SIAM J. APPL. MATH.
Vol. 61, No. 6, pp. 1857–1876

DYNAMICS OF FREE SURFACES IN RANDOM POROUS MEDIA*

DANIEL M. TARTAKOVSKY† AND C. L. WINTER‡

Abstract. We consider free surface flow in random porous media by treating hydraulic conductivity of a medium as a random field with known statistics. We start by recasting the boundary-value problem in the form of an integral equation where the parameters and domain of integration are random. Our analysis of this equation consists of expanding the random integrals in Taylor's series about the mean position of the free boundary and taking the ensemble mean. To quantify the uncertainty associated with such predictions, we also develop a set of integro-differential equations satisfied by the corresponding second ensemble moments. The resulting moment equations require closure approximations to be workable. We derive such closures by means of perturbation expansions in powers of the variance of the logarithm of hydraulic conductivity. Though this formally limits our solutions to mildly heterogeneous porous media, our analytical solutions for one-dimensional flows demonstrate that such perturbation expansions may remain robust for relatively large values of the variance of the logarithm of hydraulic conductivity.

Key words. porous media, moving boundary, random, stochastic, moment equations

AMS subject classifications. 60H30, 60H15, 60G25

PII. S0036139999358180

1. Introduction.

Moving boundary problems arise in applications ranging from the highly utilitarian, such as the wetting and drying of porous media [25], response of water tables to groundwater pumping [8], oil recovery [2], heat conduction [9], diffusion-limited aggregation [21], crystal growth and semiconductor fabrication [10], to the felicitous, such as snowflake formation [15]. For purposes of analysis, the system parameters in these problems are usually assumed to be well defined and known with certainty. In most applications, however, system parameters cannot be known precisely in all of their relevant details. This leads to uncertainty in the prediction of front dynamics and a resulting need to quantify that uncertainty. We describe methods for estimating front statistics, such as a front's mean dynamics and variance, directly from moment integro-differential equations. This approach avoids the need for large numbers of Monte Carlo simulations and allows the use of relatively coarse computational grids in numerical solutions.

We will discuss uncertain front dynamics in the context of flow in geological porous media to be concrete. Predicting flow through natural porous media is complicated by their high degree of spatial variability and the lack of detailed characterization of their hydraulic properties. In practice, parameters like hydraulic conductivity, $K(\mathbf{x})$, can at best be measured at selected locations and depth intervals where their values depend on the scale (support volume) and mode (instrumentation and procedure)

*Received by the editors July 1, 1999; accepted for publication (in revised form) November 16, 2000; published electronically April 3, 2001. This work was performed under the auspices of the U.S. Department of Energy (DOE) under the DOE/BES (Bureau of Energy Sciences) Program in the Applied Mathematical Sciences contract KC-07-01-01, Los Alamos National Laboratory under *LDRD* 98604, and the Initiatives for Proliferation Prevention Program (IPP).

http://www.siam.org/journals/siap/61-6/35818.html

†Theoretical Division (Group T-7), MS B284, Los Alamos National Laboratory, Los Alamos, NM 87545 (dmt@lanl.gov).

‡Theoretical Division (Group T-7) and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545 (winter@lanl.gov). This work made use of shared facilities supported by SAHRA (Sustainability of semi-Arid Hydrology and Riparian Areas) under the STC Program of the National Science Foundation under agreement EAR-9876800.

of measurement. Estimating the parameters at points where measurements are not available entails a random error.

Consider the motion of a fluid-fluid interface in a randomly heterogeneous porous medium Ω_T (bounded by the surface Γ_T) when gravity, capillary length, and the viscosity of one fluid are zero. In the inviscid fluid (air), the pressure is constant and may be set to zero. The viscous, incompressible fluid (water), occupies the flow domain Ω ($\Omega \subset \Omega_T$) which is bounded either entirely by a free surface γ or by a combination of γ and some segments of Γ_T. Such flow is described by a combination of Darcy's law and mass conservation,

$$\mathbf{q}(\mathbf{x},t) = -K(\mathbf{x})\nabla h(\mathbf{x},t), \qquad -\nabla\cdot\mathbf{q}(\mathbf{x},t) + f(\mathbf{x},t) = 0, \qquad \mathbf{x}\in\Omega(t), \tag{1}$$

subject to boundary conditions

$$h(\mathbf{x},t) = H(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_D, \tag{2a}$$

$$\mathbf{n}(\mathbf{x})\cdot\mathbf{q}(\mathbf{x},t) = Q(\mathbf{x},t), \qquad \mathbf{x}\in\Gamma_N, \tag{2b}$$

$$h(\mathbf{x},t) = 0, \qquad \mathbf{n}(\mathbf{x},t)\cdot\mathbf{q}(\mathbf{x},t) = V_n(\mathbf{x},t), \qquad \mathbf{x}\in\gamma(t). \tag{2c}$$

The flux, $\mathbf{q}$ $[LT^{-1}]$, flows down gradients of hydraulic head, h $[L]$, subject to constraints imposed by the hydraulic conductivity, K $[LT^{-1}]$. In principle K is a second-order tensor, but here we assume that it is a scalar function, $K = K(\mathbf{x})$, for simplicity. The randomness of the porous medium is captured by representing K as a spatially random field with mean, $\overline{K}$, variance, σ_K^2, and a correlation function, $\rho_K(\mathbf{x},\mathbf{y})$. Other sources of randomness are the source function, $f(\mathbf{x},t)$ $[T^{-1}]$, and the boundary conditions. The boundary $\Gamma = \Gamma_D\cup\Gamma_N\cup\gamma$ consists of Dirichlet segments, Γ_D, Neumann segments, Γ_N, and a moving front, γ, that is itself a dependent random process. The prescribed hydraulic head, $H(\mathbf{x},t)$ $[L]$, on Γ_D and the prescribed flux, $Q(\mathbf{x},t)$ $[LT^{-1}]$, on Γ_N are both random functions. We assume the driving forces f, H, and Q to be statistically independent for simplicity. At the free surface γ, h equals atmospheric pressure which we set equal to zero without loss of generality; $V_n(\mathbf{x},t)$ $[LT^{-1}]$ is the velocity of the moving boundary γ in the normal direction. Mass conservation requires that

$$V_n = n_e\frac{d\gamma}{dt}, \tag{3}$$

where n_e is the medium porosity. In what follows, we assume that the statistics of the random fields $K(\mathbf{x})$, $f(\mathbf{x},t)$, $H(\mathbf{x},t)$, and $Q(\mathbf{x},t)$ can be obtained from experimental data sets. We do not require these fields to be statistically homogeneous. Our aim is to derive a set of equations satisfied by the unbiased predictors $\overline{h}(\mathbf{x},t)$, $\overline{\mathbf{q}}(\mathbf{x},t)$, and $\overline{V}_n(\mathbf{x},t)$ and to estimate errors associated with these predictors.

Such a description of moving fronts implicitly ignores the presence of a transitional zone wherein the fluid saturation varies gradually from 0 to 1. This sharp interface approximation is known as the Green and Ampt model [13] and is widely used in subsurface hydrology (see, e.g., [12, 3, 8] and the references therein). Also, this is the model used in the seminal front stability analyses by Chuoke, van Meurs, and van der Poel [5] and Saffman and Taylor [25]. The absence of gravity corresponds to a horizontal displacement. Such a flow scenario was used by, among others, Green and Ampt [13], Raats [23], and Philip [22] to study wetting fronts in deterministic porous media. Descriptions of diffusive front dynamics can be found in Yortsos and Hickernell [32] and Fennemore and Xin [11].

Despite notable progress in analyzing groundwater flows in randomly heterogeneous domains with fixed boundaries, there are virtually no studies of flow with free surfaces. Stochastic averaging of flow equations has typically dealt either with flows in infinite domains (e.g., [4, 7, 14, 27, 31]) or with flows in domains bounded by Dirichlet or Neumann boundaries (e.g., [7, 20, 28]). Additionally, deriving effective hydraulic parameters for such flows has attracted considerable attention (e.g., [16, 18, 19, 29]).

