# EFFECTIVE PROPERTIES OF RANDOM COMPOSITES*

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**Abstract.** We propose a new concept of the effective properties of composites with uncertain spatial arrangements of constitutive materials and within-material properties. Rather than replacing a heterogeneous property with a constant effective parameter, we seek to preserve the internal macro structure of a composite. This general concept is used to derive the effective conductivity of composite heterogeneous media that consist of two materials whose internal geometries and conductivities are uncertain. Our analysis relies on a random domain decomposition to explicitly account for the separate effects of material and geometric uncertainties on the ensemble moments of pressure and flux. We present a general expression for the effective (apparent) conductivity of such media and analyze it in detail for one- and two-dimensional steady flows in bounded random media composed of two materials with highly contrasting conductivities.

**Key words.** composite media, random fields, stochastic, moment equations, apparent, equivalent

**AMS subject classifications.** 74Q15, 60H30, 86A05, 86A32

**1. Introduction.** Effective (upscaled) parameters have proved to be a useful tool for modeling heterogeneous systems. Such models often require assigning system parameters to large grid blocks, while experimental data are usually available at a much smaller scale. These parameters can be obtained through either deterministic approaches, such as homogenization and inverse modeling, or stochastic averaging — the approach we pursue here. A plethora of approaches used to obtain effective parameters for composites is reviewed in [3]. These and other methods seek to replace a heterogeneous system with a homogeneous system that preserves some global properties. Consider, for example, diffusion in a medium composed of several heterogeneous materials whose spatial arrangement is uncertain. Standard upscaling or homogenization techniques substitute the effective diffusion coefficient $K_{\rm eff}$ for the space varying diffusion coefficient $K(\mathbf{x})$ in a way that preserves a global mass flux induced by a global gradient of substance. While often useful, such effective parameters fail to predict important characteristics of the system behavior, e.g., the existence of preferential flow paths in porous media. Rapid advances in noninvasive data acquisition techniques, such as magnetic resonance imaging and computerized axial tomography, make it unnecessary to homogenize a system in ways that ignore the internal composition of a material. What is required instead is to derive effective parameters that account for uncertainties in both the material properties and internal boundaries. This paper takes a first step in this direction.

Stochastic approaches to upscaling are grounded in the fact that, in realistic settings, system parameters are deduced from measurements at selected locations and depth intervals, where their values depend on the scale and mode of measurement. Often, the measurement support is uncertain and data are corrupted by experimental and interpretative errors. Estimating the parameters at points where measurements are not available entails additional errors. Treating the system parameters as random fields provides a natural framework for dealing with these errors and uncertainties. Within this framework a system parameter $K(\mathbf{x})$

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is characterized by a multivariate probability density function or, equivalently, its joint ensemble moments. Thus, $K(\mathbf{x})$ varies not only across the real space coordinates $\mathbf{x}$, but also in probability space (this variation may be represented by another coordinate $\xi$, which, for simplicity of notation, is usually suppressed). Whereas spatial moments of $K$ are obtained by sampling $K(\mathbf{x})$ in real space (across $\mathbf{x}$), its ensemble moments are defined in terms of samples collected in probability space (across $\xi$).

Randomness of system parameters renders partial differential equations (PDEs) governing the dynamics of system states stochastic. Effective parameters are then defined as coefficients in the ensemble averaged stochastic PDEs. Consider, for example, Darcy's law $\mathbf{q} = -K\nabla h$, which postulates a linear relationship between the mass flux (Darcian velocity) $\mathbf{q}(\mathbf{x})$ and the pressure (hydraulic head) gradient $\nabla h(\mathbf{x})$ in porous media. Apparent conductivity $K_{\rm app}(\mathbf{x})$ is defined as a coefficient of proportionality in the ensemble averaged Darcy's law $\langle \mathbf{q} \rangle = -K_{\rm app}\nabla\langle h\rangle$. The term "apparent" was introduced in [2] to emphasize that the effective parameters thus defined are local quantities that depend not only on material's properties, but on external forces (e.g., boundary conditions) as well.

Much of the existing literature on effective conductivity is limited to mildly heterogeneous media composed of a single material whose log conductivity is treated as a statistically homogeneous (stationary) random field with small variance (e.g., [1]). One recent example of this approach [5] explores the tensorial nature of apparent transmissivity for a rectangular flow domain by the localization and perturbation expansion of the nonlocal mean flow equations in $\sigma_Y^2$, the variance of $Y = \ln K$. The requirement that $\sigma_Y^2 \ll 1$ is crucial for closing the moment differential equations. At the same time, it clearly limits the applicability of such analyses.

