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Rasetti–Regge Dirac bracket formulation of Lagrangian fluid dynamics of vortex filaments ☆

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Abstract

We review the Rasetti–Regge Dirac bracket (RRDB) for determining the constrained Hamiltonian dynamics of vortex filaments moving with an incompressible potential flow of superfluid helium-II in the Lagrangian fluid picture. We compare the equations for Lagrangian vortex filaments with their corresponding Eulerian vorticity dynamics in the examples of the Euler fluid, superfluid vortices, the local induction approximation (LIA), the Rosenhead regularization and a new class of alternative regularized theories including the Euler-alpha model. The RRDB formulation generalizes the Betchov–Da Rios equation for the transverse self-induction velocity of a vortex filament from LIA to the case of an incompressible fluid whose energy may expressed as an arbitrary functional of spatial vorticity. We also discuss the relation of RRDB to the Marsden–Weinstein bracket for vortex filaments and its implications under the Hasimoto transformation for physically meaningful nonlocal nonlinear Schrödinger (NLNLS) equations.
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1. Rasetti–Regge Dirac bracket (RRDB)

In the context of helium-II superfluid vortex filaments [15], Rasetti and Regge used Dirac's method to introduce a constrained Poisson bracket for the Hamiltonian dynamics in the Lagrangian fluid description of a massless vortex filament moving without slipping in an incompressible potential flow.

The spatial position of such a vortex filament is denoted as

$$\boldsymbol{x} = \boldsymbol{R}(\boldsymbol{a}, t) : R^3 \times R \to R^3$$

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where $\boldsymbol{a} \in R^3$ is a Lagrangian coordinate on the filament moving with the fluid and t is time. A vector tangent to the filament is defined in terms of its vorticity by the Cauchy relation

$$\boldsymbol{R}_{\mathrm{s}}(\boldsymbol{a}, t) = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s} = \omega_0(\boldsymbol{a}) \cdot \frac{\partial}{\partial \boldsymbol{a}}\boldsymbol{R} = J\omega(\boldsymbol{R}(\boldsymbol{a}, t), t), \quad \text{with} \quad J = \det\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{a}}\right).$$

Here, $\omega_0(\boldsymbol{a})$ is the vorticity at the initial position. This relation also provides the geometrical rule for transforming between Lagrangian and Eulerian descriptions of the vortex filament. Thus, the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}s} = \omega_0(\boldsymbol{a}) \cdot \frac{\partial}{\partial \boldsymbol{a}} = J\omega(\boldsymbol{R}) \cdot \frac{\partial}{\partial \boldsymbol{R}} = R_{\mathrm{s}} \cdot \frac{\partial}{\partial \boldsymbol{R}}$$

is the natural tangential derivative for the vortex filament in either the Lagrangian, or the Eulerian picture.

In this notation, the Rasetti–Regge Dirac bracket (RRDB) is expressed in the Lagrangian fluid description as

$$\begin{aligned}\{F, H\}(\boldsymbol{R}, \chi) = \int \mathrm{d}^3 a \Bigg[&\frac{1}{R_{\mathrm{s}}^2}\boldsymbol{R}_{\mathrm{s}} \cdot \frac{\delta F}{\delta \boldsymbol{R}} \times \frac{\delta H}{\delta \boldsymbol{R}} - \frac{\delta F}{\delta \boldsymbol{R}} \cdot \left(\frac{\chi}{R_{\mathrm{s}}^4}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_{\mathrm{s}}^4}\right)\frac{\delta H}{\delta \boldsymbol{R}} \\ &- \frac{\boldsymbol{R}_{\mathrm{s}}}{R_{\mathrm{s}}^2} \cdot \frac{\delta F}{\delta \boldsymbol{R}} \left(\frac{\chi}{R_{\mathrm{s}}^2}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_{\mathrm{s}}^2}\right)\frac{\boldsymbol{R}_{\mathrm{s}}}{R_{\mathrm{s}}^2} \cdot \frac{\delta H}{\delta \boldsymbol{R}} + \left(\boldsymbol{R}_{\mathrm{s}} \cdot \frac{\delta F}{\delta \boldsymbol{R}}\right)\frac{\delta H}{\delta \chi} - \frac{\delta F}{\delta \chi}\left(\boldsymbol{R}_{\mathrm{s}} \cdot \frac{\delta H}{\delta \boldsymbol{R}}\right)\Bigg],\end{aligned} \tag{1}$$

in which $R_{\mathrm{s}} = |\boldsymbol{R}_{\mathrm{s}}| = \hat{\boldsymbol{R}}_{\mathrm{s}} \cdot \boldsymbol{R}_{\mathrm{s}}$ denotes the magnitude of the tangent vector. The variable χ is the constraint degree of freedom introduced in [15] that represents longitudinal motion along the filament. The condition $\chi = 0$ is a first class constraint that eliminates this longitudinal dynamics in the RRDB. The Rasetti–Regge Dirac bracket yields the following Hamiltonian equations of motion for a vortex filament:

