Quantizing the classical cat

Ian Stewart

A mathematical analysis of a pendulum system reveals the relevance to quantum systems of the classical concept of 'monodromy' — why a falling cat always lands the right way up.

A central problem in modern physics is to find effective methods for quantizing classical dynamical systems — modifying the classical equations to incorporate the effects of quantum mechanics. One of the main obstacles is the disparity between the linearity of quantum theory and the nonlinearity of classical dynamics. Taking a big step forward, R. H. Cushman *et al.* have analysed a quantum version of the spring pendulum, whose resonant state was first discussed by Enrico Fermi and which is a standard model for the carbon dioxide molecule (*Phys. Rev. Lett.* **93,** 024302; 2004).

Cushman *et al.* show that when this system is quantized, the allowed states, or eigenstates, fail to form a perfect lattice, contrary to simpler examples. Instead, the lattice has a defect, a point at which the regular lattice structure is destroyed. They show that this defect can be understood in terms of an important classical phenomenon known as monodromy. A quantum-mechanical cliché is Schrödinger's cat, whose role is to dramatize the superposition of quantum states by being both alive and dead. Classical mechanics now introduces a second cat, which dramatizes monodromy through its ability always to land on its feet (Fig. 1). The work affords important new insights into the general problem of quantization, as well as being a beautiful example of the relation between nonlinear dynamics and quantum theory.

The underlying classical model here is the swing–spring, a mass suspended from a fixed point by a spring (Fig. 2a, overleaf). The spring is free to swing like a pendulum in any vertical plane through the fixed point, and it can also oscillate along its length by expanding and contracting. The Fermi resonance occurs when the spring frequency is twice the swing frequency. The same resonance occurs in a simplified model of the two main classical vibrational modes of the carbon dioxide molecule (Fig. 2b), and the first mathematical analysis of the swing–spring was inspired by this model.

Using a modern technique of analysis known as reduction, which exploits the rotational symmetry of a system, Cushman *et al.* show that this particular resonance has a curious implication, which manifests itself physically as a switching phenomenon. Start a slightly unstable state. The vertical 'spring mode' motion quickly becomes a 'swing mode' oscillation, just like a clock pendulum swinging in some vertical plane. However, this swing state is transient and the system returns once more to its spring mode, then back to a swing mode, and so on indefinitely. The surprise is that the successive planes in which it swings are different at each stage. Moreover, the angle through which the swing plane turns, from one occurrence to the next, depends sensitively on the amplitude of the original spring mode.

The apparent paradox here is that the initial state has zero angular momentum — the net spin about the vertical axis is zero. Yet the swing state rotates from one instance to the next. Analogously, a falling cat that starts upside down has no angular momentum about its own longitudinal axis, yet it can invert itself, apparently spinning about that axis. The resolution of the paradox, for a cat, is that the animal changes its shape by moving its paws and tail in a particular way. At each stage of the motion, angular momentum remains zero and is thus conserved, but the overall effect of the shape changes is to invert the cat. The final upright state also has zero angular momentum, so there is no contradiction of conservation. This effect is known as the 'geometric phase', or monodromy, and is important in many areas of physics and mathematics.

The central topic of the paper is this: how does monodromy show up when the system is quantized? The answer, obtained in the specific context of the carbon dioxide molecule, is both elegant and remarkable.

A molecule of carbon dioxide can be modelled classically as a central carbon atom, attached symmetrically by identical springs to two oxygen atoms, with the springs inclined at an obtuse angle (Fig. 2b). The molecule has three main vibrational modes. The two most important modes are symmetric stretching, where both springs change their lengths in synchrony, and bending, where the angle between the two springs oscillates. These modes are analogous to the spring and swing modes of a swing–spring. The third main mode, asymmetric stretching, occurs when the two springs oscillate out of phase with each other, and it can be removed from consideration by averaging over a vibrational cycle. The result is a

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100 YEARS AGO

The result of this inquiry is to prove the existence of a small number of more or less isolated hereditary centres, round which a large part of the total ability of the nation is clustered, with a closeness that rapidly diminishes as the distance of kinship from its centre increases. The materials are derived from the replies to a circular which I sent with a blank schedule, to all fellows of the Royal Society, asking for the names and achievements of their "noteworthy" kinsfolk in each degree of near kinship as specified in the schedule. Noteworthiness was defined as including any success that was, in the opinion of the sender, at least equal in its way to that in which the honour of a fellowship of the Royal Society is held by scientific men. Returns are still dropping in, and now exceed two hundred. They continue to be very acceptable, but I judged it best to content myself with the number received up to a date when I could conveniently work at them, and to publish the preliminary results without delay… the experience gained through this inquiry has strongly confirmed an opinion expressed in my lecture on Eugenics before the Sociological Society… that it would be both feasible and advantageous to make a register of gifted families. Francis Galton

From *Nature* 11 August 1904.

