neglected), the inertia of one species (the electrons) is negligible compared to the other (the ions), local charge neutrality is imposed, and drift effects (diamagnetic and Hall electric fields) are neglected. In Holm and Kupershmidt[28] this derivation has been adapted for the purpose of obtaining the non-Abelian YM–MHD theory from the equations of chromohydrodynamics (CHD), treated in Gibbons, Holm, and Kupershmidt.[27] The CHD equations describe non-Abelian Yang–Mills plasma theory, e.g., quark-gluon plasma physics, in the fluid description obtained by taking moments of the corresponding kinetic theory with particles interacting via Yang–Mills fields (i.e., Wong's equations). A consistent Hamiltonian theory of special relativistic CHD also exists (Holm and Kupershmidt[28]).

Abelian MHD may also be considered as a special case of the Hamiltonian theory of Abelian charged-fluid (ACF) motion that includes moving-material electromagnetic effects. The equations of ACF dynamics are given in the following Hamiltonian matrix form in Holm[33]:

$$\partial_t \begin{vmatrix} P_i \\ \rho \\ A_i \\ {}^*E_i \\ Q \end{vmatrix} = - \begin{vmatrix} P_k\partial_i + \partial_k P_i & \rho\partial_i & \partial_k A_i - A_{k,i} & {}^*E^k\partial_i - \partial_j {}^*E^j \delta_i^k & Q\partial_i \\ \partial_k \rho & 0 & 0 & 0 & 0 \\ A_k\partial_i + A_{i,k} & 0 & 0 & \delta_i^k & s\partial_i \\ \partial_k {}^*E^i - {}^*E^j\partial_j\delta_k^i & 0 & -\delta_k^i & 0 & 0 \\ \partial_k Q & 0 & s\partial_k & 0 & 0 \end{vmatrix} \begin{vmatrix} \delta H/\delta P_k \\ \delta H/\delta\rho \\ \delta H/\delta A_k \\ \delta H/\delta {}^*E^k \\ \delta H/\delta Q \end{vmatrix}. \tag{2.31}$$

In (2.31) the Abelian charge density Q satisfies Gauss's law,

$$a\rho = Q = \operatorname{div} {}^*\mathbf{E}, \tag{2.32}$$

which is preserved by the dynamics. In (2.32) the quantity a is the constant charge-to-mass ratio in ACF dynamics and ${}^*\mathbf{E}$ is the electric displacement vector. Actually, the Hamiltonian matrix in (2.31) is a slight extension of that in Holm[33] to include a generalized two-cocycle between Q and $\mathbf{A}$ [the terms proportional to the arbitrary constant s in (2.31)]; Holm[33] chooses $s = 0$.

The Hamiltonian matrix for Abelian MHD may be recovered either from (2.31) for $s = 1$ when the displacement vector ${}^*\mathbf{E}$ is absent, or from (2.26a) in the Abelian case, when the structure constants $t^{\alpha}_{\beta\gamma}$ vanish.

Comparing the Hamiltonian matrices (2.26a) for non-Abelian MHD and (2.31) for Abelian charged fluids suggests the following Hamiltonian matrix for the dynamics of *non-Abelian* charged fluids:

$$\partial_t \begin{vmatrix} P_i \\ \rho \\ A^{\alpha}_i \\ {}^*E^i_{\alpha} \\ G_{\alpha} \end{vmatrix} = - \begin{vmatrix} P_k\partial_i + \partial_k P_i & \rho\partial_i & \partial_k A^{\beta}_i - A^{\beta}_{k,i} & {}^*E^k_{\beta}\partial_i - \partial_j {}^*E^j_{\beta}\delta_i^k & G_{\beta}\partial_i \\ \partial_k\rho & 0 & 0 & 0 & 0 \\ A^{\alpha}_k\partial_i + A^{\alpha}_{i,k} & 0 & 0 & \delta^{\alpha}_{\beta}\delta_i^k & s\delta^{\alpha}_{\beta}\partial_i + t^{\alpha}_{\beta\gamma}A^{\gamma}_i \\ \partial_k {}^*E^i_{\alpha} - {}^*E^j_{\alpha}\partial_j\delta^i_k & 0 & -\delta^{\beta}_{\alpha}\delta^i_k & 0 & t^{\gamma}_{\alpha\beta}\, {}^*E^i_{\gamma} \\ \partial_k G_{\alpha} & 0 & s\delta^{\beta}_{\alpha}\delta_k - t^{\beta}_{\alpha\gamma}A^{\gamma}_k & t^{\gamma}_{\alpha\beta}\, {}^*E^k_{\gamma} & t^{\gamma}_{\alpha\beta}G_{\gamma} \end{vmatrix} \begin{vmatrix} \delta H/\delta P_k \\ \delta H/\delta\rho \\ \delta H/\delta A^{\beta}_k \\ \delta H/\delta {}^*E^k_{\beta} \\ \delta H/\delta G_{\beta} \end{vmatrix}, \tag{2.33}$$

where s is any real constant. This Hamiltonian matrix reduces to (2.26a) when $*\mathbf{E}$ is absent and $s = 1$, and to (2.31) in the Abelian case. In comparison with (2.20a) for YM–MHD the Hamiltonian matrix $\mathfrak{b}_3$ in (2.33) for Yang–Mills charged fluids (YMCF) has been extended by adding a row and column for the dynamics of the variable $*E^k_\beta$, the Yang–Mills analog of the electric displacement vector. The vector $*E^k_\beta$ is dual to A^α_i in both the algebraic and metric sense: while A^α_i is a one-form taking values in the gauge algebra $\mathfrak{g}$, $*E^k_\beta$ is an $(n-1)$ form taking values in the dual algebra $\mathfrak{g}^*$. The mathematical interpretation of the Poisson bracket determined from $\mathfrak{b}_3$ in (2.33) is given in Sec. III.