$$\begin{aligned}\dot{\boldsymbol{R}}(\boldsymbol{a}, t) = \{\boldsymbol{R}, H\} = &-\frac{1}{R_{\mathrm{s}}^2}\boldsymbol{R}_{\mathrm{s}} \times \frac{\delta H}{\delta \boldsymbol{R}} - \left(\frac{\chi}{R_{\mathrm{s}}^4}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_{\mathrm{s}}^4}\right)\frac{\delta H}{\delta \boldsymbol{R}} \\ &- \frac{\boldsymbol{R}_{\mathrm{s}}}{R_{\mathrm{s}}^2}\left(\frac{\chi}{R_{\mathrm{s}}^2}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_{\mathrm{s}}^2}\right)\left(\frac{\boldsymbol{R}_{\mathrm{s}}}{R_{\mathrm{s}}^2} \cdot \frac{\delta H}{\delta \boldsymbol{R}}\right) + \boldsymbol{R}_{\mathrm{s}}\frac{\delta H}{\delta \chi}, \\ \dot{\chi}(\boldsymbol{a}, t) = \{\chi, H\} = &-\boldsymbol{R}_{\mathrm{s}} \cdot \frac{\delta H}{\delta \boldsymbol{R}}.\end{aligned} \tag{2}$$

If $\boldsymbol{R}_{\mathrm{s}} \cdot \delta H/\delta \boldsymbol{R} = 0$ then χ is dynamically preserved and one may safely impose the constraint $\chi = 0$ as an initial condition, thereby simplifying the first equation for $\boldsymbol{R}$ considerably. The dependence of H on χ introduces an immaterial drift velocity along the vortex filament. This component of the velocity is equivalent to a time dependent re-parameterizarion of arc-length, which has no physical consequence. However, a nontrivial evolution in χ would disallow imposing the constraint $\chi = 0$. Dynamical preservation of the constraint $\chi = 0$ requires choosing Hamiltonians that satisfy $\boldsymbol{R}_{\mathrm{s}} \cdot \delta H/\delta \boldsymbol{R} = 0$ and, thus, Poisson-commute with χ under the RRDB. Such Hamiltonians are called "gauge invariant" in [15].

2. Examples of RRDB motions

2.1. Hamiltonians for vortex filament dynamics

If we choose Hamiltonians of the form $H = \int h(R_s)\,\mathrm{d}^3a$ depending only on the magnitude of the tangent vector, then their variations may be computed from

$$\delta H = \int h'(R_s)\delta R_s\,\mathrm{d}^3a = \int h'(R_s)\hat{\boldsymbol{R}}_s \cdot \delta \boldsymbol{R}_s\,\mathrm{d}^3a = -\int \frac{\mathrm{d}}{\mathrm{d}s}(h'(R_s)\hat{\boldsymbol{R}}_s)\cdot \delta \boldsymbol{R}\,\mathrm{d}^3a.$$

Thus,

$$\frac{\delta H}{\delta \boldsymbol{R}} = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{\delta H}{\delta R_s} = -\frac{\mathrm{d}}{\mathrm{d}s}(h'(R_s)\hat{\boldsymbol{R}}_s). \quad \left(\text{Recall that } \boldsymbol{R}_s \cdot \frac{\delta H}{\delta \boldsymbol{R}} \neq 0 \text{ implies } \dot{\chi} \neq 0.\right)$$

2.1.1. First example—longitudinal flow

We choose the Hamiltonian $H = \int \mathrm{h(R_s)}\, d^3a$, with $h(R_s) = (1/2)R_s^2$. In this case,

$$\frac{\delta H}{\delta \boldsymbol{R}} = -\frac{\mathrm{d}}{\mathrm{d}s}(h'(R_s)\hat{\boldsymbol{R}}_s) = -R_{ss}, \quad \text{for} \quad h(R_s) = \tfrac{1}{2}R_s^2,$$

and the RRDB equations of vortex motion (2) specialize to

$$\begin{aligned}\dot{\boldsymbol{R}}(\boldsymbol{a},t) &= \frac{1}{R_s^2}\boldsymbol{R}_s \times \boldsymbol{R}_{ss} + \left(\frac{\chi}{R_s^4}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_s^4}\right)\boldsymbol{R}_{ss} + \frac{\boldsymbol{R}_s}{R_s^2}\left(\frac{\chi}{R_s^4}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{\chi}{R_s^2}\right)\left(\frac{\boldsymbol{R}_s}{R_s^2}\cdot \boldsymbol{R}_{ss}\right),\\ \dot{\chi}(\boldsymbol{a},t) &= \boldsymbol{R}_s \cdot \boldsymbol{R}_{ss} \neq 0.\end{aligned}$$

The third-order $\boldsymbol{R}_{sss}$ term in the $\dot{\boldsymbol{R}}$ equation is typical for vortex filament motion with longitudinal flow. Again note that this longitudinal flow involves the co-evolution of χ.