To derive effective parameters that preserve the internal structure of a highly heterogeneous composite, we employ the random domain decomposition approach [6, 7]. It enables us to deal with uncertainty in both the spatial arrangement of composite materials and their parameters. While the approach we propose is applicable to a wide variety of physical systems, in this paper we focus on the derivation of the effective conductivity for porous media composed of two materials. The main results of our study are formulated in Sections 3 and 4, where we provide a general expression for effective conductivity and analyze it, both analytically and numerically, for one- and two-dimensional flow configurations.

**2. Problem Formulation.** Consider steady-state saturated flow in a flow domain $\Omega = \Omega_1 \cup \Omega_2$, which is composed of two disjoint sub-domains $\Omega_1$ and $\Omega_2$, separated by a contact surface $\Gamma_{12} = \Omega_1 \cap \Omega_2$. Flow is described by the combination of Darcy's law and mass conservation,

$$\mathbf{q} = -K\nabla h \qquad \text{and} \qquad -\nabla\cdot\mathbf{q} + f = 0, \tag{2.1}$$

where $f$ is a random source term. Random hydraulic head $H$ and flux $Q$ are prescribed on the Dirichlet ($\Gamma_D$) and Neumann ($\Gamma_N$) boundary segments ($\Gamma_D \cup \Gamma_N = \partial\Omega$), respectively

$$h(\mathbf{x}) = H(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_D, \tag{2.2a}$$

$$-\mathbf{q}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = Q(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_N, \tag{2.2b}$$

where $\mathbf{n}$ is the unit vector normal to the boundary.

Let the random hydraulic conductivity field belong to two distinct statistically independent populations,

$$K(\mathbf{x}) = \begin{cases} K_1(\mathbf{x}), & \mathbf{x} \in \Omega_1, \\ K_2(\mathbf{x}), & \mathbf{x} \in \Omega_2. \end{cases} \tag{2.3}$$

(Note that one can easily generalize the results obtained below to incorporate correlations between the $K_1$ and $K_2$ fields by following [7].) Then the flow equation (2.1) can be rewritten as

$$\nabla \cdot K_i \nabla h + f = 0, \qquad \mathbf{x} \in \Omega_i. \tag{2.4}$$

The boundary conditions (2.2) are now supplemented by the continuity conditions on the random interface $\Gamma_{12}$,

$$h(\mathbf{x}^-) = h(\mathbf{x}^+) \tag{2.5a}$$

and

$$K(\mathbf{x}^-)\nabla h(\mathbf{x}^-) \cdot \mathbf{n}(\mathbf{x}^-) = K(\mathbf{x}^+)\nabla h(\mathbf{x}^+) \cdot \mathbf{n}(\mathbf{x}^+). \tag{2.5b}$$

Here the superscripts $^-$ and $^+$ indicate the limits as $\mathbf{x} \to \Gamma_{12}$ from $\Omega_1$ and $\Omega_2$, respectively.

In this formulation, the randomness of $K(\mathbf{x})$ stems from two factors: small-scale within material uncertainty in $K_i(\mathbf{x})$ and large-scale uncertainty in the spatial arrangement of $\Omega_i$ or, equivalently, in the boundary $\Gamma_{12}$. Hence $p_K(k)$, the probability density function of $K$, is replaced with the joint probability density function $p_K(k,\gamma) = p_K(k|\gamma)p_\Gamma(\gamma)$. While for highly contrasting composites $p_K(k)$ is bimodal with large variance $\sigma_K^2$, the conditional distribution $p_K(k|\gamma)$ — representing the random fluctuations of conductivity within each material $\Omega_i$ — is likely to be unimodal with small variances $\sigma_{K_i}^2$. This is important because closure approximations associated with the (conditional) stochastic averaging of the flow equation (2.4) are carried out within each sub-domain $\Omega_i$ separately.