The relation of (2.33) for non-Abelian Yang–Mills charged fluids (YMCF) to the Hamiltonian matrix for CHD given in Gibbons, Holm, and Kupershmidt[27] is as follows. Let $\mathbf{M}$ define another momentum density via the map

$$M_i = P_i + \langle D^{**}_k E^k, A_i\rangle - \langle *E^k, B_{ki}\rangle = P_i + (*E^k_\alpha A^\alpha_i)_{,k} - *E^k_\alpha A^\alpha_{k,i}, \tag{2.34}$$

while the other variables $(\rho, \mathbf{A}, *\mathbf{E}, G)$ in (2.33) remain the same. The resulting Hamiltonian matrix in the new variables obtained via direct calculation using (2.25) is given (with $s = 1$) by

$$\partial_t \begin{vmatrix} M_j \\ \rho \\ A^\beta_j \\ *E^j_\beta \\ G_\beta \end{vmatrix} = - \begin{vmatrix} M_i\partial_j + \partial_i M_j & \rho\partial_j & 0 & 0 & [G_\alpha + *(\mathrm{Div}\,\mathbf{E})_\alpha]\partial_j \\ \partial_i\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^\beta_\alpha\delta^i_j & (D_j)^\beta_\alpha \\ 0 & 0 & -\delta^\alpha_\beta\delta^i_j & 0 & t^\mu_{\beta\alpha}\, *E^j_\mu \\ \partial_i[G_\beta + *(\mathrm{Div}\,\mathbf{E})_\beta] & 0 & (D^*_i)^\alpha_\beta & t^\mu_{\beta\alpha}\, *E^i_\mu & t^\mu_{\beta\alpha} G_\mu \end{vmatrix} \begin{vmatrix} \delta H/\delta M_i \\ \delta H/\delta\rho \\ \delta H/\delta A^\alpha_i \\ \delta H/\delta *E^i_\alpha \\ \delta H/\delta G_\alpha \end{vmatrix}. \tag{2.35}$$

In the extended CHD Hamiltonian matrix (2.35) we denote [see (2.9) and (2.12)]

$$(D_j)^\beta_\alpha = \partial_j\delta^\beta_\alpha + t^\beta_{\alpha\mu}A^\mu_j, \tag{2.36a}$$

$$(D^*_i)^\alpha_\beta = \partial_i\delta^\alpha_\beta - t^\alpha_{\beta\mu}A^\mu_i, \tag{2.36b}$$

and [see (2.7) and (2.15)]

$$\begin{aligned} *(\mathrm{Div}\,\mathbf{E})_\beta &= [\mathrm{Div}^*(*\mathbf{E})]_\beta = (D^*_k)^\alpha_\beta\, *E^k_\alpha, \\ [\text{by (2.36b)}] &= \partial_k\, *E^k_\beta - A^\mu_k\, *E^k_\alpha t^\alpha_{\beta\mu}, \\ [\text{by (2.7)}] &= \{[\partial_k - \mathrm{ad}^*(A_k)](*E^k)\}_\beta. \end{aligned} \tag{2.37}$$

The CHD Hamiltonian is (Gibbons, Holm, and Kupershmidt[27])

$$H = \int d^nx \left[\frac{1}{2\rho}|\mathbf{M} + G_\alpha\mathbf{A}^\alpha|^2 + U(\rho) + \frac{1}{2}\, *\mathbf{E}_\alpha\cdot\mathbf{E}^\alpha + \frac{1}{4}\, *B^{ik}_\alpha B^\alpha_{ik}\right], \tag{2.38}$$

with variational derivatives given by

$$\delta H = \int d^nx \left\{\left(-\frac{v^2}{2} + U'(\rho)\right)\delta\rho + (\mathbf{v}\cdot\mathbf{A}^\alpha)\delta G_\alpha + \mathbf{v}\cdot\delta\mathbf{M} + \mathbf{E}^\alpha\cdot\delta *\mathbf{E}_\alpha + [G_\alpha v^i + *(D_kB^{ki})_\alpha]\delta A^\alpha_i\right\}, \tag{2.39}$$

where we have integrated by parts and introduced the notation

$$\mathbf{v} = \rho^{-1}(\mathbf{M} + G_\alpha\mathbf{A}^\alpha), \tag{2.40}$$

$$\mathbf{E} = *(*\mathbf{E}), \tag{2.41a}$$

$$*B_{ij} = *(B_{ij}). \tag{2.41b}$$

The resulting Hamiltonian equations of motion for CHD are [using (2.22) to define the CHD Poisson bracket with Hamiltonian matrix $\mathfrak{b}_4$ in (2.35)]