While of considerable interest for many practical applications (such as wetting of heterogeneous porous media, or response of unconfined heterogeneous aquifers to pumping), free surface problems in random media have resisted general attempts to solve them. Dagan and Zeitoun [8] have analyzed the response of water tables (free surfaces that bound unconfined aquifers from above) to pumping by employing a restrictive Dupuit assumption and reducing heterogeneity to a perfect layering. A numerical study of water tables in a heterogeneous dam has been reported in [12]. Modeling wetting in heterogeneous porous media is further complicated by instability of the wetting fronts (free surfaces). Wetting front instability and development of fingers have been observed by, among others, Baker and Hillel [1] during their infiltration experiments in layered soils. Numerical simulations of wetting front instabilities due to pore-scale heterogeneities have been carried out by Lenormand, Touboul, and Zarcone [17]. A probabilistic criterion for the onset of wetting front instability in randomly stratified porous media has been derived by Chen and Neuman [3].

The first part of this paper is devoted to deriving a boundary-value problem which describes mean dynamics of free surfaces in random conductivity fields with known statistics. We do not require this field to be homogeneous. We also derive a set of equations for the second ensemble moments to assess the error associated with averaging. A closure for our ensemble moment equations is provided by means of a perturbation analysis. The second part of this paper is devoted to obtaining analytical solutions of the general moment equations for one-dimensional (1-D) front propagation.

2. Statement of the problem. Obviously, the boundary-value problem (1)–(2) is highly nonlinear due to the presence of the moving boundary, γ. Moreover, since the dynamics of γ depend on the random parameter K, its exact position at any given time is uncertain. Hence its normal vector $\mathbf{n}$ is also random. This complicates direct stochastic averaging of (1)–(2). To address this problem, we represent the random field K (and other random fields) through a Reynolds decomposition as a sum of its mean $\overline{K}$ and a zero-mean perturbation K', $K = \overline{K} + K'$ ($\overline{K'} \equiv 0$). This recasts (1)–(2) in the form of an integral equation,

$$
\begin{aligned}
&-\int_\Omega \nabla_{\mathbf{y}} \cdot \left[\overline{K}(\mathbf{y})\, \nabla_{\mathbf{y}}\, G(\mathbf{y},\mathbf{x})\right]\, h(\mathbf{y})\, d\mathbf{y} \;=\; \int_\Omega \nabla_{\mathbf{y}} \cdot \left[K'(\mathbf{y})\, \nabla_{\mathbf{y}}\, h(\mathbf{y})\right]\, G(\mathbf{y},\mathbf{x})\, d\mathbf{y} \\
&\quad + \int_\Gamma \overline{K}(\mathbf{y})\mathbf{n} \cdot \left[G(\mathbf{y},\mathbf{x})\nabla_{\mathbf{y}} h(\mathbf{y}) \,-\, h(\mathbf{y})\nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x})\right] d\mathbf{y} \;+\; \int_\Omega f(\mathbf{y})\, G(\mathbf{y},\mathbf{x})\, d\mathbf{y},
\end{aligned} \tag{4}
$$

where $G(\mathbf{y},\mathbf{x})$ is a deterministic Green's function satisfying

$$\nabla_{\mathbf{y}} \cdot \left[\overline{K}(\mathbf{y})\nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x})\right] \;+\; \delta(\mathbf{y}-\mathbf{x}) \;=\; 0, \qquad \mathbf{y},\mathbf{x} \,\in\, \Omega_T, \tag{5}$$

subject to the boundary conditions

$$G(\mathbf{y},\mathbf{x}) \;=\; 0, \qquad \mathbf{y} \,\in\, \Gamma_D, \tag{6a}$$

$$\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x}) = 0, \qquad \mathbf{y} \in \Gamma_N. \tag{6b}$$

Note that our Green's function $G(\mathbf{y},\mathbf{x})$ is defined for the entire domain Ω_T rather than just for the flow domain Ω, so that there are no conditions on G along the moving boundary γ. Specifying G for just the flow domain, Ω, would require recalculating G at each time as Ω evolves, which is not computationally efficient.

Applying Green's formula to the first domain integral in (4), while taking into account boundary conditions (2a)–(2c), yields

$$\begin{aligned} h(\mathbf{x},t) = &- \int_\Omega K'(\mathbf{y}) \nabla_{\mathbf{y}} h(\mathbf{y},t) \cdot \nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x}) d\mathbf{y} + \int_\Omega f(\mathbf{y},t) G(\mathbf{y},\mathbf{x}) d\mathbf{y} \\ &- \int_{\Gamma_N} Q(\mathbf{y},t)\, G(\mathbf{y},\mathbf{x}) d\mathbf{y} - \int_{\Gamma_D} H(\mathbf{y},t) \overline{K}(\mathbf{y}) \mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x}) d\mathbf{y} \\ &- \int_\gamma V_n(\mathbf{y},t) G(\mathbf{y},\mathbf{x}) d\mathbf{y}. \end{aligned} \tag{7}$$

This integral equation serves as the starting point for our stochastic analysis. The first two and last terms in this equation involve integration of random functions over the random domain, Ω, and surface, γ, respectively. Since the source function, $f(\mathbf{x},t)$, is defined on a compact support situated well within flow domain Ω, we have at all times

$$\int_\Omega f(\mathbf{y}) d\mathbf{y} \equiv \int_{\overline{\Omega}} f(\mathbf{y}) d\mathbf{y}. \tag{8}$$

In the remaining two integrals, we represent the random geometry as sums of ensemble means and zero-mean fluctuations, $\Omega = \overline{\Omega} + \Omega'$ and $\gamma = \overline{\gamma} + \gamma'$, and expand the integrals in question in Taylor series around the mean geometry $\overline{\Omega}$ and $\overline{\gamma}$. For example, the integral over γ now becomes

$$\int_\gamma V_n(\mathbf{y}) G(\mathbf{y},\mathbf{x}) d\mathbf{y} = \int_{\overline{\gamma}} V_n(\mathbf{y}) G(\mathbf{y},\mathbf{x}) d\mathbf{y} + \gamma' V_n(\overline{\gamma}) G(\overline{\gamma},\mathbf{x}) + \cdots . \tag{9}$$

The linearized solution can be obtained by retaining only the leading term in (9) and the corresponding expansion of the first integral in (7). We demonstrate in the following sections that the linearized solution can cause systematic errors in predicting mean position of the front and its variances. At the same time, a similar linearization of the nonlinear flow equation for partially saturated porous media was shown to give satisfactory results [33]. We therefore proceed by deriving linearized solutions for the first two ensemble moments of the front (its mean and variance). Then we examine the errors introduced by such a linearization.

3. Linearized moments equations. Taking the ensemble mean of the linearized version of (7) gives

$$\begin{aligned} \overline{h}(\mathbf{x},t) \;=\; &\int_{\overline{\Omega}} \overline{\mathbf{r}}(\mathbf{y},t) \cdot \nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x})\, d\mathbf{y} \;+\; \int_{\overline{\Omega}} \overline{f}(\mathbf{y},t)\, G(\mathbf{y},\mathbf{x})\, d\mathbf{y} \\ &- \int_{\Gamma_N} \overline{Q}(\mathbf{y},t)\, G(\mathbf{y},\mathbf{x})\, d\mathbf{y} \;-\; \int_{\Gamma_D} \overline{H}(\mathbf{y},t)\, \overline{K}(\mathbf{y})\, \mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{y},\mathbf{x})\, d\mathbf{y} \\ &- \int_{\overline{\gamma}} \overline{V}_n(\mathbf{y},t)\, G(\mathbf{y},\mathbf{x})\, d\mathbf{y}, \end{aligned} \tag{10}$$

where $\overline{f}(\mathbf{x},t)$, $\overline{H}(\mathbf{x},t)$, and $\overline{Q}(\mathbf{x},t)$ are prescribed ensemble means of the random forcing (source and boundary) functions $f(\mathbf{x},t)$, $H(\mathbf{x},t)$, and $Q(\mathbf{x},t)$, respectively; and $\overline{\mathbf{r}}(\mathbf{x},t) = -\overline{K'(\mathbf{x})\nabla h'(\mathbf{x},t)}$ is the mean "residual" flux. For flow through domains bounded by Dirichlet and Neumann boundaries only, implicit equations for the residual flux were derived in [20, 28]. These authors relied on random Green's functions, which, of course, are harder to evaluate than our deterministic G.