2.1.2. Second example—Betchov–Da Rios equation

We now choose the Hamiltonian $H = \int h(R_s)\,\mathrm{d}^3a$, with $h(R_s) = R_s$, itself. Thus,

$$\frac{\delta H}{\delta \boldsymbol{R}} = -\frac{\mathrm{d}}{\mathrm{d}s}(h'(R_s)\hat{\boldsymbol{R}}_s) = -\frac{\mathrm{d}}{\mathrm{d}s}\hat{\boldsymbol{R}}_s, \quad \text{for} \quad h(R_s) = R_s.$$

In this case, $\hat{\boldsymbol{R}}_s = \boldsymbol{t}$, where $\boldsymbol{t}$ is the unit tangent vector along the vortex filament, whose properties as a space curve are defined via the Serret–Frenet equations:

$$\hat{\boldsymbol{R}}_s = \boldsymbol{t}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{t} = \kappa \boldsymbol{n}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{n} = \tau \boldsymbol{n} - \kappa \boldsymbol{t}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{b} = -\tau \boldsymbol{n}.$$

Here, $\boldsymbol{n}$ is the unit vector normal to the filament, $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$ is its unit binomial vector, κ is the curvature of the filament and τ is its torsion.

Hence, we find

$$\dot{\chi} = -\boldsymbol{R}_s \cdot \frac{\delta H}{\delta \boldsymbol{R}} = R_s\,\boldsymbol{t}\cdot \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{t} = 0 \quad \text{since} \quad |\boldsymbol{t}|^2 = 1.$$

The RRDB vortex equation (Eq. (2)) in this case obtained for $\chi = 0$ is

$$\dot{\boldsymbol{R}}(s,t) = -\frac{1}{R_s^2}\boldsymbol{R}_s \times \frac{\delta H}{\delta \boldsymbol{R}} = \frac{1}{R_s}\boldsymbol{t} \times \boldsymbol{t}_s = \frac{1}{R_s}\kappa \boldsymbol{b}. \tag{3}$$

After re-parameterizing the arc-length to absorb R_s, as $R_s\,\mathrm{d}s = \mathrm{d}\ell$, Eq. (3) describes self-induced motion of a vortex filament in the *localized induction approximation* (LIA), see, e.g. [14,22]. This is the simplest of the Betchov–Da Rios equations, on which there is a great literature. See, e.g. [16,17,19,20] for modern discussions and appreciative surveys of this literature. Hasimoto [3] showed that this vortex filament equation in the LIA may be transformed to the integrable nonlinear Schrödinger equation, thereby demonstrating the coherence of waves on vortices. It is of interest to investigate vortex filament Hamiltonians $H = \int h(\boldsymbol{R}_s)\,\mathrm{d}^3a$ that produce more general Betchov–Da Rios equations via the RRDB. See Uby et al. [23] for discussions of such extensions of the Betchov–Da Rios equations in filament dynamics of plasmas and superconductors.

2.2. Hamiltonians for vorticity dynamics

We now choose Hamiltonians of the form $H = \int h(\omega)\,\mathrm{d}^3R = \int h(\omega)J\,\mathrm{d}^3a$, with $h(\omega)$ depending in a general way on the Eulerian vorticity $\omega(\boldsymbol{R})$. Using the Cauchy-theorem definition $\omega = J^{-l}\boldsymbol{R}_s$ allows one to transform these Hamiltonians into Lagrangian coordinates and compute their variations in the form needed for the RRDB:

$$\delta H = \int \frac{\delta h}{\delta\omega}\cdot\delta\omega\,\mathrm{d}^3R + \int h(\omega)\delta J\,\mathrm{d}^3a = \int\left[\frac{\delta h}{\delta\omega}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\delta\boldsymbol{R} + \left(h(\omega) - \frac{\delta h}{\delta\omega}\cdot\omega\right)\delta J\right]\mathrm{d}^3a$$
$$= \int \delta\boldsymbol{R}\cdot\left(\boldsymbol{R}_s \times \mathrm{curl}_R\frac{\delta h}{\delta\omega}(\boldsymbol{R}(\boldsymbol{a},t),t)\right)\mathrm{d}^3a.$$

In the last step, we used the identities

$$\delta J = J\,\mathrm{tr}\left[\left(\frac{\partial R}{\partial\boldsymbol{a}}\right)^{-1}\cdot\left(\frac{\partial\delta\boldsymbol{R}}{\partial\boldsymbol{a}}\right)\right] = \frac{\partial}{\partial a^A}\left(J\left(\frac{\partial\boldsymbol{R}^{-1}}{\partial\boldsymbol{a}}\right)^A_j\delta R^j\right),$$

and integrated by parts, dropping boundary terms. In the general case, we write $\delta H/\delta\omega(\boldsymbol{R})$ as the variation of $H(\omega)$ with respect to the Eulerian vorticity, *evaluated on the filament*. This yields the Lagrangian relation

$$\frac{\delta H(\omega)}{\delta\boldsymbol{R}} = \boldsymbol{R}_s \times \mathrm{curl}_R\frac{\delta H}{\delta\omega}(\boldsymbol{R}(\boldsymbol{a},t),t) = \boldsymbol{t}\times\boldsymbol{t}\times\frac{\partial}{\partial s}\frac{\delta H}{\delta\omega}(\boldsymbol{R}(s,t)t) \equiv -\hat{P}\frac{\partial}{\partial s}\frac{\partial H}{\delta\omega}(\boldsymbol{R}(s,t),t).$$