50 YEARS AGO

The chromosomes of *Mus musculus* have a high chiasma frequency, and for this reason very loose linkages are to be expected. Many of the problems of linkage and independence in this species may therefore have to be solved by cytogenetic methods rather than the breeding techniques of formal genetics. Among them is the question whether linkage group VII is carried in the pairing segment of the sex chromosome… With the object of obtaining evidence on questions such as this we have induced a number of translocations in the mouse, using X-rays, and have identified linkage groups in eleven of them… Translocation *T8* thus offers a means of settling the question whether linkage group VII is sex-linked. The translocation and the sex bivalent should be cytologically recognizable in primary spermatocytes; it should therefore be possible to establish their chromosomal independence or interdependence. T. C. Carter, Mary F. Lyon & Rita J. S. Phillips

From *Nature* 14 August 1954.

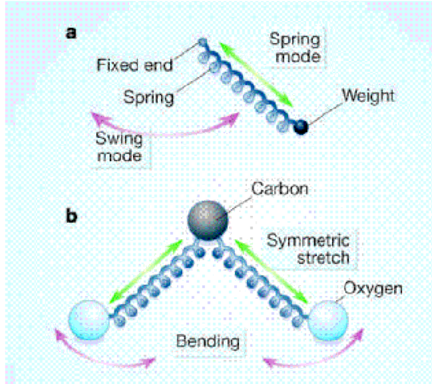

Figure 2 **The swing–spring. The swing–spring can stretch like a spring and swing like a pendulum (a), and can be used as a simple model for the carbon dioxide molecule (b).**

which is simpler than the exact Hamiltonian but is still a good model.

The quantum energy–momentum lattice of the molecule consists of the eigenstates of this Hamiltonian, that is, the pure vibrational modes. For a fixed energy, these modes correspond to two classical 'constants of motion' — angular momentum and a quantity related to the rotational symmetry. The eigenstates can be characterized by two quantum numbers, which are integers, so these eigenstates form a regular planar lattice like a chessboard.

However, there is an extra quantum number, related to another classical variable, called the 'action'. The new phenomenon here is that, because of monodromy, the action is defined only locally and cannot be consistently extended across the entire lattice. For fixed quantum numbers in the lattice, this additional quantum number can take on infinitely many values, at equally spaced points at right angles to the chessboard. The simplest structure of this kind is a three-dimensional cubic lattice — an infinite stack of chessboards, vertically above each other. Monodromy implies that the totality of all sets of quantum numbers does not form a cubic lattice. Instead, it has a single topological defect where the regularity of the lattice structure breaks down.

This analysis is important because it suggests, and supports, a general principle. The most significant features of the quantum-mechanical description of a classical system occur at its singularities. The singularities introduce defects into the ensemble of quantum eigenstates, but they also organize the structure of those defects. Everywhere else, quantization works just as in previous, simpler examples. The authors suggest several directions for future progress, mostly to develop the growing use of nonlinear dynamics in the understanding of quantization. But the most tantalizing is the possibility of detecting quantum monodromy experimentally. Maybe we will soon be able to see how Schrödinger's cat turns itself upside down. ■

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Cognitive science

Rank inferred by reason

Sara J. Shettleworth

Pinyon jays seem to work out how to behave towards an unfamiliar jay by watching it in encounters with members of their own flock. The findings provide clues about how cognition evolved in social animals.

Susan is taller than Billy. Peter is taller than Susan. Who is taller, Billy or Peter? Knowledge about pairs of objects linked by relationships such as 'taller' or 'stronger' permits conclusions to be drawn about novel pairs (here, Billy and Peter) — a process known as transitive inference. Monkeys, rats and some birds can solve transitive-inference tasks in the laboratory[1], but why would this ability evolve? A plausible answer is that transitive inference is an evolutionary adaptation in certain kinds of social group. For example, suppose I know from bitter experience that Bob always beats me in contests (that is, he dominates me). I now observe some new individual, Andy, dominating Bob. If I reason, "Andy dominates Bob, and Bob dominates me, therefore Andy will dominate me", I can avoid fights by has been no well-controlled evidence that animals actually use transitive inference in social situations. In the study reported on page 778 of this issue, Paz-y-Miño and colleagues[2] provide this.

In effect, the authors staged the Andy-and-Bob scenario using pinyon jays (*Gymnorhinus cyanocephalus*; Fig. 1), a highly social member of the crow family. These birds live in large, permanent flocks with clear pecking orders. Paz-y-Miño and colleagues created groups of captive pinyon jays that were previously unknown to each other, and allowed stable dominance relationships to develop in each group. Then jays from each group were allowed to observe individuals from other groups interacting over a peanut, and later interacted with some of those same birds. In the experiment, the