For statistically independent forcing functions, operating on the linearized version of (7) with the stochastic differential operator $K'(\mathbf{x})\nabla_{\mathbf{x}}$ gives the residual flux $\overline{\mathbf{r}}(\mathbf{x},t)$,

$$\begin{aligned}\overline{\mathbf{r}}(\mathbf{x},t) &= \int_{\overline{\Omega}} C_K(\mathbf{x},\mathbf{y})\,\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}^T G(\mathbf{y},\mathbf{x})\,\nabla_{\mathbf{y}}\overline{h}(\mathbf{y},t)\,d\mathbf{y}\\ &+ \int_{\overline{\Omega}} \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}^T G(\mathbf{y},\mathbf{x})\,\overline{K'(\mathbf{x})K'(\mathbf{y})\nabla_{\mathbf{y}}h'(\mathbf{y},t)}\,d\mathbf{y}\\ &+ \int_{\overline{\gamma}} C_{KV}(\mathbf{x};\mathbf{y},t)\,\nabla_{\mathbf{x}}G(\mathbf{y},\mathbf{x})\,d\mathbf{y}\,.\end{aligned}\tag{11}$$

Here $C_K(\mathbf{x},\mathbf{y}) = \overline{K'(\mathbf{x})K'(\mathbf{y})}$ is the covariance of K. The linearized integral equation for cross-covariance $C_{KV}(\mathbf{x};\mathbf{y},t) = \overline{K'(\mathbf{x})V_n'(\mathbf{y})}$ is obtained by evaluating the linearized version of (7) on the front, linearizing the resulting expression around $\overline{\gamma}$, multiplying by $K'(\mathbf{x})$, and taking the mean,

$$\begin{aligned}\int_{\overline{\gamma}} C_{KV}(\mathbf{x};\mathbf{y},t)\,G(\mathbf{y},\overline{\gamma})\,d\mathbf{y} &= -\int_{\overline{\Omega}} C_K(\mathbf{x},\mathbf{y})\,\nabla_{\mathbf{y}}G(\mathbf{y},\overline{\gamma})\cdot\nabla_{\mathbf{y}}\overline{h}(\mathbf{y},t)\,d\mathbf{y}\\ &- \int_{\overline{\Omega}} \nabla_{\mathbf{y}}G(\mathbf{y},\overline{\gamma})\cdot\overline{K'(\mathbf{x})K'(\mathbf{y})\nabla_{\mathbf{y}}h'(\mathbf{y},t)}\,d\mathbf{y}\,.\end{aligned}\tag{12}$$

To evaluate (11) and (12) one needs a closure approximation for dealing with the third mixed moments. We derive such a closure below by means of a perturbation analysis in a small parameter σ_Y^2, the variance of log hydraulic conductivity $Y = \ln K$.

Before proceeding any further we notice that (11) reveals that the residual flux $\overline{\mathbf{r}}(\mathbf{x},t)$ is nonlocal (i.e., depends on more than one point in space). Therefore the mean flux $\overline{\mathbf{q}}(\mathbf{x},t)$ is likewise nonlocal (depends on averaged head gradients at points other than $\mathbf{x}$), and thus an effective hydraulic conductivity tensor $\mathbf{K}_{eff}$ does not generally exist. This finding is in line with previous investigations of flow in bounded and unbounded randomly heterogeneous domains [6, 7, 14, 20, 28].

3.1. Closure by perturbation expansion. In (10), expanding $K(\mathbf{x})$, $h(\mathbf{x},t)$, $\mathbf{q}(\mathbf{x},t)$, $V_n(\mathbf{x},t)$, and $G(\mathbf{y},\mathbf{x})$ in powers of $Y'(\mathbf{x})$ and collecting terms of the same powers of σ_Y^2 yields ith-order sequential approximations of the mean head (here we consider only the first two terms in this expansion, $i = 0, 1$),

$$\begin{aligned}\overline{h}^{(0)}(\mathbf{x},t) &= \int_{\overline{\Omega}} \overline{f}(\mathbf{y},t)\,G^{(0)}(\mathbf{y},\mathbf{x})\,d\mathbf{y} - \int_{\Gamma_N} \overline{Q}(\mathbf{y},t)\,G^{(0)}(\mathbf{y},\mathbf{x})\,d\mathbf{y}\\ &- \int_{\Gamma_D} \overline{H}(\mathbf{y},t)\,K_g(\mathbf{y})\,\mathbf{n}\cdot\nabla_{\mathbf{y}}G^{(0)}(\mathbf{y},\mathbf{x})\,d\mathbf{y} - \int_{\overline{\gamma}^{(0)}} \overline{V}_n^{(0)}(\mathbf{y},t)\,G^{(0)}(\mathbf{y},\mathbf{x})\,d\mathbf{y}\end{aligned}\tag{13a}$$

and

$$\begin{aligned}\overline{h}^{(1)}(\mathbf{x},t) =& \int_{\overline{\Omega}^{(0)}} \overline{\mathbf{r}}^{(1)}(\mathbf{y},t)\cdot\nabla_{\mathbf{y}}G^{(0)}(\mathbf{y},\mathbf{x})d\mathbf{y} + \int_{\overline{\Omega}} \overline{f}(\mathbf{y},t)G^{(1)}(\mathbf{y},\mathbf{x})d\mathbf{y}\\ &- \int_{\Gamma_N} \overline{Q}(\mathbf{y},t)G^{(1)}(\mathbf{y},\mathbf{x})d\mathbf{y}\\ &- \sum_{j=0}^{1}\int_{\Gamma_D} \overline{H}(\mathbf{y},t)K_g(\mathbf{y})\left[\frac{\sigma_Y^2(\mathbf{y})}{2}\right]^j \mathbf{n}\cdot\nabla_{\mathbf{y}}G^{(1-j)}(\mathbf{y},\mathbf{x})d\mathbf{y}\\ &- \sum_{j=0}^{1}\int_{\overline{\gamma}^{(j)}} \overline{V}_n^{(j)}(\mathbf{y},t)G^{(i-j)}(\mathbf{y},\mathbf{x})d\mathbf{y},\end{aligned} \tag{13b}$$

where $K_g = \exp\left(\overline{Y}\right)$ is the geometric mean of K, and the first-order approximations of $\overline{\mathbf{r}}(\mathbf{x},t)$ and $C_{KV}(\mathbf{x};\mathbf{y},t)$ are given by

$$\begin{aligned}\overline{\mathbf{r}}^{(1)}(\mathbf{x},t) =& K_g(\mathbf{x})\int_{\overline{\Omega}^{(0)}} K_g(\mathbf{y})\,C_Y(\mathbf{x},\mathbf{y})\,\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}^T G^{(0)}(\mathbf{y},\mathbf{x})\,\nabla_{\mathbf{y}}\,\overline{h}^{(0)}(\mathbf{y},t)\,d\mathbf{y}\\ &+\int_{\overline{\gamma}^{(0)}} C_{KV}^{(1)}(\mathbf{x};\mathbf{y},t)\,\nabla_{\mathbf{x}}G^{(0)}(\mathbf{y},\mathbf{x})d\mathbf{y}\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\int_{\overline{\gamma}^{(0)}} C_{KV}^{(1)}(\mathbf{x};\mathbf{y},t)G^{(0)}(\mathbf{y},\overline{\gamma}^{(0)})d\mathbf{y}\\ &= -K_g(\mathbf{x})\int_{\overline{\Omega}^{(0)}} K_g(\mathbf{y})C_Y(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}G^{(0)}(\mathbf{y},\overline{\gamma}^{(0)})\cdot\nabla_{\mathbf{y}}\overline{h}^{(0)}(\mathbf{y},t)d\mathbf{y},\end{aligned} \tag{15}$$

respectively, $C_Y(\mathbf{x},\mathbf{y}) = \overline{Y'(\mathbf{x})Y'(\mathbf{y})}$ being the covariance of Y. The zeroth- and first-order approximations of the mean front velocity $\overline{V}_n^{(i)}$ $(i = 1,2)$ is obtained by evaluating (13a) and (13b) at $\mathbf{x}\in\gamma$ and linearizing the resulting expressions around $\overline{\gamma}$. Once the linearized approximations $\overline{V}_n^{(i)}$ are found, the mean position of the front follows from (3),

$$n_e\frac{d\overline{\gamma}^{(i)}}{dt} = \overline{V}_n^{(i)}, \qquad i = 1,2. \tag{16}$$

For weakly homogeneous random fields K, evaluation of $G^{(i)}(\mathbf{y},\mathbf{x})$ up to any order is trivial once the zeroth-order approximation, $G^{(0)}(\mathbf{y},\mathbf{x})$, is found. Indeed, for $\overline{K}(\mathbf{x}) = \text{const}$ it is enough to solve (5)–(6b) for $g(\mathbf{y},\mathbf{x}) = \overline{K}G(\mathbf{y},\mathbf{x})$. When K is log-normal, $\overline{K} = K_g\exp(\sigma_Y^2/2)$ and $G_K^{(j)} \equiv K_gG^{(j)} = (-1)^j/j!(\sigma_Y^2/2)^j g(\mathbf{y},\mathbf{x})$ for all $j\geq 0$.