Here, we have used the chain-rule identity

$$\mathrm{curl}_R\frac{\delta H}{\delta\omega}(\boldsymbol{R}(s)) = \boldsymbol{t}(\boldsymbol{t}\cdot\nabla_R)\times\frac{\delta H}{\delta\omega}(\boldsymbol{R}(s)) = \boldsymbol{t}\times\frac{1}{R_s}\frac{\partial}{\partial s}\frac{\partial H}{\delta\omega}(\boldsymbol{R}(s)),$$

and introduced the operator $\hat{P} \equiv -\boldsymbol{t}\times\boldsymbol{t}\times$ which projects any vector onto the transverse plane normal to the vortex filament at a given spatial point. Consequently,

$$\dot{\chi} \equiv -\boldsymbol{R}_s\cdot\frac{\delta H}{\delta\boldsymbol{R}} = 0,$$

and the variations in $\boldsymbol{R}$ of Eulerian vorticity functionals $H(\omega)$ have no components along the vortex filament. Hence, $\{\chi, H(\omega)\} = 0$ under RRDB for any $H(\omega)$ and such vorticity Hamiltonians preserve the constraint $\chi = 0$.

The vorticity functionals $H(\omega)$ thus comprise one class of the gauge-invariant dynamical variables that were sought in [15]. The RRDB equation (Eq. (2)) in this case yields the following expression for the induced velocity of a vortex filament:

$$\dot{\boldsymbol{R}}(s,t) = -\frac{1}{R_{\rm s}^2}\boldsymbol{R}_{\rm s} \times \left(\boldsymbol{R}_{\rm s} \times \mathrm{curl}_R \frac{\delta H}{\delta \omega}\right) = \hat{P}\,\mathrm{curl}_R \frac{\delta H}{\delta \omega} = \frac{1}{R_{\rm s}^2}\boldsymbol{R}_{\rm s} \times \frac{\partial}{\partial s}\frac{\delta H}{\delta \omega}(\boldsymbol{R}(s,t),t). \tag{4}$$

This form of the RRDB equation (Eq. (2)) generalizes the Betchov–Da Rios formula (3) for the transverse induced velocity of a vortex filament to the case of an incompressible fluid whose energy functional $H(\omega)$ depends *arbitrarily* on spatial vorticity. The remainder of this paper discusses the interpretation of the generalized Betchov–Da Rios formula (4) in several applications.

2.2.1. RRDB implies the Lie–Poisson bracket for vorticity functionals

For two functionals $F(\omega)$ and $H(\omega)$ that depend only on the *Eulerian* vorticity, substituting the previous mixed Eulerian and Lagrangian expression for $\delta H/\delta \boldsymbol{R}$ into the RRDB and evaluating at $\chi = 0$ gives the following Lie–Poisson bracket relation:

$$\begin{aligned}\{F,H\}(\omega) &= \int \mathrm{d}^3 a \frac{1}{R_{\rm s}^2}\boldsymbol{R}_{\rm s} \cdot \frac{\delta F}{\delta \boldsymbol{R}} \times \frac{\delta H}{\delta \boldsymbol{R}} = \int \mathrm{d}^3 a \frac{1}{R_{\rm s}^2}\boldsymbol{R}_{\rm s} \cdot \left(\boldsymbol{R}_{\rm s} \times \mathrm{curl}_R \frac{\delta F}{\delta \omega}\right) \times \left(\boldsymbol{R}_{\rm s} \times \mathrm{curl}_R \frac{\delta H}{\delta \omega}\right)\\ &= \int (J\,\mathrm{d}^3 a)\omega \cdot \mathrm{curl}_R \frac{\delta F}{\delta \omega} \times \mathrm{curl}_R \frac{\delta H}{\delta \omega} = \int \mathrm{d}^3 R\,\omega(\boldsymbol{R}) \cdot \mathrm{curl}_R \frac{\delta F}{\delta \omega}(\boldsymbol{R}) \times \mathrm{curl}_R \frac{\delta H}{\delta \omega}(\boldsymbol{R}),\end{aligned} \tag{5}$$

and Eulerian vorticity dynamics

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} = -\mathrm{curl}\left(\omega \times \mathrm{curl}\frac{\delta H}{\delta \omega}\right). \tag{6}$$

This Lie–Poisson bracket for Eulerian continuous vorticity ω appears in [7] and has been discussed in the literature many times since then. A version of it also appears in [24]. This Lie–Poisson bracket was also discussed recently again by [8–10] from a viewpoint that is logically opposite to the present one, which follows [15] in making the Lagrangian picture primary. As is well known, the reduction of a Poisson bracket from the Lagrangian picture to the Eulerian picture is not an invertible procedure. See, e.g. [1,12] for references to the literature and authoritative discussions of such reductions.