$$\partial_t\rho = \{H, \rho\} = -\partial_i(\rho v^i), \tag{2.42a}$$

$$\partial_t A^\beta_j = \{H, A^\beta_j\} = -E^\beta_j - (D_j)^\beta_\alpha(\mathbf{v}\cdot\mathbf{A}^\alpha), \tag{2.42b}$$

$$\partial_t\, *E^j_\beta = \{H, *E^j_\beta\} = G_\beta v^j + *(D_kB^{kj})_\beta - t^\mu_{\beta\alpha}\, *E^j_\mu(\mathbf{v}\cdot\mathbf{A}^\alpha), \tag{2.42c}$$

$$\begin{aligned} \partial_t G_\beta = \{H, G_\beta\} = &-\partial_i([G_\beta + *(\mathrm{Div}\,\mathbf{E})_\beta]v^i) \\ &- (D^*_i)^\alpha_\beta(G_\alpha v^i) - *(D_iD_kB^{ki})_\beta \\ &- t^\mu_{\beta\alpha}\, *E^i_\mu E^\alpha_i - t^\mu_{\beta\alpha}G_\mu(\mathbf{v}\cdot\mathbf{A}^\alpha). \end{aligned} \tag{2.42d}$$

Upon using (2.36b), (2.41a), and antisymmetry of B^{ki}, the G_β equation (2.42d) takes the form

$$\partial_t G_\beta = -\partial_i([G_\beta + *(\mathrm{Div}\,\mathbf{E})_\beta]v^i) - \partial_i(G_\beta v^i). \tag{2.43}$$

This becomes simply the equation for gauge charge conservation upon setting

$$\text{GAUSS:} = G + *(\mathrm{Div}\,\mathbf{E}) = 0, \tag{2.44}$$

and noting that this relation is preserved by the dynamics of (2.42b)–(2.42d), since

$$\begin{aligned} \partial_t(\text{GAUSS}) &= -\mathrm{div}[(\text{GAUSS})\mathbf{v}] + \mathrm{ad}^*(\mathbf{v}\cdot\mathbf{A})(\text{GAUSS}) \\ &= -D^*_i[(\text{GAUSS})v^i]. \end{aligned} \tag{2.45}$$

The proof of relation (2.45) is by direct computation, as follows. Using (2.44) we have

$$\begin{aligned} \partial_t(\text{GAUSS})_\beta &= \partial_t[G_\beta + (\partial_i\delta^\alpha_\beta - t^\alpha_{\beta\gamma}A^\gamma_i)*E^i_\alpha] \\ &= \partial_t G_\beta - t^\alpha_{\beta\gamma}(\partial_t A^\gamma_i)*E^i_\alpha + (D^*_i)^\alpha_\beta\partial_t\, *E^i_\alpha, \\ &= -\partial_i[(\text{GAUSS})_\beta v^i] - \partial_i(G_\beta v^i) \\ &\quad + t^\alpha_{\beta\gamma}\, *E^i_\alpha(D_i)^\gamma_\mu(\mathbf{v}\cdot\mathbf{A}^\mu) + (D^*_i)^\alpha_\beta(G_\alpha v^i) \\ &\quad - (D^*_i)^\alpha_\beta[t^\mu_{\alpha\gamma}\, *E^i_\mu(\mathbf{v}\cdot\mathbf{A}^\gamma)] \\ &\quad [\text{by (2.42b)–(2.42d)}]. \end{aligned} \tag{2.46}$$

Thus, in the shorter notation of (2.7) and (2.11), and using (2.15), we have

$$\partial_t(\mathrm{GAUSS}) = \mathrm{Div}^*(G\mathbf{v}) - \partial_i[(\mathrm{GAUSS})v^i] - \partial_i(Gv^i) + \mathrm{ad}^*(\mathbf{v}\cdot\mathbf{A})^*(\mathrm{Div}\,\mathbf{E}),$$
$$= -\mathrm{div}[(\mathrm{GAUSS})\mathbf{v}] + \mathrm{ad}^*(\mathbf{v}\cdot\mathbf{A})(\mathrm{GAUSS}) \quad [\text{by (2.37a)}], \tag{2.47}$$

which recovers relation (2.45). Consequently, (2.43) becomes

$$\partial_t G_\beta = -\mathrm{div}(G_\beta \mathbf{v}), \tag{2.48}$$

upon using the nondynamical constraint (2.44), which may be regarded as an initial condition by virtue of (2.45).