The zeroth-order approximation of the mean hydraulic head, $\overline{h}^{(0)}(\mathbf{x},t)$, satisfies a standard boundary-value problem with moving boundaries for a medium with known properties, driven by mean source and boundary functions. Nonlocality of the averaged boundary-value problem manifests itself solely in first-order (and higher) terms.

The above systems of deterministic moment equations involve relatively smooth parameters and dependent variables. As such, these moment equations can be solved either analytically, as we do below, or, more generally, by standard numerical methods, such as those proposed in [15, 26]. These authors used the Laplace equation with free boundaries to describe snowflake growth and diffusion-limited aggregation, respectively.

The error introduced by stochastic averaging of (7) can be estimated through the variance of hydraulic head $\sigma_h^2(\mathbf{x},t) = \overline{h'(\mathbf{x},t)h'(\mathbf{x},t)}$. For simplicity, we do so for the case where all forcing terms, f, H, and Q, are deterministic. This limitation can be easily overcome [30]. The first-order linearized approximation of the hydraulic head covariance, $C_h(\mathbf{x},t;\mathbf{y},t) = \overline{h'(\mathbf{x},t)h'(\mathbf{y},t)}$, is obtained by multiplying the linearized version of (7) with $h'(\mathbf{x},t)$, taking the ensemble mean, and retaining the terms of σ_Y^2-order,

$$C_h^{(1)}(\mathbf{x},t;\mathbf{y},t) = -\int_{\overline{\Omega}^{(0)}} C_{Kh}^{(1)}(\mathbf{z};\mathbf{y},t)\nabla_{\mathbf{z}}\overline{h}^{(0)}(\mathbf{z},t)\cdot\nabla_{\mathbf{z}}G^{(0)}(\mathbf{z},\mathbf{x})d\mathbf{z} - \int_{\overline{\gamma}^{(0)}} C_{hV}^{(1)}(\mathbf{y},t;\mathbf{z},t)G^{(0)}(\mathbf{z},\mathbf{x})d\mathbf{z}, \tag{17}$$

where the first-order linearized approximations of cross-covariances $C_{Kh}(\mathbf{x};\mathbf{y},t) = \overline{K'(\mathbf{x})h'(\mathbf{y},t)}$ and $C_{hV}(\mathbf{x},t;\mathbf{y},t) = \overline{h'(\mathbf{x},t)V_n'(\mathbf{y},t)}$ are

$$C_{Kh}^{(1)}(\mathbf{x};\mathbf{y},t) = -K_g(\mathbf{x})\int_{\overline{\Omega}^{(0)}} K_g(\mathbf{z})C_Y(\mathbf{x},\mathbf{z})\nabla\overline{h}^{(0)}(\mathbf{z},t)\cdot\nabla_{\mathbf{z}}G^{(0)}(\mathbf{z},\mathbf{y})d\mathbf{z} - \int_{\overline{\gamma}^{(0)}} C_{KV}^{(1)}(\mathbf{x};\mathbf{z},t)G^{(0)}(\mathbf{z},\mathbf{y})d\mathbf{z} \tag{18}$$

and

$$C_{hV}^{(1)}(\mathbf{x},t;\mathbf{y},t) = -\int_{\overline{\Omega}^{(0)}} C_{KV}^{(1)}(\mathbf{z};\mathbf{y},t)\nabla_{\mathbf{z}}\overline{h}^{(0)}(\mathbf{z},t)\cdot\nabla_{\mathbf{z}}G^{(0)}(\mathbf{z},\mathbf{x})\,d\mathbf{z} - \int_{\overline{\gamma}^{(0)}} C_V^{(1)}(\mathbf{y},\mathbf{z},t)G^{(0)}(\mathbf{z},\mathbf{x})d\mathbf{z}. \tag{19}$$

The front velocity covariance, C_V, is obtained from (19) by setting $\mathbf{x}\in\gamma$, noting that the first boundary condition in (2c) implies that $C_{hV}(\gamma,\mathbf{y})\equiv 0$, and linearizing around $\overline{\gamma}$. A similar procedure applied to the linearized version of (7) leads to the integral equation for the first-order approximation of the front variance, σ_γ^2,

$$\frac{n_e}{2}\int_{\overline{\gamma}^{(0)}} \frac{d\left[\sigma_\gamma^2\right]^{(1)}}{dt}G^{(0)}(\mathbf{y},\overline{\gamma}^{(0)})d\mathbf{y} = -\int_{\overline{\Omega}^{(0)}} C_{K\gamma}^{(1)}(\mathbf{z};\mathbf{y},t)\nabla_{\mathbf{z}}\overline{h}^{(0)}(\mathbf{z},t)\cdot\nabla_{\mathbf{z}}G^{(0)}(\mathbf{z},\overline{\gamma}^{(0)})d\mathbf{z}. \tag{20}$$

It follows from (3) that cross-covariances $C_{K\gamma}$ and C_{KV} are related by

$$C_{KV} = n_e\frac{dC_{K\gamma}}{dt}. \tag{21}$$

Once the first-order approximation of the head covariance C_h in (17) is found, one can evaluate the first-order approximation of the head variance $\sigma_h^2(\mathbf{x},t)$ by taking the limit $\mathbf{y}\to\mathbf{x}$.

At this stage, it becomes obvious that the linearization of (7) introduces errors into the first-order expressions (13b)–(15). This is so, since both the second term in the Taylor expansion of the first integral in (7) and the second and third terms in the Taylor expansion of the last integral in (7) contribute to the proper first-order approximations $\overline{h}^{(1)}$, $\overline{\mathbf{r}}^{(1)}$, and $C_{KV}^{(1)}$. These errors are investigated in detail in the next section.

4. Dynamics of 1-D fronts. The remainder of this paper is devoted to the development and exploration of approximate solutions for the averaged boundary-value problem with free surfaces in one dimension. In particular, we employ perturbation analysis to obtain an analytical solution for 1-D front propagation in a porous medium column of length L. To emphasize the 1-D nature of the problem we use x_f to denote the front, instead of γ, which was employed in our general analysis. We consider two scenarios: (i) constant deterministically prescribed flux Q at the boundary $x = 0$, and (ii) constant deterministically prescribed hydraulic head H at the same boundary. In both cases, zero hydraulic head is maintained at the boundary $x = L$. Extensions to random Q and H are straightforward. It is assumed that log-hydraulic conductivity $Y(x)$ of the porous medium is a statistically homogeneous (stationary) multivariate Gaussian and random field with constant mean $\overline{Y}$ and an exponential covariance function

$$C_Y(\mid x-y \mid) \;=\; \sigma_Y^2 \, \exp\left(-\frac{\mid x-y \mid}{l_Y} \right), \tag{22}$$

where l_Y is the spatial autocorrelation scale of Y.

For the domain under consideration, the deterministic auxiliary function $G_K^{(0)}(x,y) = K_g G^{(0)}(x,y)$ satisfies, on the open interval $(0,L)$, the equation

$$\frac{d^2\, G_K^{(0)}(x,y)}{d\,x^2} \;+\; \delta(x-y) \;=\; 0, \qquad x\,,y \;\in\; (0,L), \tag{23}$$

and is given by [24]

$$G_K^{(0)}(x,y) \;=\; -\,(x-y)\,\mathcal{H}(x-y) \;+\; \alpha(y)\,x \;+\; \beta(y). \tag{24}$$

Here $\mathcal{H}(a) = 1$ when $a \geq 0$ (and $= 0$ otherwise) is the Heaviside function, and $\alpha(y)$ and $\beta(y)$ are arbitrary functions to be determined from the corresponding boundary conditions. Also,

$$G_K^{(1)}(x,y) \;=\; -\,\frac{\sigma_Y^2}{2}\, G_K^{(0)}(x,y). \tag{25}$$

4.1. Flux-driven front propagation. Consider a front driven by a constant flux Q at the boundary $x = 0$, i.e., $K dh(0)/dx = -Q$. We also set $h(L) = 0$. Then

$$G_K^{(0)}(x,y) \;=\; -\,(x-y)\,\mathcal{H}(x-y) \;+\; L \;-\; y, \tag{26}$$

and evaluating the 1-D version of (13b) with $i = 0$ at the front, $x = x_f$, leads to

$$\overline{V}^{(0)} \;=\; Q. \tag{27}$$

Substituting (26) into the 1-D version of (14) and (15) yields, after some algebraic manipulations,

$$\overline{r}^{(1)}(x) \equiv -Q\sigma_Y^2. \tag{28}$$

It then follows from the 1-D version of (13b) with $i = 1$ that $\overline{V}_n^{(1)} \equiv 0$. By the same token, substituting (26) into the 1-D versions of (18) with $x = x_f$ and (19)–(20) leads to $C_{KV}^{(1)}(x,x_f) = C_{hV}^{(1)}(x,x_f) = \sigma_V^2(x_f) \equiv 0$. These results can be obtained directly from mass conservation arguments. Indeed, prescribing constant flux Q at the boundary requires that, with probability 1, the front propagates with the same deterministic velocity $V = \overline{V} = Q$. While trivial by itself, this correspondence indicates that our averaged boundary-value problem is free of internal contradictions. Since deterministic V implies deterministic dynamics of the moving front $x_f = \overline{x}_f$, the linearization of the governing equation about $\overline{x}_f$ does not introduce any additional errors.