**3. Apparent Conductivity.** Applying the Reynolds decomposition to represent random fields $\mathcal{R} = \langle \mathcal{R} \rangle + \mathcal{R}'$ as the sum of their ensemble means $\langle \mathcal{R} \rangle$ and zero-mean fluctuations $\mathcal{R}'$ and taking the ensemble mean of (2.4) yields the mean Darcy's law,

$$\langle \mathbf{q}(\mathbf{x}) \rangle = -\langle K_i(\mathbf{x}) \rangle \nabla \langle h(\mathbf{x}) \rangle + \langle \mathbf{r}_i(\mathbf{x}) \rangle, \qquad \mathbf{x} \in \Omega_i, \tag{3.1}$$

where "residual" flux $\langle \mathbf{r}_i \rangle = -\langle K_i' \nabla h' \rangle$ represents the single point cross-covariance between the fluctuations of head gradient and hydraulic conductivity. The ensemble mean of a random field $\mathcal{R}$ is given by

$$\langle \mathcal{R}(\mathbf{x}) \rangle = \int\int \mathcal{R}(k,\gamma;\mathbf{x}) p_K(k,\gamma) dk d\gamma = \int \langle \mathcal{R}(\gamma;\mathbf{x}) \rangle_\Gamma p_\Gamma(\gamma) d\gamma, \tag{3.2a}$$

where

$$\langle \mathcal{R}(\gamma;\mathbf{x}) \rangle_\Gamma = \int \mathcal{R}(k,\gamma;\mathbf{x}) p_K(k|\gamma) dk \tag{3.2b}$$

is the ensemble mean of $\mathcal{R}$ conditioned on the location of the internal boundary $\Gamma_{12}$.

Let $\mathcal{G}$ be a random Green's function for (2.4) subject to (2.2) and (2.5), but with the fixed (known) boundary $\Gamma_{12}$. (Realizations of $\Gamma_{12}$ come from its distribution $p_\Gamma$.) Then conditional residual flux can be found as a solution of the integral equation [7]

$$\langle \mathbf{r}_i(\gamma;\mathbf{x}) \rangle_\Gamma = \int_{\Omega_i} \mathbf{a}_i(\gamma;\mathbf{y},\mathbf{x}) \nabla \langle h(\gamma;\mathbf{y}) \rangle_\Gamma d\mathbf{y} + \int_{\Omega_i} \mathbf{b}_i(\gamma;\mathbf{y},\mathbf{x}) \langle \mathbf{r}_i(\gamma;\mathbf{y}) \rangle_\Gamma d\mathbf{y} \tag{3.3}$$

with the kernels $\mathbf{a}_i$ and $\mathbf{b}_i$ taking the form of second-rank tensors

$$\mathbf{a}_i = \langle K_i'(\mathbf{x}) K_i'(\mathbf{y}) \nabla_\mathbf{y} \nabla_\mathbf{x}^T \mathcal{G}(\gamma;\mathbf{y},\mathbf{x}) \rangle_\Gamma, \qquad \mathbf{b}_i = \langle K_i'(\mathbf{x}) \nabla_\mathbf{y} \nabla_\mathbf{x}^T \mathcal{G}(\gamma;\mathbf{y},\mathbf{x}) \rangle_\Gamma. \tag{3.4}$$

It follows from (3.1) and (3.3) that the mean Darcy's flux $\langle \mathbf{q}(\mathbf{x})\rangle$ is nonlocal, i.e., depends on the mean head gradient $\nabla\langle h\rangle$ at points other than $\mathbf{x}$. Hence the apparent conductivity does not exist, in general. This finding is in line with numerous previous investigations, e.g., [1] and references therein.

Assume that both the mean pressure gradient and the residual flux vary slowly in space [5] within $\Omega_i$ ($i = 1, 2$). Then (3.3) can be localized, leading to an approximate expression

$$\langle \mathbf{r}_i(\gamma;\mathbf{x}\rangle_\Gamma \approx \mathbf{A}_i(\gamma;\mathbf{x})\nabla\langle h(\gamma;\mathbf{y})\rangle_\Gamma + \mathbf{B}_i(\gamma;\mathbf{x})\langle \mathbf{r}_i(\gamma;\mathbf{y})\rangle_\Gamma, \tag{3.5}$$

where

$$\mathbf{A}_i = \int_{\Omega_i} \mathbf{a}_i(\gamma;\mathbf{y},\mathbf{x})d\mathbf{y} \qquad \text{and} \qquad \mathbf{B}_i = \int_{\Omega_i} \mathbf{b}_i(\gamma;\mathbf{y},\mathbf{x})d\mathbf{y}. \tag{3.6}$$

Substituting (3.5) into a conditional version of (3.1) yields the conditional mean Darcy's law

$$\langle \mathbf{q}(\mathbf{x})\rangle_\Gamma = -\mathbf{K}_{\mathrm{app}_i}(\gamma;\mathbf{x})\nabla\langle h(\mathbf{x})\rangle_\Gamma, \qquad \mathbf{x}\in\Omega_i. \tag{3.7}$$