Finally, we have the momentum equation

$$\partial_t M_j = \{H, M_j\} = -M_i\partial_j v^i - \partial_i(M_j v^i) - \rho\partial_j[-v^2/2 + U'(\rho)]. \tag{2.49}$$

Substituting (2.40) in the form

$$M_j = \rho v_j - \langle G, A_j\rangle, \tag{2.50}$$

into the momentum equation, (2.49), readily gives the velocity equation,

$$\rho[\partial_t v_j + v^i v_{j,i} + U'(\rho)_{,j}]$$
$$= -\partial_t\langle G, A_j\rangle - \partial_i\langle Gv^i, A_j\rangle - \langle G, A_i v^i_{,j}\rangle,$$
$$= \langle \partial_t G, A_j\rangle + \langle G, \partial_t A_j\rangle - \langle D_i^*(Gv^i), A_j\rangle - \langle Gv^i, D_i A_j\rangle - \langle G, (\mathbf{A}\cdot\mathbf{v})_{,j}\rangle - \langle G, v^i A_{i,j}\rangle \quad [\text{by (2.12)}],$$
$$= \langle G, \partial_t A_j + v^i(D_i A_j - D_j A_i) + (\mathbf{A}\cdot\mathbf{v})_{,j}\rangle \quad [\text{by (2.36) and (2.48)}],$$
$$= \langle G, -E_j - v^i B_{ij}\rangle \quad [\text{by (2.42b)}]. \tag{2.51}$$

Hence we recover precisely the motion equation for the fluid velocity in CHD. Namely, with $(\mathbf{v}\times\mathbf{B})_j := v^i B_{ij}$, in vector form,

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla U'(\rho) - \rho^{-1}\langle G, \mathbf{E} + \mathbf{v}\times\mathbf{B}\rangle. \tag{2.52}$$

This completes the derivation of the CHD equations (2.42a)–(2.42d) and (2.52) from the extended CHD Hamiltonian matrix in (2.35) and the CHD Hamiltonian H in (2.38). The physical interpretation of the CHD equations (2.42a)–(2.42d) and (2.52), and their derivation from kinetic theory is discussed in Gibbons, Holm, and Kupershmidt.[27]

The Yang–Mills "displacement vector" $^*\mathbf{E}$ has an interesting interpretation in spin-glass theory. Namely, $\mathbf{E} = \delta H/\delta{}^*\mathbf{E}$ is the disclination flux density, so that $^*\mathbf{E}$ is the disclination current density. More discussion of this interpretation is given in the concluding section.

III. MATHEMATICAL DISCUSSION

In this section we explain the general mathematical facts underlying the various Hamiltonian matrices appearing in the preceding section. This will supply the proof that the Jacobi identity is satisfied for all of the Poisson brackets in the preceding section.

A. General notation

Let $K = C^\infty(\mathbb{R}^n)$; $D = D(\mathbb{R}^n)$: Lie algebra of vector fields on $\mathbb{R}^n$; $\Lambda^k = \Lambda^k(\mathbb{R}^n)$: K module of differential k forms on $\mathbb{R}^n$; $X(\xi)$ denotes the Lie derivative of $\xi\in\Lambda^k$ with respect to $X\in D$; $\mathfrak{g}$: a finite-dimensional Lie algebra over $\mathbb{R}$; $\mathfrak{g}^*$: its dual; (,): a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$;

$$(e_1,\ldots,e_M)\text{: basis in } \mathfrak{g}, \text{ satisfying } [e_\alpha, e_\beta] = t^\gamma_{\alpha\beta} e_\gamma,$$

where $t^\gamma_{\alpha\beta}$ are the structure constants of $\mathfrak{g}$; $(e^1,\ldots,e^M)$, the dual basis in $\mathfrak{g}^*$; if $\sigma:\mathfrak{g}\to\mathrm{End}\,V$ is a representation of $\mathfrak{g}$, then $\sigma(a)(v)$ is denoted simply by $a.v$, for $a\in\mathfrak{g}$ and $v\in V$.

B. Lie algebra

We start with the Lie algebra $\mathfrak{g}_1$ (2.27). Its commutator is given by the formula

$$\left[\begin{matrix} X^1 \\ f^1\otimes a^1 \\ \omega^1\otimes b^1 \\ g^1 \end{matrix}\;,\;\begin{matrix} X^2 \\ f^2\otimes a^2 \\ \omega^2\otimes b^2 \\ g^2 \end{matrix}\right] = \begin{matrix} [X^1, X^2] \\ X^1(f^2)\otimes a^2 - X^2(f^1)\otimes a^1 + f^1 f^2\otimes[a^1,a^2] \\ X^1(\omega^2)\otimes b^2 - X^2(\omega^1)\otimes b^1 + f^1\omega^2\otimes a^1.b^2 - f^2\omega^1\otimes a^2.b^1 \\ X^1(g^2) - X^2(g^1) \end{matrix}\;, \tag{3.1}$$

where $X^i\in D$; $f^i, g^i\in K$; $\omega^i\in\Lambda^{n-1}$; $a^i\in\mathfrak{g}$; $b^i\in\mathfrak{g}^*$; $i = 1,2$; and, e.g., for $h\in\mathfrak{g}$ and pairing $\langle\,,\rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$,

$$\langle a^1.b^2, h\rangle := -\langle b^2, [a^1, h]\rangle.$$

Claim: The commutator (3.1) defines a Lie algebra.

Proof: This results from the following general fact.