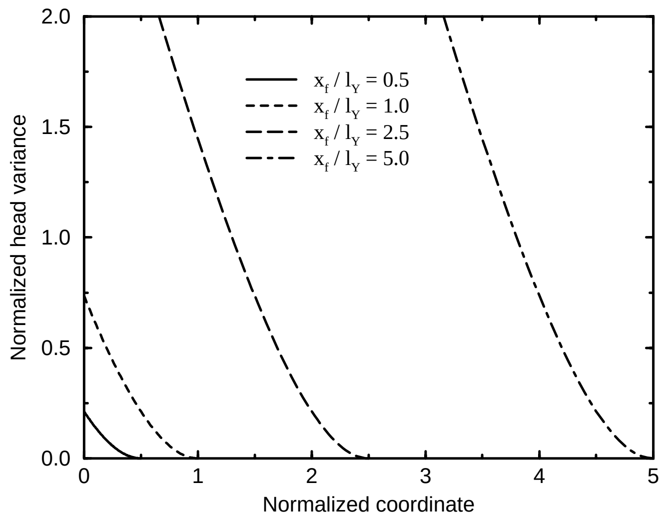

FIG. 1. *Normalized head variance versus normalized coordinate for different numbers of correlation scales passed by the front.*

While the front moves through a random porous medium with deterministic velocity, the hydraulic head associated with this front remains random. Its predictor, obtained by substituting (26) into the 1-D version of (13b), is given by

$$\overline{h}^{[1]}(x) = \overline{h}^{(0)}(x) + \overline{h}^{(1)}(x) = \left(1 + \frac{\sigma_Y^2}{2}\right) \frac{Q}{K_g} \left(\overline{x}_f^{(0)} - x\right), \tag{29}$$

where the bracket superscript $^{[1]}$ indicates an approximation through order 1. The quality of the hydraulic head prediction can be estimated by, among other means, the head variance $\sigma_h^2(x)$. Its first-order approximation is obtained by substituting (26) into (18) and (17) and taking the limit $y \to x$,

$$\frac{K_g^2}{Q^2\, l_Y^2} \left[\sigma_h^2(x,t)\right]^{(1)} = 2\sigma_Y^2 \left[-1 + \frac{\overline{x}_f^{(0)} - x}{l_Y} + \exp\left(\frac{x - \overline{x}_f^{(0)}}{l_Y}\right)\right]. \tag{30}$$

As expected, $\sigma_h^2(x_f; x_f) \equiv 0$. Figure 1 shows the dependence of the hydraulic head variance, normalized by $Q^2\sigma_Y^2 l_Y^2/K_g^2$, on dimensionless coordinate $x\,/\,l_Y$ for several front's positions relative to the correlation length l_Y. As the flow domain grows, so does the head variance. The head variance is directly proportional to squared dimensionless velocity of the front propagation, Q^2/K_g^2.

4.1.1. Comparison with the exact analytical solution. We now compare the general perturbation solutions derived above with the corresponding solutions obtained by direct integration of the 1-D flow equation,

$$\frac{\partial}{\partial x}\left[K(x)\frac{\partial h}{\partial x}\right] = 0, \qquad 0 < x < x_f(t), \tag{31}$$

subject to the boundary condition at the inlet, $x = 0$,

$$K\partial h/\partial x = -Q \tag{32}$$

and to the conditions on the moving front $x = x_f(t)$,

$$h(x,t) = 0, \qquad K(x)\frac{\partial h}{\partial x} = -V, \qquad V = n_e\frac{dx_f}{dt}. \tag{33}$$

Then direct integration gives

$$h(x,t) = Q\int_x^{x_f}\frac{ds}{K(s)}, \tag{34}$$

from which it follows immediately that $V = Q$. This corresponds exactly to our perturbation solution (27). Moreover, since V is deterministic, all its higher moments and cross-covariances $\sigma_V^2(x_f) = C_{KV}(x,x_f) = C_{hV}(x,x_f) \equiv 0$, which is in exact agreement with our perturbative solutions. Also, deterministic V implies deterministic dynamics of the moving front, $x_f(t) = (V/n_e)t$.

It follows from (34) that the residual flux is given exactly by

$$\overline{r} \equiv -\overline{K'(x)\frac{dh}{dx}} = Q\overline{\frac{K'(x)}{K(x)}} = Q\left(1 - e^{\sigma_Y^2}\right). \tag{35}$$

Thus, indeed, (28) is the true first-order perturbation solution.

Taking the ensemble mean of the derivative of (34) shows that the mean gradient is inversely proportional to the harmonic mean, $K_h = K_g\exp(-\sigma_Y^2/2)$, of hydraulic conductivity,

$$\frac{d\overline{h}}{dx} = -\frac{Q}{K_h}. \tag{36}$$

Then mean head distribution conditioned on the position of the moving front, $\langle h|x_f\rangle$, is given by

$$\langle h|x_f\rangle = \frac{Q}{K_h}(x_f - x) \qquad \text{and} \qquad \overline{h} = \frac{Q}{K_h}(\overline{x}_f - x). \tag{37}$$

Since in this case, $\overline{x}_f = \overline{x}_f^{(0)}$ ($\overline{x}_f^{(i)} \equiv 0$ for $i \geq 1$), (29) is indeed the first-order approximation of the mean head distribution (37).

It follows from (34) that cross-covariance $C_{Kh}(y,x)$ is given as the solution of

$$\frac{dC_{Kh}(y,x)}{dx} = -Q\overline{\frac{K'(y)}{K(x)}} \tag{38}$$

subject to the boundary condition $C_{Kh}(y,x_f) = C_{Kh}(y,\overline{x}_f) = 0$. For Gaussian K, this solution is

$$C_{Kh}(y,x) = -Qe^{\sigma_Y^2}\int_x^{\overline{x}_f}\left[1 - e^{-\sigma_Y^2\rho_Y(y,z)}\right]dz. \tag{39}$$

It is easy to verify that $C_{Kh}(y,x)$ in (39) corresponds exactly to the first-order perturbation solution, $C_{Kh}^{(1)}(y,x)$, obtained by substituting (26) into (18). By the same token, it follows from (34) that head variance, σ_h^2, is given exactly by

$$\sigma_h^2(x) = \frac{Q^2}{K_h^2} \int_x^{\overline{x}_f} \int_x^{\overline{x}_f} \left[e^{\sigma_Y^2 \rho_Y(z,s)} - 1 \right] dz ds. \tag{40}$$

The first order in the perturbation expansion of σ_h^2 is identical to that derived from (17) and given by (30).

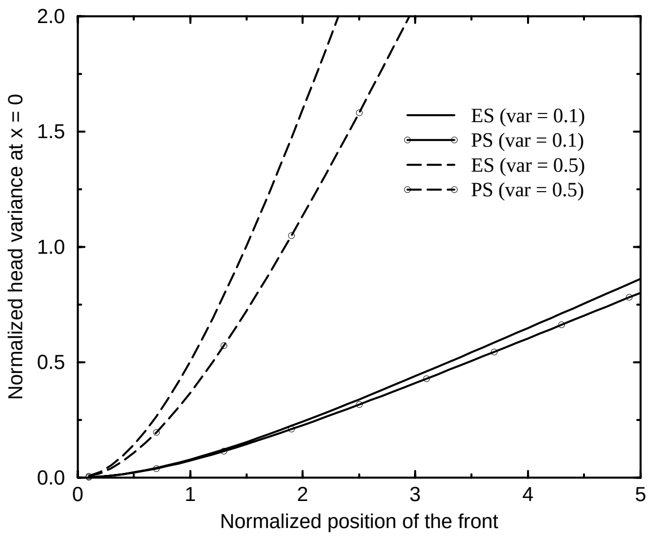

FIG. 2. *Comparison of the exact and perturbation solutions for the normalized head variance evaluated at the origin $x = 0$.*

Figure 2 compares the first-order perturbation solution for the head variance (30), normalized by $Q^2 l_Y^2 / K_g^2$, with the exact solution (40) for two values of the log-hydraulic conductivity variance, $\sigma_Y^2 = 0.1$ and 0.5. Both solutions are evaluated at the inlet $x = 0$. Not surprisingly, the quality of the perturbation solution deteriorates as the perturbation parameter σ_Y^2 increases. Nevertheless, it can be considered satisfactory even for relatively large values of σ_Y^2, especially taking into account that (30) represents only the leading term in the perturbation expansion of head variance.