The conditional apparent conductivity tensor in (3.7) is given by

$$\mathbf{K}_{\mathrm{app}_i} = \langle K_i(\mathbf{x})\rangle_\Gamma \mathbf{I} - \mathbf{k}_i(\gamma;\mathbf{x}), \tag{3.8a}$$

where $\mathbf{I}$ is the identity tensor and

$$\mathbf{k}_i(\mathbf{x}) = \left[\mathbf{I} - \mathbf{B}_i(\mathbf{x})\right]^{-1}\mathbf{A}_i(\mathbf{x}). \tag{3.8b}$$

Evaluation of the conditional apparent conductivity requires a closure approximation for the tensors $\mathbf{a}_i$ and $\mathbf{b}_i$ in (3.4). Following [5], we obtain such a closure by using perturbation expansions in $\sigma^2_{Y_i}$, the variances of log-hydraulic conductivities $Y_i = \ln K_i$ ($i = 1, 2$). Consider asymptotic expansions

$$\langle K\rangle_\Gamma = T_{\mathrm{g}_i}(1 + \sigma^2_{Y_i}/2 + \ldots) \qquad\qquad \mathcal{T} = \mathcal{T}^{(0)} + \mathcal{T}^{(1)} + \ldots \tag{3.9}$$

where $K_{\mathrm{g}_i} = \exp(\langle Y_i\rangle)$ is the geometric mean of the conductivity of the $i$th material and $\mathcal{T}$ stands for $h$, $\mathbf{q}$, $\mathbf{r}$, and other relevant random fields. The superscript $(n)$ denotes the $n$th-order terms, i.e., the terms that are proportional $\sigma^{2n}_{Y_i}$. It follows from (3.5) and (3.6) that the first-order (in $\sigma^2_{Y_i}$) approximation of the localized conditional residual flux $\langle \mathbf{r}_i\rangle_\Gamma$ is given by $\langle \mathbf{r}_i\rangle_\Gamma = \mathbf{A}^{(1)}_i \nabla\langle h\rangle^{(0)}_\Gamma$, where [7]

$$\mathbf{A}^{(1)}_i = \int_{\Omega_i} \mathbf{a}^{(1)}_i(\gamma;\mathbf{y},\mathbf{x})d\mathbf{y} \qquad \text{and} \qquad \mathbf{a}^{(1)}_i = \sigma^2_{Y_i} K^2_{\mathrm{g}_i}\rho_{Y_{ii}}\nabla_{\mathbf{y}}\nabla^T_{\mathbf{x}} G. \tag{3.10}$$

Here $\rho_{Y_{ii}}(\mathbf{x},\mathbf{y})$ is the two-point correlation function of $Y$ for $\mathbf{x},\mathbf{y}\in\Omega_i$ and $G = \langle \mathcal{G}\rangle^{(0)}_\Gamma$. In analogy to [5], it then follows from (3.8) that, up to the first order in $\sigma^2_{Y_i}$, the conditional apparent conductivity tensor is given by $\mathbf{K}^{[1]}_{\mathrm{app}_i} = \mathbf{K}^{(0)}_{\mathrm{app}_i} + \mathbf{K}^{(1)}_{\mathrm{app}_i}$,

$$\mathbf{K}^{[1]}_{\mathrm{app}_i}(\gamma;\mathbf{x}) = K_{\mathrm{g}_i}\left(1 + \frac{\sigma^2_{Y_i}}{2}\right)\mathbf{I} - \sigma^2_{Y_i}K^2_{\mathrm{g}_i}\int_{\Omega_i}\rho_{Y_{ii}}(\mathbf{y},\mathbf{x})\nabla_{\mathbf{y}}\nabla^T_{\mathbf{x}}G(\gamma;\mathbf{y},\mathbf{x})d\mathbf{y}. \tag{3.11}$$

The final step in obtaining the apparent conductivity consists of the ensemble averaging of (3.11) by computing (3.2b).

The perturbation approximation in (3.11) is carried out in terms of the variances within the materials $\sigma^2_{Y_i}$, which are small in most natural formations. However, if they are not small enough for (3.11) to remain accurate, one can generalize this expression by means of the Matheron-Landau conjecture [4].

