THE LIE-TRANSFORMED VLASOV ACTION PRINCIPLE: RELATIVISTICALLY COVARIANT WAVE PROPAGATION AND SELF-CONSISTENT PONDEROMOTIVE EFFECTS ☆

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A covariant action principle is formulated for the Vlasov–Maxwell system. Upon subjecting the particle phase space and invariant hamiltonian to a Lie transform for interaction with an eikonal wave, we obtain a new action principle for the invariant oscillation-center distribution and the self-consistent wave propagation, with a common kernel for the invariant ponderomotive hamiltonian and the linear susceptibility.

Some years ago, Dewar [1] showed how the concepts of *oscillation-center* and *ponderomotive force* followed naturally from a canonical transformation eliminating the linear nonresonant interaction of a particle with a given wave. Since then, there has been an evident need to introduce *self-consistency*, so that the wave propagation is described at the same conceptual level. Previous hamiltonian treatments, such as those of Dewar [1], of Johnston [2], of Cary [3], and of Dubin, Krommes, Oberman and Lee [4], have been constructive, requiring deep insight, and sometimes leading to ambiguity. Recently we have discovered a deductive approach [5,6], by means of applying the standard Lie transform to the phase space and hamiltonian of the Vlasov *action principle* of Lewis and Symon [7]. Our first results dealt with the problem of Coulomb interaction, either at the gyrokinetic level [5] (as posed by Dubin et al. [4]), or for eikonal waves [6]. This work explicated the intimate relation [8,9,3] between the linear susceptibility and the ponderomotive hamiltonian. We have also developed an action principle for investigating the ponderomotive stabilization of a mirror-confined plasma [10,11].

Motivated by the upsurge of interest in relativistic plasmas, we have formulated a *covariant* treatment, which we present here, for the self-consistent nonresonant interaction of an electromagnetic wave (in eikonal form) and an unmagnetized distribution of oscillation centers. In subsequent papers, we plan to discuss resonance [12], magnetized plasma [13], and wave–wave interactions.

Our first step is to self-consistently derive (a) the Ignatiev equation [14]:

$$\{f(z), H(z)\} = 0\,, \tag{1}$$

for the invariant Vlasov distribution $f(z)$ in terms of the invariant particle hamiltonian $H(z)$, and (b) the Maxwell equation $F^{\mu\nu}{}_{,\nu} = 4\pi j^{\mu}$. Here z is a point in eight-dimensional phase space $(\boldsymbol{r},\boldsymbol{p})$, with $\boldsymbol{r} = (r, t)$, $\boldsymbol{p} = (p, -h)$; $H = [\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r})]^2/2m$ yields the correct covariant hamiltonian equation: $\mathrm{d}r^{\mu}/\mathrm{d}\tau = \partial H/\partial p_{\mu}$, $\mathrm{d}p_{\mu}/\mathrm{d}\tau = -\partial H/\partial r^{\mu}$, implying $\mathrm{d}u_{\mu}/\mathrm{d}\tau = (e/M)F_{\mu\nu}u^{\nu}$; the Poisson bracket is the canonical covariant expres-

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sion $\{a,b\} = (\partial a/\partial r^\mu)(\partial b/\partial p_\mu) - (\partial a/\partial p_\mu)(\partial b/\partial r^\mu)$; and the four-current density is $j^\mu(x;f) = \int d^8z\, f(z) j^\mu(x;z)$, while $j^\mu(x;z) = eu^\mu \delta^4(x-r)$ is the contribution of a particle at z. (We omit the obvious species labels and sums.)

Consider the family of particle world lines in phase space, each parameterized by its proper time τ. With a smooth but arbitrary assignment of $\tau = 0$ on each line, we denote the seven-dimensional hypersurface $\tau = 0$ as the "initial-condition surface". Introducing seven arbitrary coordinates η on that surface, we let $g(\eta)d^7\eta$ represent the particle density at $\tau = 0$. The particle orbits $z(\tau,\eta)$ are thus an eight-component field on the eight-dimensional space (τ,η).

We now introduce the action functional:

$$S[z(\tau,\eta), A(x)] = \int d^7\eta\, g(\eta) \times \int d\tau\, [p_\mu(\tau,\eta) dr^\mu(\tau,\eta)/d\tau - H(p,r;A)] - \int d^4x\, F_{\mu\nu}F^{\mu\nu}/16\pi , \tag{2}$$

the sum of the particle action (in phase-space form) and the Maxwell action. Requiring that S be stationary with respect to $z(\tau,\eta)$ yields the hamiltonian equations stated above. The invariant Vlasov distribution is defined as

$$f(\bar{z}) = \int d^7\eta\, g(\eta) \int d\tau\, \delta^8(\bar{z} - z(\tau,\eta)) . \tag{3}$$

To obtain (1), we introduce the intermediate distribution $\bar{g}(\bar{z};\tau) = \int d^7\eta\, g(\eta)\delta^8(\bar{z} - z(\tau,\eta))$, and take its τ-derivative, using the hamiltonian equations. We find $\partial\bar{g}/\partial\tau = -\{\bar{g}, H\}$. Integration over τ then yields (1), since g vanishes for finite z and infinite τ.