2.2.2. Helicity and knottedness

The helicity $\Lambda(\omega) = \int \omega \cdot \mathrm{curl}^{-1}\,\omega\,\mathrm{d}^3 x$ is a *Casimir* for the Lie–Poisson vorticity bracket (5). That is, $\{\omega, \Lambda\} = 0$ so that $\{H, \Lambda\}(\omega) = 0$ and, hence, helicity is conserved for every vorticity Hamiltonian $H(\omega)$. In terms of Lagrangian vortex filaments, the helicity is expressed as

$$\Lambda(\boldsymbol{R}) = \int \frac{\partial \boldsymbol{R}}{\partial s} \cdot \left(\frac{\partial}{\partial \boldsymbol{R}(s)} \times \int \frac{1}{4\pi|\boldsymbol{R}(s) - \boldsymbol{R}(s')|}\frac{\partial \boldsymbol{R}}{\partial s'}\,\mathrm{d}s'\right)\mathrm{d}s,$$

and is interpreted as the total linkage number, or knottedness, of the vortex filament distribution. The helicity does not Poisson-commute under RRDB with the Lagrangian vortex filament position parameterized by $\boldsymbol{R}(s)$. Instead, by either Eq. (2) or Eq. (4), one computes the Poisson bracket:

$$\{\boldsymbol{R}, \Lambda\} = -\frac{1}{R_s^2}\boldsymbol{R}_s \times \frac{\delta \Lambda}{\delta \boldsymbol{R}} = \frac{2}{R_s^2}\boldsymbol{R}_s \times \frac{\partial}{\partial s}\left(\frac{\partial}{\partial \boldsymbol{R}(s)} \times \int \frac{1}{4\pi|\boldsymbol{R}(s) - \boldsymbol{R}(s')|}\frac{\partial \boldsymbol{R}}{\partial s'}\,\mathrm{d}s'\right),$$

of helicity with the position of a vortex filament in free space. Hence, under RRDB the conserved helicity Poisson-generates an infinitesimal transformation that shifts the vortex filament's Lagrangian position $\boldsymbol{R}(s)$ in the plane normal to $\boldsymbol{R}_s$ at s, but preserves its Eulerian vorticity field $\omega(\boldsymbol{x})$ at every point. Conservation of helicity Λ by the Eulerian vorticity dynamics for any vorticity Hamiltonian $H(\omega)$ means that the corresponding motion of the Lagrangian vortex filament distribution must preserve its total linkage number, or knottedness. This and other topological properties of vortex filament dynamics are discussed and explained in [1], for example, and are surveyed recently in [18].

2.2.3. Euler equations

For the Euler equations, the vorticity Hamiltonian is

$$H(\omega) = \frac{1}{2}\int \omega \cdot (-\Delta)^{-1}\omega\,\mathrm{d}^3R.$$

The divergence-free Eulerian fluid velocity is given by the expression $\boldsymbol{u} = \mathrm{curl}_R\,\psi$ with vector stream function $\psi = \delta H/\delta\omega = (-\Delta)^{-1}\omega$ in which the Laplacian operator is inverted using proper boundary conditions. For a vortex filament in free space, one may also choose to transform the vector stream function for the Euler fluid to the Lagrangian picture as

$$\psi(\boldsymbol{x}) = \frac{\delta H}{\delta \omega} = \frac{1}{4\pi}\int \frac{\omega(\boldsymbol{R})\,\mathrm{d}^3R}{|\boldsymbol{x} - \boldsymbol{R}|} = \frac{1}{4\pi}\int \frac{\boldsymbol{R}_s\mathrm{d}^3a}{|\boldsymbol{x} - \boldsymbol{R}(\boldsymbol{a}, t)|},$$

with $\mathrm{div}\,\psi = 0$ so, that $\mathrm{curlcurl}\,\psi = -\Delta\psi = \omega$ in analogy to the classical electromagnetic theory of currents. Then, the Hamiltonian for an Euler fluid in the Lagrangian representation is (cf. [15]):

$$H = \frac{1}{8\pi}\int \frac{1}{|\boldsymbol{R}(\boldsymbol{a}, t) - \boldsymbol{Q}(\boldsymbol{a}', t)|}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}s'}\,\mathrm{d}^3a\,\mathrm{d}^3a',$$

where $\mathrm{d}/\mathrm{d}s' = \omega_0(\boldsymbol{a}')\cdot\partial/\partial\boldsymbol{a}'$ is the arc-length derivative in the primed Lagrangian coordinates. The vectors $\boldsymbol{R}$ and $\boldsymbol{Q}$ may refer to the positions of either different, or the same vortex filaments, although the well-known singularity at $|\boldsymbol{R} - \boldsymbol{Q}| = 0$ for the Euler fluid requires special treatment.

2.2.4. Superfluid helium-II

If the vorticity Hamiltonian is $H = \int |\omega|\,\mathrm{d}^3R = \int \hat{\omega}\cdot\omega\,\mathrm{d}^3R$, then one finds the associated stream function $\psi = \delta H/\delta\omega = \hat{\omega}$, where $\hat{\omega}$ is the vorticity unit vector. The corresponding filament velocity is $\boldsymbol{u} = \mathrm{curl}\,\psi = \mathrm{curl}\,\hat{\omega}$ and the vorticity itself satisfies

$$\frac{\partial\omega}{\partial t} + \mathrm{curl}(\omega \times \mathrm{curl}\,\hat{\omega}) = 0.$$

This equation is a special case in the Hall–Vinen–Bekarevich–Khalatnikov (HVBK) theory of superfluid helium-II. See [2] for the classic discussion. The corresponding Hamiltonian for the superfluid vortex filament is $H = \int R_s \,\mathrm{d}^3 a$ treated above. This produces the Betchov–Da Rios equation (Eq. (3)) familiar from LIA. A similar vorticity Hamiltonian is introduced by [15] in their discussion of superfluid helium-II. Recently, Kuznetsovand and Ruban [8–10] also revisited this special case of the HVBK theory from the viewpoint of Rasetti–Regge Dirac brackets. The theoretical basis for the Lie–Poisson bracket formulation of the full HVBK theory is discussed in [5] and reviewed in [4].