The comparison of our perturbation and exact solutions demonstrates that the perturbation solutions remain asymptotic as long as the Taylor expansions of the corresponding exponents are asymptotic. In particular, the perturbation solution for the mean head distribution, $\overline{h}(x)$, is asymptotic for $\sigma_Y^2 < 2$. At the same time, the perturbation solutions for the second moments of head, C_{Kh} and σ_h^2, are asymptotic for $\sigma_Y^2 < 1$.

4.2. Front propagation by fixed head boundaries. Consider now a front driven by a hydraulic head gradient. To do so, we prescribe a constant head $h = H$ at the inlet $x = 0$, while maintaining $h = 0$ at the outlet $x = L$. These boundary conditions lead to

$$G_K^{(0)}(x,y) \; = \; -(x-y)\,\mathcal{H}(x-y) \; + \; \frac{L-y}{L}x. \tag{41}$$

4.2.1. Linearized perturbation solution. Substituting (41) into the 1-D versions of (13a)–(20) leads to the linearized solutions for the zeroth- and first-order approximations of the mean front dynamics,

$$\overline{x}_f^{(0)} = \sqrt{\frac{2HK_g}{n_e}}\sqrt{t}, \qquad \left(\frac{\overline{V}^{(0)}}{K_g} = \frac{H}{\overline{x}_f^{(0)}}\right), \tag{42a}$$

(42b)

$$\overline{x}_f^{(1)} = 2\sigma_Y^2 l_Y \left[-\frac{1}{4}\frac{\overline{x}_f^{(0)}}{l_Y} + \ln\left(\frac{\overline{x}_f^{(0)}}{l_Y}\right) + \frac{l_Y}{\overline{x}_f^{(0)}} - \frac{l_Y}{\overline{x}_f^{(0)}}e^{-\overline{x}_f^{(0)}/l_Y} + E_1\left(\frac{\overline{x}_f^{(0)}}{l_Y}\right) - 1 + \gamma\right],$$

where γ is the Euler's constant and $E_1(x)$ is the elliptic integral, and for the first-order approximation of the front velocity variance,

$$\frac{\left[\sigma_V^2\right]^{(1)}}{K_g^2} = 2\sigma_Y^2\frac{H^2}{l_Y^2}\left(\frac{l_Y}{\overline{x}_f^{(0)}}\right)^4\left[-1 + \frac{\overline{x}_f^{(0)}}{l_Y} + \exp\left(-\frac{\overline{x}_f^{(0)}}{l_Y}\right)\right]. \tag{43}$$

Other quantities of interest which are necessary for deriving (42)–(43) include the zeroth-order approximation of the mean hydraulic head, the first-order approximation of the cross-covariance C_{KV},

$$\frac{C_{KV}^{(1)}}{K_g^2} = \sigma_Y^2\frac{H}{l_Y}\left(\frac{l_Y}{\overline{x}_f^{(0)}}\right)^2\left[2 - \exp\left(-\frac{x}{l_Y}\right) - \exp\left(\frac{x-\overline{x}_f^{(0)}}{l_Y}\right)\right], \tag{44}$$

and the first-order approximation of the residual flux,

$$\frac{\overline{r}^{(1)}}{K_g} = \sigma_Y^2\frac{H}{l_Y}\left(\frac{l_Y}{\overline{x}_f^{(0)}}\right)^2\left[2 - \frac{\overline{x}_f^{(0)}}{l_Y} - \exp\left(-\frac{x}{l_Y}\right) - \exp\left(\frac{x-\overline{x}_f^{(0)}}{l_Y}\right)\right]. \tag{45}$$

Figure 3 compares zeroth-order, $\overline{x}_f^{(0)}$, and first-order, $\overline{x}_f^{[1]} = \overline{x}_f^{(0)} + \overline{x}_f^{(1)}$, approximations of the mean front dynamics for several values of the log-hydraulic conductivity variance, $\sigma_Y^2 = 0.1$, 0.5, and 1.0. The mean front is normalized by correlation length, $\overline{x}_f/l_Y$, and dimensionless time is defined as $t_d = tHK_g/(n_e l_Y^2)$. This comparison suggests that the expansion $\overline{x}_f = \overline{x}_f^{(0)} + \overline{x}_f^{(1)} + \cdots$ is indeed asymptotic. A more rigorous analysis of the asymptotics of this expansion is conducted in the next section. One can see that the mean position of the front scales as $\sqrt{t}$.

$$\frac{\overline{h}^{(0)}(x;x_f)}{H} \; = \; \frac{x_f - x}{x_f}, \tag{46}$$

Figure 4 shows how the velocity variance, normalized by $K_g^2\sigma_Y^2$, varies with the dimensionless distance $\overline{x}_f^{(0)}/l_Y$ for several values of the normalized boundary head,

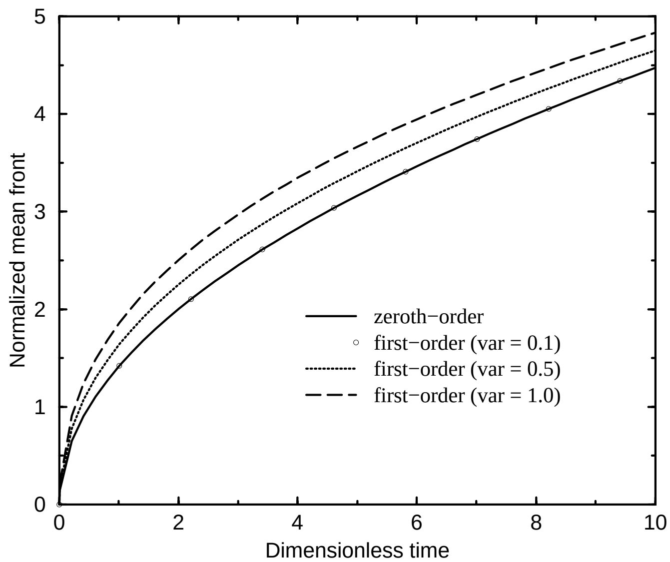

FIG. 3. *Zeroth- and first-order approximations of the mean front dynamics for several values of the log-hydraulic conductivity variance,* σ_Y^2.

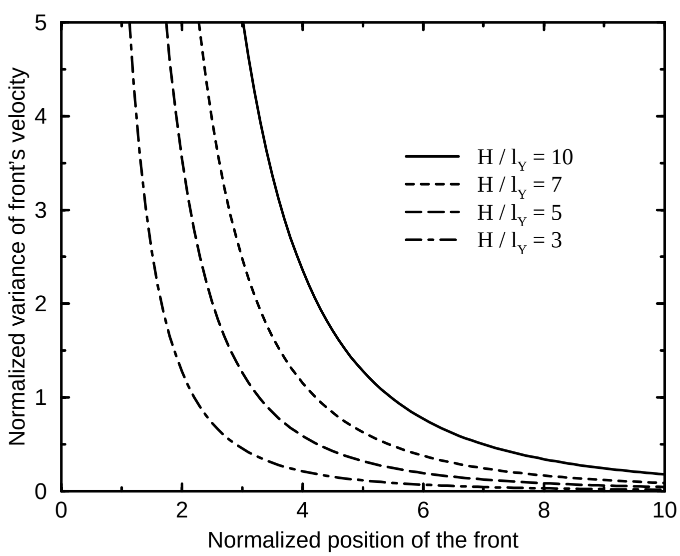

FIG. 4. *Normalized variance of velocity of the front versus normalized position of the front.*

H/l_Y. The velocity variance increases with the mean velocity of the front, which in turn increases with H. It also decreases as the front "samples" more and more correlation scales, as expressed by $\overline{x}_f/l_Y$.