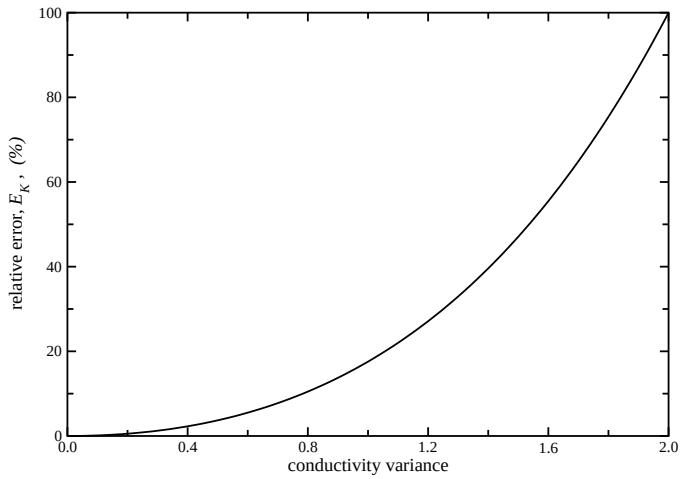


FIG. 4.1. *Relative errors between the exact and perturbation expressions for the apparent conductivity* $K_{\rm app}$ *as a function of the variance of log conductivity* $\sigma^2_{Y_1} = \sigma^2_{Y_1} = \sigma^2_Y$.

**4. Computational Examples.** To analyze our general expression for the apparent conductivity of composite media in detail, we consider one- and two-dimensional flows in layered media. The one-dimensional example is amenable to analytical analysis, while the two-dimensional example relies on numerical evaluation of the Green's function and quadratures in (3.11).

**4.1. One-Dimensional Flow.** Consider the one-dimensional version of (2.4) with $f \equiv 0$, which is defined on the interval $x \in \Omega = (0, 1)$. The boundary conditions are

$$K\frac{dh}{dx} = -Q \qquad \text{for} \qquad x = 0 \tag{4.1a}$$

and

$$h(x) = 0 \qquad \text{for} \qquad x = 1. \tag{4.1b}$$

The flow domain $\Omega$ is composed of two materials $\Omega_1 = [0, \beta]$ and $\Omega_2 = [\beta, 1]$ joined at the point $x = \beta$. The continuity conditions (2.5) at the interface $x = \beta$ become

$$h(\beta^-) = h(\beta^+) \qquad \text{for} \qquad K_1(\beta^-)\frac{dh(x=\beta^-)}{dx} = K_2(\beta^+)\frac{dh(x=\beta^+)}{dx}. \tag{4.2}$$

The sub-domains $\Omega_1$ and $\Omega_2$ are characterized by random conductivity fields $K_1$ and $K_2$, respectively. These fields are assumed to be log-normal, statistically homogeneous, and mutually uncorrelated. The fields $Y_i = \ln K_i$ are described by their geometric means $K_{{\rm g}_i} = \exp(\langle Y_i \rangle)$, variances $\sigma^2_{Y_i}$, and correlation functions $\rho_{Y_i}(y, x)$. The contact point $\beta$ is assumed to have a truncated Gaussian distribution with mean $\langle \beta \rangle$ and variance $\sigma^2_\beta$, so that its probability density function has the form

$$p(\beta) = \frac{1}{\mathcal{W}} \exp\left[-\frac{1}{2}\left(\frac{\beta - \langle\beta\rangle}{\sigma_\beta}\right)^2\right], \tag{4.3a}$$

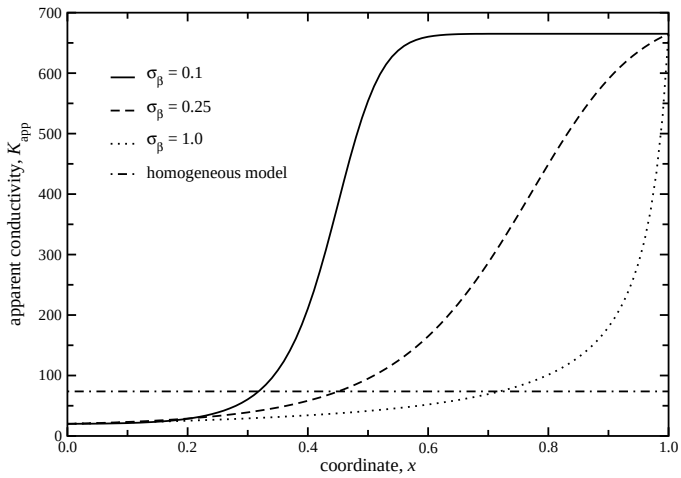


FIG. 4.2. *Apparent conductivity $K_{\rm app}$ for the one-dimensional random composite with uncertain internal geometry $\beta$ and conductivities $K_1$ and $K_2$.*

where

$$\mathcal{W}(\langle\beta\rangle, \sigma_\beta) = \int_0^1 \exp\left[-\frac{1}{2}\left(\frac{\beta - \langle\beta\rangle}{\sigma_\beta}\right)^2\right] d\beta. \tag{4.3b}$$