2.3. *Alternative regularized fluid models*

2.3.1. *The Euler-alpha model*

The Euler-alpha model of incompressible fluid flow introduced in [6] is defined in terms of an auxiliary quantity $\boldsymbol{q} = \operatorname{curl} \boldsymbol{v}$ that plays a role similar to vorticity in the Hamiltonian as

$$H_\alpha = \frac{1}{2} \int \boldsymbol{q} \cdot (-\Delta)^{-1} (1 - \alpha^2 \Delta)^{-1} \boldsymbol{q} \,\mathrm{d}^3 R,$$

for a constant length-scale, α. The corresponding vector stream function is

$$\psi_\alpha = \frac{\delta H}{\delta \boldsymbol{q}} = (-\Delta)^{-1} (1 - \alpha^2 \Delta)^{-1} \boldsymbol{q},$$

and the divergence-free Euler-alpha fluid velocity is defined by

$$\boldsymbol{u}_\alpha = \operatorname{curl} \psi_\alpha = (1 - \alpha^2 \Delta)^{-1} \boldsymbol{v}.$$

Thus, the Euler-alpha fluid velocity $\boldsymbol{u}_\alpha$, is a smoothed version of the velocity $\boldsymbol{v}$. The quantity $\boldsymbol{q} = \operatorname{curl} \boldsymbol{v}$ plays the role of vorticity in the Cauchy relation. Thus, an Euler-alpha vortex filament moving with velocity $\dot{\boldsymbol{R}}(\boldsymbol{a}, t) = \boldsymbol{u}_\alpha(\boldsymbol{R}, t)$ has tangent vector

$$\boldsymbol{R}_s(\boldsymbol{a}, t) = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s} = \boldsymbol{q}_0(\boldsymbol{a}) \cdot \frac{\partial}{\partial \boldsymbol{a}} \boldsymbol{R} = J \boldsymbol{q}(\boldsymbol{R}(\boldsymbol{a}, t), t), \quad \text{with} \quad J = \det\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{a}}\right).$$

The corresponding Hamiltonian for the Lagrangian motion of Euler-alpha vortex filament is given by

$$H = \frac{1}{2} \int G(\boldsymbol{R}(\boldsymbol{a}, t) - \boldsymbol{Q}(\boldsymbol{a}', t)) \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s} \cdot \frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}s'} \,\mathrm{d}^3 a \,\mathrm{d}^3 a', \quad \text{with}$$
$$G(\boldsymbol{R} - \boldsymbol{Q}) = \frac{1 - \exp(-|\boldsymbol{R} - \boldsymbol{Q}|/\alpha)}{4\pi |\boldsymbol{R} - \boldsymbol{Q}|}. \tag{7}$$

This is the Green's function in free space for the operator product $(-\Delta)(1 - \alpha^2 \Delta)$ of the three-dimensional Laplacian with the Helmholtz operator. For $r \equiv |\boldsymbol{R} - \boldsymbol{Q}|$, the quantities

$$g(r) = \frac{1}{4\pi r} \quad \text{and} \quad h(r) = \frac{1}{4\pi\alpha^2} \frac{\mathrm{e}^{-r/\alpha}}{r}$$

satisfy, respectively,

$$-\Delta g(r) = \delta(r) \quad \text{and} \quad (1 - \alpha^2 \Delta) h(r) = \delta(r).$$

Consequently, $G(r)$ in Eq. (7) is the Green's function for the operator $-\Delta(1 - \alpha^2 \Delta)$ This may be checked by direct substitution of $g(r) - \alpha^2 h(r) = (1 - \mathrm{e}^{-r/\alpha})/(4\pi r)$. A Taylor expansion of the exponential shows that this Green's function is regular at $|\boldsymbol{R} - \boldsymbol{Q}| = 0$.

The Biot–Savart law for the Euler-alpha model is, thus, $\boldsymbol{u}_\alpha = \text{curl}\,\psi_\alpha$, with vector stream function:

$$\psi_\alpha(\boldsymbol{x}) = \frac{\delta H}{\delta \boldsymbol{q}} = \int G(|\boldsymbol{x}-\boldsymbol{R}|)\boldsymbol{q}(\boldsymbol{R})\,\mathrm{d}^3R = \int G(|\boldsymbol{x}-\boldsymbol{R}(\boldsymbol{a},t)|)\boldsymbol{R}_\mathrm{s}\,\mathrm{d}^3a,$$

and Green's function $G(r) = (1-\mathrm{e}^{-r/\alpha})/(4\pi r)$ for free boundaries.