The limitations of the linearized solution become obvious when one tries to evaluate the front variance, σ_f^2. Indeed it follows from (44) that

$$\frac{dC^{(1)}_{Kx_f}}{d\overline{x}_f^{(0)}} = \sigma_Y^2 K_g \frac{l_Y}{\overline{x}_f^{(0)}} \left[2 - \exp\left(-\frac{x}{l_Y}\right) - \exp\left(\frac{x-\overline{x}_f^{(0)}}{l_Y}\right) \right], \tag{47}$$

which is clearly nonintegrable on the interval $[0, \overline{x}_f^{(0)}]$. Consequently, the first-order approximation of σ_f^2 in (20) is not defined. We now proceed with deriving the first-order solution without linearization.

4.2.2. Perturbation solution without linearization. For the problem under consideration, substituting (41) into the 1-D version of (7), evaluated at $x = x_f$, yields

$$0 = -\int_0^{x_f} K'(y) \frac{dh}{dy} dy + H\overline{K} - V x_f. \tag{48}$$

Expanding the integral in Taylor's series around $\overline{x}_f$ gives

$$0 = -\int_0^{\overline{x}_f} K'(y) \frac{dh}{dy} dy - x_f' K'(\overline{x}_f) \frac{dh}{dy}(\overline{x}_f) + \cdots + H\overline{K} - V x_f. \tag{49}$$

Note that in the linearized solution, the second term in the Taylor expansion is absent, and x_f is replaced with $\overline{x}_f$. The recursive approximations of the mean front dynamics, $\overline{x}_f^{(i)}$ $(i = 1, 2)$, are obtained be taking the ensemble mean of (49) and retaining the terms of ith order in σ_Y^2; the first-order approximations of front variance, σ_f^2, and cross-covariance C_{Kx_f} are derived upon multiplying (49) with x_f' and $K'(x)$, respectively, taking the mean and retaining the first-order terms. This results in the zeroth- and first-order approximations of the mean front,

$$\overline{x}_f^{(0)}(t) = \sqrt{\frac{2HK_g}{n_e}} \sqrt{t} \tag{50a}$$

and

$$\begin{aligned} \frac{\overline{x}_f^{(1)}}{l_Y} = -\frac{\sigma_Y^2}{4} \frac{\overline{x}_f^{(0)}}{l_Y} + \sigma_Y^2 \frac{l_Y}{\overline{x}_f^{(0)}} \Bigg(& -\frac{1}{2} + \frac{2}{3} \frac{\overline{x}_f^{(0)}}{l_Y} - \left\{ \frac{l_Y}{\overline{x}_f^{(0)}} \right\}^2 \\ & + \left[1 + \frac{l_Y}{\overline{x}_f^{(0)}} + \left\{ \frac{l_Y}{\overline{x}_f^{(0)}} \right\}^2 \right] e^{-\overline{x}_f^{(0)}/l_Y} \Bigg), \end{aligned} \tag{50b}$$

and the first-order approximation of the front variance,

$$\left[\sigma_f^2\right]^{(1)} = 2\sigma_Y^2 l_Y^2 \left(-\frac{1}{2} + \frac{1}{3} \frac{\overline{x}_f^{(0)}}{l_Y} + \left\{ \frac{l_Y}{\overline{x}_f^{(0)}} \right\}^2 - \frac{l_Y}{\overline{x}_f^{(0)}} \left[1 + \frac{l_Y}{\overline{x}_f^{(0)}} \right] e^{-\overline{x}_f^{(0)}/l_Y} \right). \tag{51}$$

Other quantities of interest which are necessary for deriving (50)–(51) include the zeroth-order approximation of the mean hydraulic head,

$$\overline{h}^{(0)}(x,t) = -H \frac{x - \overline{x}_f^{(0)}}{\overline{x}_f^{(0)}}, \tag{52}$$

the first-order approximation of the cross-covariance C_{Kx_f},

(53)
$$C^{(1)}_{Kx_f}(x,t) = K_g \sigma_Y^2 l_Y \left(2\left[1 - \frac{x}{\overline{x}_f^{(0)}}\right] - \left[1 + \frac{l_Y}{\overline{x}_f^{(0)}}\right] e^{-x/l_Y} + \frac{l_Y}{\overline{x}_f^{(0)}} e^{-(\overline{x}_f^{(0)}-x)/l_Y} \right),$$

and the first-order approximation of the residual flux,

(54)
$$\overline{r}^{(1)}(x,t) = -K_g \sigma_Y^2 \frac{H}{\overline{x}_f^{(0)}} + \frac{n_e}{K_g} \frac{\partial C^{(1)}_{Kx_f}(x,t)}{\partial t}.$$

These were obtained from Taylor's expansion of the 1-D version of (7).

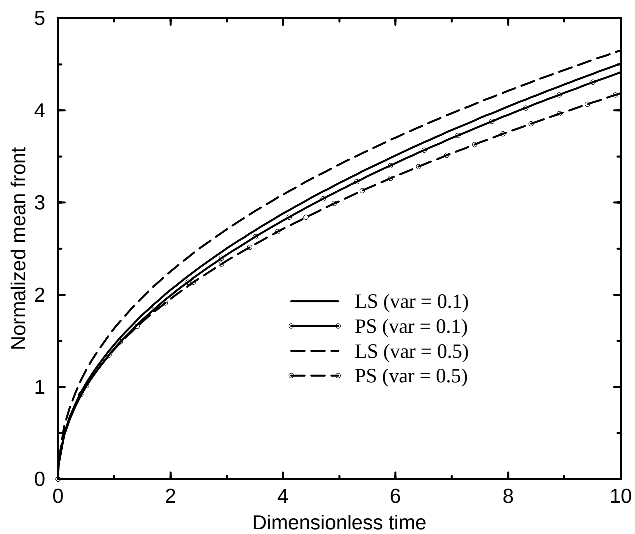

FIG. 5. *Comparison of the linearized perturbation solutions (LS) and the perturbation solution without linearization (PS) for the mean front dynamics.*

Figure 5 compares the linearized perturbation solution (42) for the mean front dynamics (plain lines) with the perturbation solution without linearization (50) (lines with circles). The mean position of the front, $\overline{x}_f^{[1]} = \overline{x}_f^{(0)} = \overline{x}_f^{(1)}$, is normalized with l_Y, and dimensionless time is defined as $t_d = tHK_g/(n_e l_Y^2)$. One can see that for relatively small variances of log-hydraulic conductivity ($\sigma_Y^2 = 0.1$) the two perturbation solutions are in a good agreement. The errors introduced by the linearization of our perturbation solution increase with σ_Y^2, with the linearized solution consistently overestimating the true perturbation solution.

4.2.3. Comparison with the exact analytical solution. If constant head is maintained at the inlet, the pressure head distribution is given by

(55)
$$h(x,t) = H\left[1 - \int_0^x \frac{ds}{K(s)} \left(\int_0^{x_f} \frac{ds}{K(s)} \right)^{-1} \right].$$

Hence

$$V(t) = n_e \frac{dx_f}{dt} = H \left(\int_0^{x_f} \frac{ds}{K(s)} \right)^{-1}. \tag{56}$$

The exact solution for the mean dynamics of the front, $\overline{x}_f(t)$, can now be obtained by taking $K(s) \equiv K$ to be a random constant, which corresponds to a perfectly correlated medium, i.e., $l_Y \to \infty$. Indeed,

$$x_f^2 = 2\frac{KH}{n_e} t. \tag{57}$$

When $K(x) \equiv K$, the exact solution for the mean dynamics of the front, $\overline{x}_f$, can be obtained from (57). The mean position of the front scales as $\sqrt{t}$:

$$\overline{x}_f = \sqrt{\frac{2H}{n_e}} \overline{K^{1/2}} \sqrt{t} = e^{\sigma_Y^2/8} \sqrt{\frac{2HK_g}{n_e}} \sqrt{t} = e^{\sigma_Y^2/8} \overline{x}_f^{(0)}. \tag{58}$$

The last equality holds if K is Gaussian. The corresponding linearized and nonlinear perturbation solutions are obtained by taking the limit as $l_Y \to \infty$ in (42b) and (50b), respectively. It is easy to verify that the linearized perturbation expansion becomes (we use the "˜" sign to distinguish between the linearized perturbation solution and the perturbation solution without localization)

$$\tilde{x}_f^{[1]} \equiv \overline{x}_f^{(0)} + \tilde{x}_f^{(1)} = \left(1 + \frac{\sigma_Y^2}{2}\right) \sqrt{\frac{2HK_g}{n_e}} \sqrt{t}, \tag{59}$$

while the perturbation solution without linearization is

$$\overline{x}_f^{[1]} \equiv \overline{x}_f^{(0)} + \overline{x}_f^{(1)} = \left(1 + \frac{\sigma_Y^2}{8}\right) \sqrt{\frac{2HK_g}{n_e}} \sqrt{t}. \tag{60}$$

The perturbation solution without linearization corresponds to the expansion of the exponent in the exact solution (58). As such, the nonlinearized perturbation solution remains asymptotic as long as $\sigma_Y^2 < 8$. At the same time, the linearized solution, while providing the correct expression for the leading term in the expansion, overestimates the true solution.