Theorem 3.1: Let Ω be a tensor field on $\mathbb{R}^n$, i.e., a K and D module, so that

$$X(f\omega) = fX(\omega) + X(f)\omega, \quad X\in D, \quad f\in K, \quad \omega\in\Omega. \tag{3.2}$$

Let $\sigma:\mathfrak{g}\to\mathrm{End}\,V$ be a representation of $\mathfrak{g}$. Then the following formula defines a Lie algebra $\overline{\mathfrak{g}}(\Omega,\sigma)$:

$$\left[\begin{matrix} X^1 \\ f^1\otimes a^1 \\ \omega^1\otimes v^1 \\ g^1 \end{matrix}\;,\;\begin{matrix} X^2 \\ f^2\otimes a^2 \\ \omega^2\otimes v^2 \\ g^2 \end{matrix}\right] = \begin{matrix} [X^1, X^2] \\ X^1(f^2)\otimes a^2 - X^2(f^1)\otimes a^1 + f^1 f^2\otimes[a^1,a^2] \\ X^1(\omega^2)\otimes v^2 - X^2(\omega^1)\otimes v^1 + f^1\omega^2\otimes a^1.v^2 - f^2\omega^1\otimes a^2.v^1 \\ X^1(g^2) - X^2(g^1) \end{matrix}\;, \tag{3.3}$$

where $X^i \in D$, f^i and $g^i \in K$, $\omega^i \in \Omega$, $a^i \in \mathfrak{g}$, $v^i \in V$, $i = 1,2$. A straightforward computation reduces the Jacobi identity for (3.3) to a set of identities of the form (3.2).

C. Generalized two-cocycles on the Lie algebra $\mathfrak{g}_1$

We now turn to the generalized two-cocycle on the Lie algebra $\mathfrak{g}_1$ responsible for the field-independent terms in the matrix (2.26a).

Proposition 3.2: The following formula defines a (generalized) two-cocycle ν_1 on $\mathfrak{g}_1 = \bar{\mathfrak{g}}(\Lambda^{n-1}, \mathrm{ad}^*)$:

$$\nu_1(1,2) = s(f^1\omega^2_{i,i}\langle b^2,a^1\rangle - f^2\omega^1_{i,i}\langle b^1,a^2\rangle), \quad s\in\mathbb{R}, \tag{3.4}$$

where $\langle\ ,\ \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$; and the notation $\nu_1(1,2)$ is shorthand for

$$\nu_1\begin{pmatrix} X_1 & X_2 \\ \vdots & \vdots \end{pmatrix}.$$

Recall (Kupershmidt,[32] Chap. viii) that a bilinear form ν on a Lie algebra $\mathfrak{g}'$ over K is called a (generalized) two-cocycle if

$$\nu(X,Y) \sim -\nu(Y,X), \quad X,Y\in\mathfrak{g}', \tag{3.5}$$

$$\nu([X,Y],Z) + \text{c.p.} \sim 0, \quad X,Y,Z\in\mathfrak{g}', \tag{3.6}$$

where "c.p." stands for "cyclic permutation"; and $a \sim b$ means $(a-b)\in\Sigma_i \operatorname{Im}\partial_i$, i.e., $(a-b)$ is a "divergence." One checks directly that ν_1 in (3.4) is indeed a two-cocycle on $\mathfrak{g}_1$.

D. Poisson bracket

The Poisson bracket associated to the two-cocycle ν_1 on the Lie algebra $\mathfrak{g}_1$ is computed by the standard rules of the general theory described in Kupershmidt,[32] Chap. viii (with n-dimensional volume element $d^n x$)

$$\begin{aligned}\{H,F\}_1 = -\int d^n x \Big\{ &\frac{\delta F}{\delta P_i}\left[(P_k\partial_i + \partial_k P_i)\left(\frac{\delta H}{\delta P_k}\right) + G_\beta\partial_i\left(\frac{\delta H}{\delta G_\beta}\right) + (\partial_k A^\beta_i - A^\beta_{k,i})\left(\frac{\delta H}{\delta A^\beta_k}\right) + \rho\partial_i\left(\frac{\delta H}{\delta\rho}\right)\right] \\ &+ \frac{\delta F}{\delta G_\alpha}\left[\partial_k G_\alpha\left(\frac{\delta H}{\delta P_k}\right) + t^\gamma_{\alpha\beta}G_\gamma\frac{\delta H}{\delta G_\beta} + (-t^\beta_{\alpha\gamma}A^\gamma_k + \delta^\beta_\alpha s\partial_k)\left(\frac{\delta H}{\delta A^\beta_k}\right)\right] \\ &+ \frac{\delta F}{\delta A^\alpha_i}\left[(A^\alpha_k\partial_i + A^\alpha_{i,k})\left(\frac{\delta H}{\delta P_k}\right) + (t^\alpha_{\beta\gamma}A^\gamma_i + \delta^\alpha_\beta s\partial_i)\left(\frac{\delta H}{\delta G_\beta}\right)\right] + \frac{\delta F}{\delta\rho}\partial_k\rho\left(\frac{\delta H}{\delta P_k}\right)\Big\},\end{aligned} \tag{3.7}$$

where dual coordinates on $\mathfrak{g}_1^*$ are chosen to be

P_k dual to $\partial_k\in D$; G_α to $1\otimes e_\alpha\in K\otimes g^\alpha$; A_i to $(\partial_i \lrcorner\, d^n x)\otimes e^\alpha$; ρ to $1\in K$.