Variation of S with respect to $A_\mu(x)$ yields the Maxwell equation, with $j^\mu(x) = \int d^7\eta\, g(\eta) \int d\tau (-) \times \delta H(z(\eta,\tau))/\partial A_\mu(x)$. Use of (3) and H then produces $j^\mu(x) = \int d^8z\, f(z)\, eu^\mu\delta^4(x-r)$, as desired. Thus the action principle yields the self-consistent Maxwell and Ignatiev equations.

We now restrict the Maxwell potential to represent an eikonal wave: $A_\mu(x) = \widetilde{A}_\mu(x) \exp i\Theta(x)/\epsilon$ + c.c., where $\widetilde{A}$ and Θ are the slowly varying amplitude and phase, while the eikonal infinitesimal ϵ will be omitted from subsequent formulas. The Maxwell field is thus $F_{\mu\nu}(x) = i\widetilde{F}_{\mu\nu}(x) \exp i\Theta(x)$ + c.c., with $\widetilde{F}_{\mu\nu} = k_\mu\widetilde{A}_\nu - k_\nu\widetilde{A}_\mu$, and $k_\mu(x) = \partial\Theta(x)/\partial x^\mu = (\boldsymbol{k}, -\omega)$. The Maxwell action in (2) is thus $-\int d^4x\, \widetilde{F}^*_{\mu\nu}\widetilde{F}^{\mu\nu}/8\pi$.

The Lie generator $w(z)$ is determined [15] by $\{w, H^{(0)}\} = -H^{(1)}$, where $H^{(0)} = p^2/2m$ is the zero-order hamiltonian, and $H^{(1)} = \widetilde{H} \exp i\Theta(r)$ + c.c., with $\widetilde{H} = -(e/m)p\cdot\widetilde{A}(r)$. We find the solution $w(z) = -ie[p\cdot\widetilde{A}(r)/p\cdot k(r)] \exp i\Theta(r)$ + c.c., and proceed to the new hamiltonian $K = [\exp i\{w,\ \}]H = p^2/2m + \Psi(z)$ to second order, with $\Psi = \frac{1}{2}\{w, H^{(1)}\} + H^{(2)}$, where $H^{(2)} = (e^2/m)|\widetilde{A}(r)|^2$ and oscillatory terms are omitted (being eliminated by a second Lie transform). Evaluation of the *relativistically invariant ponderomotive hamiltonian* $\Psi(z;A)$ yields

$$\Psi(z) = (e^2/m)|p\cdot\widetilde{F}|^2/(p\cdot k)^2 , \tag{4}$$

where $|p\cdot\widetilde{F}|^2 = p_\mu\widetilde{F}^{\mu\nu}(r)\,p^\sigma\widetilde{F}^*_{\sigma\nu}(r)$, and the denominator is the familiar resonance $p\cdot k = m\gamma(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)$. To this order, Ψ has the interesting gauge-invariant expression $\Psi = m|d\widetilde{u}/d\tau|^2\,[d\Theta(r)/d\tau]^{-2}$, and reduces, in the unperturbed rest frame, to the familiar $e^2|\widetilde{E}|^2/m\omega^2$. We express the dependence of $\Psi(z)$ on $\widetilde{A}_\mu(x)$ explicitly:

$$\Psi(z;A) = \int d^4x\, \widetilde{A}^*_\mu(x)\, \Psi^\mu{}_\nu(x;z)\widetilde{A}^\nu(x) , \tag{5}$$

by (4), the kernel is [remember that $k = k(x)$]

$$\Psi^\mu{}_\nu(x;z) = \delta^4(x-r)(e^2/m)(p\cdot k)^{-2} \times [k^2p^\mu p_\nu + \delta^\mu_\nu(k\cdot p)^2 - (k\cdot p)(p^\mu k_\nu + k^\mu p_\nu)] . \tag{6}$$