2.3.2. The Rosenhead regularization

In their paper, Rasetti and Regge [15] suggested another regularized kernel originally due to Rosenhead [21]:

$$G_{\lambda_0}(\boldsymbol{R}-\boldsymbol{Q}) = \lim_{\lambda\to 0}\left[(|\boldsymbol{R}-\boldsymbol{Q}|^2+\lambda^2)^{-1/2} + 2\ln\left(\frac{\lambda}{\lambda_0}\right)\delta(|\boldsymbol{R}-\boldsymbol{Q}|)\right],$$

with a cut-off length-scale λ_0, interpreted as the vortex core size. Various other suggestions for this sort of regularization have been made since Rasetti and Regge [15]. See, for example [22] and the last chapter of [14] for insightful explanations, history going back to Lord Kelvin and current references to other approaches for regularizing Euler vortex dynamics.

2.4. Dynamics of a single vortex filament

The vorticity Hamiltonians $H(\omega)$ for Euler's equations, Euler-alpha equations and the Rosenhead regularization may each be expressed in either the Eulerian, or the Lagrangian representation as a quadratic convolution with a Green's function kernel, namely

$$H = \frac{1}{2}\int G(|\boldsymbol{x}-\boldsymbol{y}|)\omega(\boldsymbol{x},t)\cdot\omega(\boldsymbol{y},t)\,\mathrm{d}^3x\,\mathrm{d}^3y = \frac{1}{2}\int G(|\boldsymbol{R}(\boldsymbol{a},t)-\boldsymbol{Q}(\boldsymbol{a}',t)|)\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s}\cdot\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}s'}\,\mathrm{d}^3a\,\mathrm{d}^3a'.$$

As before, the vectors $\boldsymbol{R}$ and $\boldsymbol{Q}$ may refer to the positions of either different, or the same vortex filaments. For all these theories, the variational derivative is

$$\frac{\delta H}{\delta\omega} = \int G(|\boldsymbol{x}-\boldsymbol{y}|)\omega(\boldsymbol{y},t)\,\mathrm{d}^3y \equiv (G*\omega)(\boldsymbol{x},t),$$

for a vorticity distribution in the Eulerian representation, and[1]

$$\frac{\delta H}{\delta\omega}(\boldsymbol{R}(s,t)) = \int G(|\boldsymbol{R}(s,t)-\boldsymbol{R}(s',t)|)\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s'}\,\mathrm{d}s' \equiv (G*\boldsymbol{R}_\mathrm{s})(s,t), \tag{8}$$

for a single vortex filament in the Lagrangian representation with arc-length parameter s. The latter expression implies the following RRDB dynamics using Eq. (2) with $\chi = 0$ for vortex filaments whose Hamiltonians $H(\omega)$ are expressible as quadratic convolutions with vorticity

$$\dot{\boldsymbol{R}}(s,t) = -\frac{1}{R_\mathrm{s}^2}\boldsymbol{R}_\mathrm{s}\times\frac{\delta H}{\delta\boldsymbol{R}} = \frac{1}{R_\mathrm{s}^2}\boldsymbol{R}_\mathrm{s}\times\frac{\partial}{\partial s}(G*\boldsymbol{R}_\mathrm{s}). \tag{9}$$

[1] We may drop the transverse Lagrangian coordinates for a single line vortex.

This is the nonlocal generalization of the classical Betchov–Da Rios formula (3) for the case of quadratic vorticity Hamiltonians of the Eulerian (resp. Lagrangian) form:

$$H = \frac{1}{2}\int \omega \cdot (G * \omega)\,\mathrm{d}^3x = \frac{1}{2}\int \boldsymbol{R}_\mathrm{s} \cdot (G * \boldsymbol{R}_\mathrm{s})\,\mathrm{d}s. \tag{10}$$

In the classical Betchov–Da Rios formula (3), the nonlocal expression $G * \boldsymbol{R}_\mathrm{s}$ in Eq. (9) is replaced by (twice) the local unit vector $\boldsymbol{t} = \hat{\boldsymbol{R}}_\mathrm{s}$ tangent to the vortex filament. Consequently, $H_{\mathrm{BDR}} = \int R_\mathrm{s}\,\mathrm{d}s$. and the dynamics of the tangent vector $\boldsymbol{R}$, along a single vortex filament in this case recovers the Local Induction Approximation, cf. the Betchov–Da Rios formula (Eq. (3)):

$$\dot{\boldsymbol{R}}_\mathrm{s} = \frac{\partial}{\partial s}\left(\frac{1}{R_\mathrm{s}^2}\boldsymbol{R}_\mathrm{s} \times \frac{\partial}{\partial s}\left(\frac{\delta H_{\mathrm{BDR}}}{\delta \boldsymbol{R}}\right)\right) = \frac{\partial}{\partial s}\left(\frac{1}{R_\mathrm{s}}\boldsymbol{t} \times \boldsymbol{t}_\mathrm{s}\right). \tag{11}$$

This equation implies $\boldsymbol{R}_\mathrm{s} \cdot \dot{\boldsymbol{R}}_\mathrm{s} = 0$, so the LIA admits solutions with uniform strength $R_\mathrm{s} = \Gamma$. In this case, the LIA equation reduces to the Landau–Lifschitz equation for a continuum approximation of a ferromagnetic Heisenberg spin chain, namely

$$\Gamma^2 \dot{\boldsymbol{t}} = \boldsymbol{t} \times \boldsymbol{t}_{\mathrm{ss}}. \tag{12}$$