**4.1.1. First-order approximation.** It is easy to verify that the conditional mean Green's function $G(\beta; y, x)$ is given by

$$G(y, x \le \beta) = \begin{cases} \frac{x-y}{K_{\rm g_1}}\mathcal{H}(y-x) + \frac{\beta-x}{K_{\rm g_1}} + \frac{1-\beta}{K_{\rm g_2}}, & 0 < y \le \beta, \\ \frac{1-y}{K_{\rm g_2}}, & \beta < y < 1 \end{cases} \tag{4.4a}$$

and

$$G(y, x > \beta) = \begin{cases} \frac{1-x}{K_{\rm g_2}}, & 0 < y \le \beta, \\ \frac{x-y}{K_{\rm g_2}}\mathcal{H}(y-x) + \frac{1-x}{K_{\rm g_2}}, & \beta < y < 1. \end{cases} \tag{4.4b}$$

where $\mathcal{H}(z)$ is the Heaviside function. Substituting (4.4) into the one-dimensional version of (3.11) yields, for an arbitrary $\rho_{Y_i}(y, x)$,

$$K_{\rm app}^{[1]}(\beta; x) = K_{\rm g_1}\left(1 - \frac{\sigma_{Y_1}^2}{2}\right)\mathcal{H}(\beta - x) + K_{\rm g_2}\left(1 - \frac{\sigma_{Y_2}^2}{2}\right)\mathcal{H}(x - \beta). \tag{4.5}$$

To ascertain the accuracy of the perturbation approximation of the conditional apparent conductivity, we derive in the next section the corresponding exact expression.

**4.1.2. Exact solution.** Integrating the flow equation once and taking the conditional mean yields

$$\frac{d\langle h\rangle}{dx} = -Q\left[\frac{\mathcal{H}(\beta - x)}{K_{\rm h_1}} + \frac{\mathcal{H}(x - \beta)}{K_{\rm h_2}}\right] \tag{4.6}$$

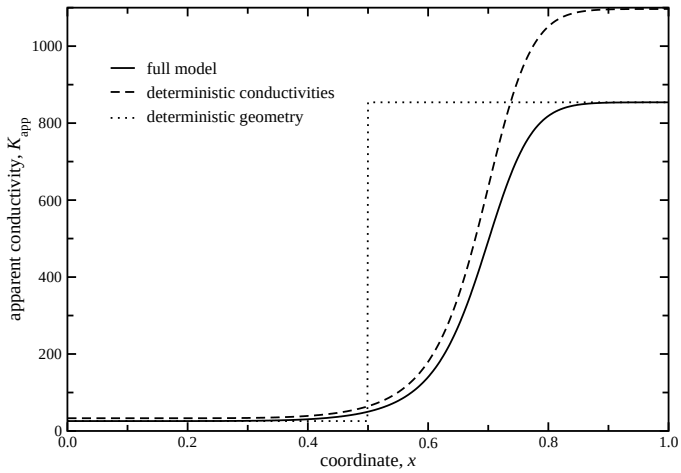


FIG. 4.3. *A relative impact of the two sources of uncertainty on apparent conductivity $K_{\rm app}$: uncertain geometry $\beta$ and uncertain conductivities $K_1$ and $K_2$.*

where $K_{{\rm h}_i} = K_{{\rm g}_i} \exp(-\sigma_{Y_i}^2/2)$ is the harmonic mean of $K_i$ ($i = 1, 2$). Hence, the conditional apparent conductivity is given by

$$K_{\rm app}^{-1}(\beta; x) = \frac{\mathcal{H}(\beta - x)}{K_{\rm h_1}} + \frac{\mathcal{H}(x - \beta)}{K_{\rm h_2}}. \tag{4.7}$$

Comparing (4.5) and (4.7), while recalling the definition of the harmonic mean, shows that (4.5) is indeed the first-order approximation of the exact expression (4.7). Since the approximation (4.5) is analogous to the expansion $\exp(-\sigma) \approx 1 - \sigma$, it remains valid as long as $\sigma_{Y_i}^2 < 2$.