We now use the property of the phase-space lagrangian action, of invariance under canonical transformations [16]:

$$\int d\tau\, [p_\mu dr^\mu/d\tau - H(p,r)] = \int d\tau\, [\bar{p}_\mu d\bar{r}^\mu/d\tau - K(\bar{p},\bar{r})] , \tag{7}$$

where the overbar (omitted subsequently) denotes oscillation-center variables. Thus the action now reads

$$S = \int d^7\eta\, g(\eta) \int d\tau\, [p_\mu dr^\mu/d\tau - p^2/2m - \Psi(z;A)] - \int d^4x\, \widetilde{F}^*_{\mu\nu}\widetilde{F}^{\mu\nu}/8\pi . \tag{8}$$

Variation with respect to the oscillation-center orbit $z(\tau,\eta)$ yields $dr^\mu/d\tau = \partial K/\partial p_\mu$, $dp_\mu/d\tau = -\partial K/\partial r^\mu$, for the covariant ponderomotive effects. Introducing the *invariant oscillation-center distribution* (not to be confused with $F_{\mu\nu}$): $F(z) = \int d^7\eta\, g(\eta) \int d\tau\, \delta^8(z - \bar{z}(\tau, \eta))$, we obtain, in analogy to the steps leading to (1), the corresponding Ignatiev equation:

$$\{F(z), p^2/2m + \Psi(z; A)\} = 0\,. \tag{9}$$

In order to vary S with respect to $\tilde{A}_\mu(x)$ and $\Theta(x)$, we first substitute (5) into (8), obtaining (for the terms bilinear in A)

$$S^{(2)} = \int d^4x\, \tilde{A}^*_\mu(x) D^\mu{}_\nu(x; F) \tilde{A}^\nu(x)\,, \tag{10}$$

where the dielectric matrix $D^\mu{}_\nu$ is the sum of the vacuum part

$$D_{\text{vac}}{}^\mu{}_\nu(k(x)) = (k^\mu k_\nu - k^2 \delta^\mu_\nu)/4\pi \tag{11}$$

and the susceptibility matrix

$$\chi^\mu{}_\nu(x; F) = -\int d^8z\, F(z)\, \Psi^\mu{}_\nu(x; z)\,. \tag{12}$$

This relation (12) is the "K–χ theorem" [8,9,3], which thus is the essential ingredient of the action functional, coupling $F(z)$ to $A_\mu(x)$.

It is convenient to express the dielectric matrix in terms of its local eigenvalues and eigenvectors. Since $S^{(2)}$ is a scalar and $\tilde{A}^\nu \tilde{A}^*_\mu$ is hermitian, $D^\mu{}_\nu(x)$ is hermitian, with real eigenvalues $D_\alpha(x)$ (labeled by α) and with orthonormal (complex) eigenvectors $\hat{e}_\alpha(x)$: $e^{\alpha *}_\mu \times e^\mu_\beta = \delta^\alpha_\beta$. Thus we express

$$D^\mu{}_\nu(x) = \sum_\alpha D_\alpha(x) e^\mu_\alpha(x) e^{\alpha *}_\nu(x)\,, \tag{13}$$

so that

$$S^{(2)} = \int d^4x\, D_\alpha(x) |A_\alpha(x)|^2\,, \tag{14}$$

where $A_\alpha(x) = e^\alpha{}_\nu{}^*(x) \tilde{A}^\nu(x)$ is the projection of $\tilde{A}$ on the α eigenvector of $D^\mu{}_\nu(x)$.

Noting that $D_\alpha(x) = D_\alpha(x, k(x))$ [see (11) and (6)], we proceed to vary $S^{(2)}$, first with respect to $A_\alpha(x)$, obtaining the eikonal equation for the phase:

$$D_\alpha(x, k = \partial\Theta/\partial x) = 0\,, \tag{15}$$

associated with polarization e_α. This yields the covariant ray equations:

$$dx^\mu/d\sigma = -\partial D_\alpha/\partial k_\mu, \quad dk_\mu/d\sigma = \partial D_\alpha/\partial x^\mu\,. \tag{16}$$

Variation with respect to $\Theta(x)$ (the phase for a given polarization) yields the wave-action conservation law [17]: $\partial J^\mu(x)/\partial x^\mu = 0$, where the action density four-vector is

$$J^\mu(x) = -|A_\alpha(x)|^2 \partial D_\alpha(x, k)/\partial k_\mu\,, \tag{17}$$

[evaluated at $k(x)$]. Since the local eigenvalues $D_\alpha(x)$ are functionals of the oscillation-center distribution $F(z)$, by (12), we have now obtained a closed self-consistent set of coupled equations for $F(z)$ and the wave amplitude and phase.

In a future paper, we shall investigate the conservation laws [18] associated with these action principles.

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