E. Spin-glass Hamiltonian matrix

The Hamiltonian matrix $\mathsf{b} = \mathsf{b}(\mathfrak{g}_1,\nu_1)$ associated to the Poisson bracket (3.7) via the standard rule

$$\{H,F\} \sim \frac{\delta F}{\delta u_i} b_{ij} \frac{\delta H}{\delta u_j}$$

is given by

$$\begin{array}{c|cccc|} & P_k & G_\beta & A^\beta_k & \rho \\ P_i & P_k\partial_i + \partial_k P_i & G_\beta\partial_i & \partial_k A^\beta_i - A^\beta_{k,i} & \rho\partial_i \\ G_\alpha & \partial_k G_\alpha & t^\gamma_{\alpha\beta}G_\gamma & -t^\beta_{\alpha\gamma}A^\gamma_k + \delta^\beta_\alpha s\partial_k & 0 \\ A^\alpha_i & A^\alpha_k\partial_i + A^\alpha_{i,k} & t^\alpha_{\beta\gamma}A^\gamma_i + \delta^\alpha_\beta s\partial_i & 0 & 0 \\ \rho & \partial_k\rho & 0 & 0 & 0 \end{array}. \tag{3.8}$$

This is b_1 in (2.26a) when $s = 1$.

F. Origin of the generalized two-cocycle

Since the two-cocycle (3.4) plays a crucial role in what follows, we explain its origin and unique features. Let $\tilde{\Omega}$ be an additional tensor field on $\mathbb{R}^n$, and let ϑ: $\Omega\to\tilde{\Omega}$ be a homomorphism of D modules, i.e.,

$$\vartheta(X(\omega)) = X(\vartheta(\omega)), \quad X\in D, \quad \omega\in\Omega. \tag{3.9}$$

(For example, $\Omega = \Lambda^k$, $\tilde{\Omega} = \Lambda^{k+1}$, $\vartheta = d$.) Then ϑ induces a natural Lie algebra homomorphism

$$\vartheta: \bar{\mathfrak{g}}(\Omega,\sigma) \to \bar{\mathfrak{g}}(\tilde{\Omega},\sigma). \tag{3.10}$$

Therefore, from ϑ one obtains a Hamiltonian (i.e., canonical) map ϕ: $C_{\bar{\mathfrak{g}}}(\Omega,\sigma)\to C_{\bar{\mathfrak{g}}}(\tilde{\Omega},\sigma)$ on $C_{\bar{\mathfrak{g}}}$, the ring of functions on the dual to the Lie algebras $\bar{\mathfrak{g}}(\Omega,\sigma)$ and $\bar{\mathfrak{g}}(\tilde{\Omega},\sigma)$ [see Chap. viii (3.42) in Kupershmidt[32]]. In particular, take

$$\Omega = \Lambda^{n-1},\ \tilde{\Omega} = \Lambda^n,\ \vartheta = -d,\ V = \mathfrak{g}^*,\ \sigma = \mathrm{ad}^*,$$

and denote by η^α coordinates on $\bar{\mathfrak{g}}_2^* = \bar{\mathfrak{g}}(\Lambda^n,\mathrm{ad}^*)^*$ dual to $d^nx \otimes e^\alpha$. Then by formula viii (3.42) in Kupershmidt,[32] the map ϕ^* can be written in the form

$$P_i = P_i;\quad G_\alpha = G_\alpha;\quad A_i^\alpha = \eta^\alpha_{,i};\quad \rho = \rho; \tag{3.11}$$

and ϕ^* is a Hamiltonian map between the cocycles Poisson bracket $(3.7)|_{s=0}$ and the Poisson bracket on $\mathfrak{g}_2^* = \bar{\mathfrak{g}}(\Lambda^n,\mathrm{ad}^*)^*$,

$$\begin{aligned}\{H,F\} = &-\int d^nx \left\{\frac{\delta F}{\delta P_i}\left[(P_k\partial_i + \partial_k P_i)\left(\frac{\delta H}{\delta P_k}\right) + G_\beta \partial_i\left(\frac{\delta H}{\delta G_\beta}\right) - \eta^\beta_{,i}\frac{\delta H}{\delta \eta^\beta} + \rho\partial_i\left(\frac{\delta H}{\delta\rho}\right)\right]\right.\\ &\left.+\frac{\delta F}{\delta G_\alpha}\left[\partial_k G_\alpha\left(\frac{\delta H}{\delta P_k}\right) + t^\gamma_{\alpha\beta}G_\gamma\frac{\delta H}{\delta G_\beta} + t^\beta_{\alpha\gamma}\eta^\gamma\frac{\delta H}{\delta\eta^\beta}\right] + \frac{\delta F}{\delta\eta^\alpha}\left(\eta^\alpha_{,k}\frac{\delta H}{\delta P_k} - t^\alpha_{\beta\gamma}\eta^\gamma\frac{\delta H}{\delta G_\beta}\right) + \frac{\delta F}{\delta\rho}\partial_k\rho\left(\frac{\delta H}{\delta P_k}\right)\right\}.\end{aligned} \tag{3.12}$$

Now, as a K module, $\mathfrak{g}_2 = \bar{\mathfrak{g}}(\Lambda^n,\mathrm{ad}^*)$, has inside it two submodules: $K\otimes\mathfrak{g}$ and $\Lambda^n\otimes\mathfrak{g}^*$, which are mutually dual as D modules, i.e.,

$$\langle X(\bar{b}),\bar{a}\rangle + \langle\bar{b},X(\bar{a})\rangle \sim 0,\quad \bar{a}\in K\otimes\mathfrak{g},\quad \bar{b}\in\Lambda^n\otimes\mathfrak{g}^*. \tag{3.13}$$