It is important to contrast our expression for the (conditional) apparent conductivity with the traditional apparent conductivity that effectively homogenizes the medium. One can easily verify that the latter is given by the weighted sum of the harmonic means of $K_1$ and $K_2$,

$$K_{\rm hom}^{-1} = \frac{\beta}{K_{\rm h_1}} + \frac{1 - \beta}{K_{\rm h_2}}. \tag{4.8}$$

Of course, the traditional definition of apparent conductivity is constant in space.

The final step in obtaining the apparent conductivity is to average the conditional apparent conductivity (4.7) in the probability space of $\beta$. For $\beta$ whose probability density function is given by (4.3), the apparent conductivity takes the form

$$K_{\rm app}^{-1}(x) = \frac{{\rm erf}(u) - {\rm erf}(u_0)}{{\rm erf}(u_1) - {\rm erf}(u_0)} \left[ \frac{1}{K_{\rm h_2}} - \frac{1}{K_{\rm h_1}} \right] + \frac{1}{K_{\rm h_1}}, \tag{4.9}$$

where

$$u = \frac{x - \langle \beta \rangle}{\sqrt{2}\,\sigma_\beta}, \qquad u_0 = -\frac{\langle \beta \rangle}{\sqrt{2}\,\sigma_\beta}, \qquad u_1 = \frac{1 - \langle \beta \rangle}{\sqrt{2}\,\sigma_\beta}. \tag{4.10}$$

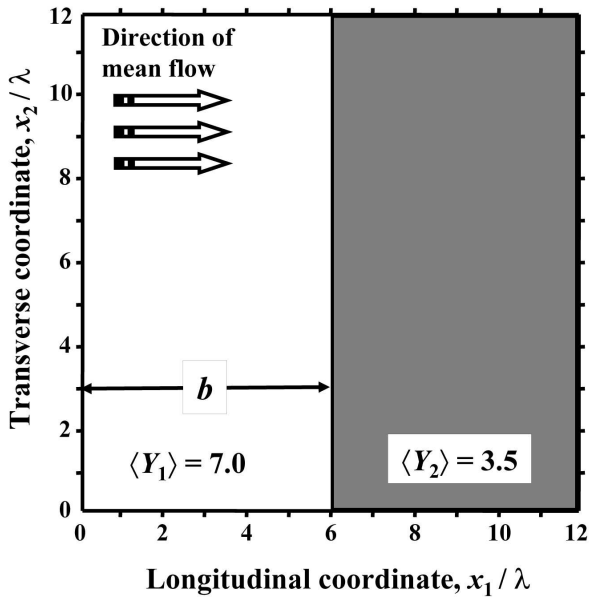


FIG. 4.4. *Two-dimensional flow domain.*

By the same token, the first-order approximation of the apparent conductivity is obtained by averaging (4.5). Relative errors between the two solutions are shown in Fig. 4.1. These errors are uniform in space and, as expected, increase exponentially with the variance of log conductivities $\sigma^2_{Y_1} = \sigma^2_{Y_2} = \sigma^2_Y$.

Figure 4.2 shows the spatial variation of the apparent conductivity $K_{\rm app}(x)$ in (4.9) for $\langle Y_1\rangle = 3.5$, $\langle Y_2\rangle = 7.0$, $\sigma^2_{Y_1} = \sigma^2_{Y_2} = 1$, $\langle\beta\rangle = 0.25$ and several values of $\sigma_\beta$. Also shown in this figure is the constant $K_{\rm app}$ corresponding to the homogeneous model (4.8). As can be seen from Fig. 4.2, and follows directly from (4.9), the apparent conductivity $K_{\rm app}(x)$ is given by the harmonic means $K_{\rm h_1}$ or $K_{\rm h_2}$, when $x$ is deep withing the sub-domains $\Omega_1$ or $\Omega_2$, respectively. The width of a transitional zone between these two harmonic means increases with uncertainty in $\beta$, i.e., with $\sigma_\beta$. If $\beta$ is known with certainty ($\sigma_\beta = 0$), $K_{\rm app}(x)$ becomes a step function, and (4.9) reduces to (4.7).

Figure 4.3 elucidates a relative impact of the two sources of uncertainty (randomness) on apparent conductivity $K_{\rm app}$. The full model (solid line) corresponds to the random $\beta$ ($\langle\beta\rangle = 0.5$ and $\sigma_\beta = 0.1$) and random log conductivities ($\langle Y_1\rangle = 3.5$, $\langle Y_2\rangle = 7.0$, and $\sigma^2_{Y_1} = \sigma^2_{Y_2} = \sigma^2_Y = 0.5$). The simplified models assume that either $Y_i$, $i = 1, 2$, (broken line), or $\beta$ (dotted line) are deterministic, i.e., that either $\sigma^2_Y = 0$ or $\sigma_\beta = 0$ respectively. One can see that, in general, both sources of uncertainty have to be accounted for in deriving expressions for apparent conductivity.