2.5. Relating RRDB to the Marsden–Weinstein bracket for vortex filaments

The Rasetti–Regge Dirac bracket (1) is expressed in vortex filament notation as

$$\{F, H\}(\boldsymbol{R}) = \int \frac{1}{R_\mathrm{s}^2}\boldsymbol{R}_\mathrm{s} \cdot \frac{\delta F}{\delta \boldsymbol{R}} \times \frac{\delta H}{\delta \boldsymbol{R}}\,\mathrm{d}s. \tag{13}$$

We shall re-parameterize the arc-length to absorb R_s as $R_\mathrm{s}\,\mathrm{d}s = \mathrm{d}\ell$, and define new dynamical variables by $\mathrm{d}\boldsymbol{Z} = R_\mathrm{s}\,\mathrm{d}\boldsymbol{R}$, so that $\boldsymbol{Z}_\ell = \boldsymbol{R}_\mathrm{s}$ and $Z_\ell = R_\mathrm{s}$. After this invertible transformation, the RRDB for vortex filaments (13) becomes

$$\{F, H\}(\boldsymbol{Z}) = \int \frac{\boldsymbol{Z}_\ell}{Z_\ell} \cdot \frac{\delta F}{\delta \boldsymbol{Z}} \times \frac{\delta H}{\delta \boldsymbol{Z}}\,\mathrm{d}\ell. \tag{14}$$

This is the vortex filament bracket in [13], which is now seen to be equivalent to the RRDB under this invertible change of variables, when acting in the space of vorticity functionals. The quadratic vorticity Hamiltonians (10) transform as

$$H = \frac{1}{2}\int \omega \cdot (G * \omega)\,\mathrm{d}^3x = \frac{1}{2}\int \boldsymbol{R}_\mathrm{s} \cdot (G * \boldsymbol{R}_\mathrm{s})\,\mathrm{d}s = \frac{1}{2}\int \boldsymbol{T} \cdot ((G \circ R) * T)\,\mathrm{d}\ell, \tag{15}$$

where $\boldsymbol{T}(\ell) \equiv \boldsymbol{Z}_\ell / Z_\ell$ is the unit tangent vector under the new parameterization and the convolution in Eq. (8) transforms as

$$G * \boldsymbol{R}_\mathrm{s}(s) = \int G(|\boldsymbol{R}(\boldsymbol{Z}(\ell)) - \boldsymbol{R}(\boldsymbol{Z}(\ell'))|)\boldsymbol{T}(\ell')\,\mathrm{d}\ell' \equiv (G \circ R) * \boldsymbol{T}(\ell). \tag{16}$$

Consequently, for the quadratic vorticity Hamiltonians, we find

$$\frac{\delta H}{\delta \boldsymbol{Z}} = -\frac{\partial}{\partial \ell}\frac{\delta H}{\delta \boldsymbol{T}} = -\frac{\partial}{\partial \ell}((G \circ R) * \boldsymbol{T}), \tag{17}$$

and the Marsden–Weinstein bracket (14) transforms to

$$\{F, H\}(\boldsymbol{T}) = \int \boldsymbol{T} \cdot \frac{\partial}{\partial \ell} \frac{\delta F}{\delta \boldsymbol{T}} \times \frac{\partial}{\partial \ell} \frac{\delta H}{\delta \boldsymbol{T}} \, \mathrm{d}\ell. \tag{18}$$

The corresponding vortex filament dynamics is given by

$$\dot{\boldsymbol{T}} = \{\boldsymbol{T}, H\} = \frac{\partial}{\partial \ell}\left(\boldsymbol{T} \times \frac{\partial}{\partial \ell} \frac{\delta H}{\delta \boldsymbol{T}}\right) = \frac{\partial}{\partial \ell}\left(\boldsymbol{T} \times \frac{\partial}{\partial \ell}((G \circ R) * \boldsymbol{T})\right). \tag{19}$$

Langer and Perline [11] discovered that the Hasimoto transformation which takes solutions of the Landau–Lifschitz equation (Eq. (12)) to solutions of the nonlinear Schrödinger (NLS) equation is also a Poisson map. This Poisson map transforms the Marsden–Weinstein bracket (14) to the fourth (noncanonical) Poisson bracket for NLS. The application of the Hasimoto–Langer–Perline transformation to the quadratic vorticity Hamiltonians in Eq. (15) can be expected to produce physically meaningful *nonlocal* NLS equations for future study.

3. Conclusion

Many other geometrical properties (and other interesting approximations!) of fluid vortex dynamics remain to be explored using the RRDB equation (Eq. (2)). There are still several fundamental open problems. For example, in using the Cauchy definition of arc-length, one assumes the vortex filament does not slip relative to the ambient fluid. Thus, the RRDB formulation leaves out the reactive forces that are known to cause a vortex filament in helium-II to move relative to *both* the superfluid and the normal fluid components. This reactive force was recently given a Hamiltonian formulation in the Eulerian fluid picture in [4]. Expressing the Dirac bracket formulation for this reactive slipping of a vortex filament through its ambient fluid in the Lagrangian picture is still an open problem.

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