This means that we may have a symplectic two-cocycle ν_2 on $\mathfrak{g}_2$

$$\nu_2(1,2) = -s(f^1\psi^2\langle b^2,a^1\rangle - f^2\psi^1\langle b^1,a^2\rangle),\quad s\in\mathbb{R},\quad \psi^i\in\Lambda^n. \tag{3.14}$$

And indeed, ν_2 is a two-cocycle on $\mathfrak{g}_2$, as one verifies by a direct computation. [A verification is required since, for non-Abelian $\mathfrak{g}$, $K\otimes\mathfrak{g}$ acts nontrivially on $\Lambda^n\otimes\mathfrak{g}^*$; otherwise (3.13) would have guaranteed that (3.14) is a two-cocycle.] Now, since the map ϕ^* in (3.11) is constant coefficient, it transforms two-cocycles on $\mathfrak{g}_2$ into two-cocycles on $\mathfrak{g}_1$; in particular, ν_2 is transformed into ν_1. From this discussion, one concludes that if $\Omega\otimes V \neq \Lambda^{n-1}\otimes\mathfrak{g}^*$, then one cannot have a two-cocycle on $\bar{\mathfrak{g}}(\Omega,\sigma)$ similar to ν_1, since a symplectic two-cocycle of the type ν_1 exists only on $\bar{\mathfrak{g}}(\Lambda^n,\mathrm{ad}^*)$. This observation saves us from a futile search for new two-cocycles in the extended YMCF case, when $^*\mathbf{E}$ variables (dual to $\Lambda^1\otimes\mathfrak{g}$) come into the picture.

G. Lie algebra $\mathfrak{g}_3$

The Lie algebra $\mathfrak{g}_3 = \bar{\mathfrak{g}}(\Lambda^{n-1}\oplus\Lambda^1,\mathrm{ad}^*\oplus\mathrm{ad})$ ($^*\mathbf{E}$ is included), has commutator (cf. Theorem 3.1)

$$\begin{bmatrix} X^1 & X^2 \\ f^1\otimes a^1 & f^2\otimes a^2 \\ \omega^1\otimes b^1 & \omega^2\otimes b^2 \\ \mu^1\otimes\tilde{a}^1 & \mu^2\otimes\tilde{a}^2 \\ g^1 & , \quad g^2\end{bmatrix} = \begin{matrix}[X^1,X^2] \\ X^1(f^2)\otimes a^2 - X^2(f^1)\otimes a^1 + f^1f^2\otimes[a^1,a^2] \\ X^1(\omega^2)\otimes b^2 - X^2(\omega^1)\otimes b^1 + f^1\omega^2\otimes a^1.b^2 - f^2\omega^1\otimes a^2.b^1 \\ X^1(\mu^2)\otimes\tilde{a}^2 - X^2(\mu^1)\otimes\tilde{a}^1 + f^1\mu^2\otimes[a^1,\tilde{a}^2] - f^2\mu^1\otimes[a^2,\tilde{a}^1] \\ X^1(g^2) - X^2(g^1)\end{matrix}\ , \tag{3.15}$$

where $X^i\in D, f^i,g^i\in K, \omega^i\in\Lambda^{n-1}, \mu^i\in\Lambda^1; a^i,\tilde{a}^i\in\mathfrak{g}; b^i\in\mathfrak{g}^*; i = 1,2.$

H. Remarks

(a) $\mathfrak{g}_3$ contains $\mathfrak{g}_1$ as a subalgebra, and $\mathfrak{g}_3$ itself is a semidirect product of $\mathfrak{g}_1$ and $\Lambda^1\otimes\mathfrak{g}$. Hence, there is a two-cocycle $\tilde{\nu}_1$ on $\mathfrak{g}_3$, which coincides with ν_1 on $\mathfrak{g}_1$ and vanishes when one of its arguments belongs to $\Lambda^1\otimes\mathfrak{g}$:

$$\tilde{\nu}_1(1,2) = s(f^1\omega^2_{i,i}\langle b^2,a^1\rangle - f^2\omega^1_{i,i}\langle b^1,a^2\rangle),\quad s\in\mathbb{R}. \tag{3.16}$$

(b) There is also a new symplectic two-cocycle on $\mathfrak{g}_3$,

$$\nu_3(1,2) = c_1(\omega^1\wedge\mu^2\langle b^1,\tilde{a}^2\rangle - \omega^2\wedge\mu^1\langle b^2,\tilde{a}^1\rangle),\quad c_1\in\mathbb{R}. \tag{3.17}$$