**4.2. Two-Dimensional Flow.** Consider flow in a square domain composed of two materials separated by an uncertain boundary (Figure 4.4). The materials are characterized by log conductivities $Y_i = \ln K_i$, which are treated as statistically homogeneous Gaussian random fields with means $\langle Y_1\rangle = 3.5$ and $\langle Y_2\rangle = 7.0$, variances $\sigma^2_{Y_1} = \sigma^2_{Y_2} = 1$ and two-point exponential correlation functions $\rho_{Y_i}$ of unit correlation lengths, $\lambda_{Y_1} = \lambda_{Y_2} = 1$. A random location of the internal boundary between the two materials $x_1 = \beta$ is taken to be Gaussian with mean $\langle\beta\rangle = L/2$ and variance $\sigma^2_\beta$, where $L$ is the square's size.

The Dirichlet boundary conditions are prescribed on the vertical boundaries, $h(0, y) =$

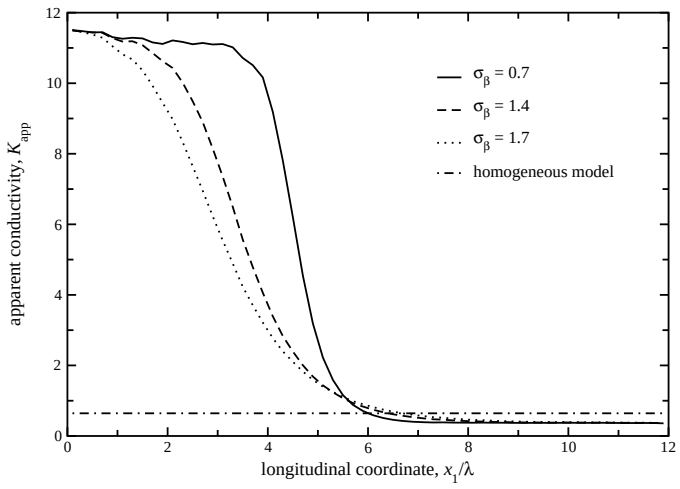


FIG. 4.5. *A horizontal cross-section $x_2 = L/2$ of the apparent conductivity $K_{\rm app}$ for the two-dimensional random composite with uncertain internal boundary $x_1 = \beta$ and conductivities $K_1$ and $K_2$.*

$H_a$ and $h(L, y) = H_b$, while the remaining two boundaries ($y = 0, L$) are assumed to be impermeable. In the reported simulations, we set $H_a = 1.6$, $H_b = 1.0$, and $L = 12$.

The apparent conductivity $K_{\rm app}$ in (3.11) is obtained by evaluating numerically (i) the conditional mean Green's functions for each realization of $\beta$, (ii) the quadratures in (3.11), and (iii) the weighted averages of the conditional apparent conductivities, whose weights are determined from the distribution of $\beta$. Figure 4.5 shows a horizontal cross-section $x_2 = L/2$ of $K_{\rm app}$ for several values of $\sigma_\beta$. The apparent conductivity of the two-dimensional composite exhibits the same general behavior as its one-dimensional counterpart.

**5. Conclusions.** We derived a general expression for the apparent conductivity of materials composed of multiple materials, whose internal geometries and conductivities are uncertain. This study leads to the following major conclusions:

1. Apparent properties of composite materials should preserve their internal structure whenever possible. This is crucial for probabilistic analyses of the critical behavior of physical systems, such as the existence of preferential flow paths in natural porous media.
2. For steady-state flow in bounded heterogeneous composite media, we used perturbation expansions in the variances of log conductivities to derive a general expression for the apparent conductivity. Since the conductivity of each material is more uniform than that of a composite as a whole, this expression is more accurate and robust than its homogeneous counterpart.
3. The general perturbation expression for apparent conductivity is analyzed in detail for one- and two-dimensional steady flow in the bounded porous medium composed of two materials. Both log conductivities and the internal boundaries between materials are assumed to be Gaussian. Away from the internal boundaries, the apparent conductivity is given by the harmonic means of the corresponding conductivities of each material. Within a transitional zone around the boundary, the apparent conductivity varies smoothly between these harmonic means. The width of the transitional

zone increases with the degree of uncertainty about the internal boundary.

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