(c) The new Poisson bracket associated to the two-cocycle $\tilde{\nu}_1 + \nu_3$ on $\mathfrak{g}_3$ equals

$$\begin{aligned}\{H,F\}_3 = \{H,F\}_1 &+ \int d^nx\left[\frac{\delta F}{\delta P_i}({}^*E^k_\beta\partial_i - \partial_j{}^*E^j_\beta\delta^i_k) + \frac{\delta F}{\delta G_\alpha}t^\gamma_{\alpha\beta}{}^*E^k_\gamma + \frac{\delta F}{\delta A^\alpha_i}c_1\delta^\alpha_\beta\delta^k_i\right]\left(\frac{\delta H}{\delta{}^*E^k_\beta}\right)\\ &+\frac{\delta F}{\delta{}^*E^i_\alpha}\left[(\partial_k{}^*E^i_\alpha - {}^*E^j_\alpha\partial_j\delta^i_k)\left(\frac{\delta H}{\delta P_k}\right) + t^\gamma_{\alpha\beta}{}^*E^i_\gamma\frac{\delta H}{\delta G_\beta} - c_1\delta^\beta_\alpha\delta^i_k\frac{\delta H}{\delta A^\beta_k}\right],\end{aligned} \tag{3.18}$$

where $\{H,F\}_1$ is the Poisson bracket corresponding to $\mathfrak{b}_1$ in (3.8), and $^*E^k_\beta$ is dual to $dx^k\otimes e_\beta$, in both the metric, and the Lie-algebraic senses. For $c_1 = 1$, the Hamiltonian matrix associated to (3.18) is given in (2.33).

I. Lie algebra $\mathfrak{g}_4$

Let $\mathfrak{g}_4 = \bar{\mathfrak{g}}(\Lambda^{n-2},\mathrm{ad}^*)$. This is the Lie algebra with the commutator (3.3) for $\omega^i\in\Lambda^{n-2}$ and $v^i\in\mathfrak{g}^*$, $i = 1,2$. The corresponding Hamiltonian matrix is given by formula (2.28), provided one lets B^α_{ij} be the coordinate dual to $(\partial_i \lrcorner\, \partial_j \lrcorner\, d^nx)\otimes e^\alpha\in\Lambda^{n-2}\otimes\mathfrak{g}^*$.

IV. CONCLUSIONS

We have considered the analogy between spin glasses and Yang–Mills fluids (CHD) within the Hamiltonian framework. Our results complete this analogy, according to the following "dictionary."

spin glass		Yang–Mills fluid	
ρ,	defect inertial-mass density	ρ,	mass density of fluid carrying gauge charge
$\mathbf{v}$,	fluid velocity	$\mathbf{v}$,	fluid velocity
$\mathbf{K}$,	hydrodynamic momentum density of defects	$\mathbf{M}$,	total momentum density, including YM field momentum
B_{ij},	disclination density	B_{ij},	Yang–Mills magnetic field
F_{i0},	disclination current density	$^*E^i$,	Yang–Mills electric displacement vector

Along the way, we have noticed an interesting phenomenon in YM–MHD and CHD, namely, the existence of two different Poisson brackets for the non-Abelian case and a one-parameter family of Poisson brackets for the Abelian case, in the **A** representation for CHD, see Eq. (2.33).

Physically, our conclusion is that the analogy between spin-glass theory and Yang–Mills charged fluids is very close, on the level of the Hamiltonian formalism. Specifically, the Hamiltonian matrices are identical for the Volovik–Dotsenko spin-glass theory and Yang–Mills MHD. In addition, the Hamiltonian matrix (2.33) in the Yang–Mills charged-fluid representation provides a potentially interesting extension of the Volovik–Dotsenko spin-glass theory, by providing a dynamical equation for the disclination current density $^*\mathbf{E}$, which is the spin-glass analog of the Yang–Mills electric displacement vector.

Our basic mathematical observations are these: the highly nonlinear candidate (2.20) for the Poisson bracket in Volovik and Dotsenko,[24] when transformed to appropriate (natural) variables, becomes of affine type and is thus associated to a certain Lie algebra, called $\mathfrak{g}_1$, and a two-cocycle, called ν_1, on $\mathfrak{g}_1$. It turns out that $\mathfrak{g}_1$ is a subalgebra of another Lie algebra, $\mathfrak{g}_3$, which closely resembles the chromohydrodynamics Lie algebra $\mathfrak{g}_2$. The Lie algebra $\mathfrak{g}_2$ is, in turn, another subalgebra of $\mathfrak{g}_3$. Moreover, the two-cocycle ν_1 on g_1 is a restriction on $\mathfrak{g}_1 \subset \mathfrak{g}_3$ of a certain two-cocycle $\tilde{\nu}_1$ on $\mathfrak{g}_3$. Furthermore, there is another, canonical, two-cocycle ν_3 on $\mathfrak{g}_3$, whose restriction on $\mathfrak{g}_1$ vanishes and whose restriction on $\mathfrak{g}_2$ produces precisely the canonical $^*\mathbf{E}$-$\mathbf{A}$ structure in CHD.

Roughly speaking, the absence of a dynamical equation for $^*\mathbf{E}$ in Volovik and Dotsenko[24] is of the same nature as the absence of displacement current. The dynamical equation for $^*\mathbf{E}$ is present only in the full electromagnetic or Yang–Mills field equations, or in an extended theory of spin-glass dynamics accounting for time dependence of the disclination current density, F_{i0}. In that case, the present theory would provide the dynamics by using Poisson bracket (2.33), in conjunction with an appropriate choice for the Hamiltonian.

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