It follows from (57) that the exact solution for cross-covariance C_{Kx_f} is

$$C_{Kx_f} = \sqrt{\frac{2Ht}{n_e}} K_g^{3/2} \left(e^{9\sigma_Y^2/8} - e^{5\sigma_Y^2/8} \right) = K_g \left(e^{9\sigma_Y^2/8} - e^{5\sigma_Y^2/8} \right) \overline{x}_f^{(0)}. \tag{61}$$

The first-order approximation of C_{Kx_f} without linearization is obtained by taking the limit as $l_Y \to \infty$ in (53):

$$C_{Kx_f}^{(1)} = K_g \frac{\sigma_Y^2}{2} \overline{x}_f^{(0)}. \tag{62}$$

This corresponds exactly to the first-order solution obtained by expanding the exponents in (61). On the other hand, taking the limit as $l_Y \to \infty$ in the linearized solution (47) gives

$$\tilde{C}_{Kx_f}^{(1)} = K_g \sigma_Y^2 \overline{x}_f^{(0)}, \tag{63}$$

which misses the mark by a factor of 1/2.

The exact solution for the front variance follows directly from (57):

$$\sigma_f^2 = \left(e^{\sigma_Y^2/2} - e^{\sigma_Y^2/4}\right)\frac{2HK_g}{n_e}t = \left(e^{\sigma_Y^2/2} - e^{\sigma_Y^2/4}\right)\overline{x}_f^{(0)^2}. \tag{64}$$

Taking the limit as $l_Y \to \infty$ in (51) yields the first-order nonlinear approximation of the front variance

$$\left[\sigma_f^2(t)\right]^{(1)} = \frac{\sigma_Y^2}{4}\overline{x}_f^{(0)^2}, \tag{65}$$

which coincides with the first-order approximation of the exact solution. The first-order linearized expression for the front variance follows from (20):

$$\left[\tilde{\sigma}_f^2(t)\right]^{(1)} = \sigma_Y^2\overline{x}_f^{(0)^2}. \tag{66}$$

Once again, the linearized solution overestimates the true solution.

4.2.4. Comparison with the "mean field" solution. A naive approach to dealing with random environments is to replace a random parameter, such as the medium hydraulic conductivity $K(\mathbf{x})$, with its averaged counterpart, $\overline{K}$. Such an approximation, often referred to as the mean field solution, is attractive due to its simplicity but is often in error, since it ignores all cross-product terms.

If $K(x)$ is a log-normal statistically homogeneous field, its ensemble mean, $\overline{K}$, is given by the arithmetic mean, $\overline{K} = K_g \exp(\sigma_Y^2/2)$. Replacing random K with $\overline{K}$ in (55)–(57) leads to

$$\hat{x}_f = e^{\sigma^2/4}\sqrt{\frac{2K_gH}{n_e}}\sqrt{t}, \tag{67}$$

where the "$\hat{}$" sign denotes the mean field solution. Comparison with (59) and (60) reveals that

$$\overline{x}_f^{[1]}(t) \le \hat{x}_f^{[1]}(t) \le \tilde{x}_f^{[1]}(t). \tag{68}$$

Thus, despite its simplicity, the mean field solution $\hat{x}_f(t)$ provides a tighter bound for the mean dynamics of the front $\overline{x}_f(t)$ than does the linearized solution $\tilde{x}_f(t)$. Not surprisingly, the mean field solution $\hat{x}_f(t)$ overestimates the true mean solution $\overline{x}_f(t)$.

5. Summary. We have described flow in a random porous medium through domains that are saturated by a fluid but are at least partially bounded by a free surface. The randomness in our model arises from uncertainty about details of the medium's hydraulic conductivity, which we treat as a random field. It would be straightforward to include random fixed boundary and initial conditions, although we do not consider those additional complications here. Random conductivity renders the hydraulic pressure head, the fluid flux, and the position of the free surface random too.

Our initial result is an integral equation for the pressure head from which we can derive the mean fluid flux and thus the mean position of the free surface boundary. Unlike solutions for flow in random media with fixed boundaries, however, the domains of integration depend on the random position of the free boundary. Hence, we have expanded the random integrals in Taylor's series about the mean position of the free boundary. In principle, we could obtain mean head, as well as other quantities of interest, to any desired accuracy by taking the ensemble average of the Taylor series to high enough order. However, a low-order expansion is desirable because higher-order terms in the Taylor series are quite complex.

We have obtained a linearized equation (LE) for mean head by averaging the zeroth-order term in the Taylor series for each integral. From this we have determined a linearized mean flux approximation and thus predict the (linearized) mean location of the free surface. To quantify the uncertainty associated with such predictions, we have also developed a set of LEs satisfied by the corresponding second moments. The LEs are themselves fairly complicated and require closure approximations to be workable. We have based our closures on perturbation expansions in powers of σ_Y^2, the variance of the logarithm of hydraulic conductivity. The perturbation analysis leads to recursive equations that can be solved analytically, as we have been able to do here, or by standard numerical methods in other cases.

All of this begs the question, How good are the linearized approximations? We answer this question in 1-D media confined to the interval $[0, L]$. We have analyzed two cases: fronts that are driven either by deterministic constant boundary fluxes at $x = 0$ or by deterministically prescribed constant boundary head, also at $x = 0$. We suppose head is zero at $x = L$ and conductivity is a random field in the first example and a constant but random variable in the second. In these cases we can obtain exact solutions to the free surface problem and compare them to the LE approximations. For the prescribed flux boundary, the first-order perturbation expansions of the LE are correct to order 1. However, in the other case when head is fixed at $x = 0$, the first-order LE overestimates the position of the front and of the covariance between the front and (random, but constant) conductivity. Finally, a mean field solution for the averaged front dynamics overestimates the expected position of the front, but it provides a tighter bound than the linearized estimate.

Appendix. Perturbation expansions. To render (11) workable, we employ a perturbation analysis in small parameter σ_Y representing a measure of the standard deviation of $Y'(\mathbf{x}) = Y(\mathbf{x}) - \overline{Y}(\mathbf{x})$, where $Y(\mathbf{x}) = \ln K(\mathbf{x})$. In doing so, the state variables $\mathcal{A}$ are formally expanded in the asymptotic series, $\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \cdots + \mathcal{A}^{(2[n-1])} + O\left(\sigma_Y^{2n}\right)$, and the following identities are used:

$$\overline{K} \;=\; \overline{e^Y} \;=\; K_g \overline{e^{Y'}} \;=\; K_g \left[1 + \frac{\sigma_Y^2}{2} + O\left(\overline{Y'^3}\right)\right], \tag{69}$$

$$K' \;=\; K - \overline{K} \;=\; K_g\left[e^{Y'} - \overline{e^{Y'}}\right] \;=\; K_g\left[Y' + \frac{Y'^2}{2} - \frac{\sigma_Y^2}{2} + O\left(Y'^3\right)\right], \tag{70}$$

where $K_g = \exp\left(\overline{Y}\right)$ is the geometric mean of K, and $\sigma_Y^2 = \overline{Y'Y'}$ is the variance of Y.

Acknowledgments. The authors are grateful to Alexandre M. Tartakovsky (University of Arizona) and the two anonymous referees, whose comments and suggestions greatly improved the original version of the paper.

REFERENCES

[1] R. S. Baker and D. Hillel, *Laboratory tests of a theory of fingering during infiltration into layered soils*, Soil Sci. Soc. Amer. J., 54 (1990), pp. 20–30.

[2] R. C. Ball, *Stochastic simulation and viscous fingering*, in Mathematics in Oil Production, S. Edwards and P. R. King, eds., Oxford University Press, London, 1988, pp. 211–216.

[3] G. Chen and S. P. Neuman, *Wetting front instability in randomly stratified soils*, Phys. Fluids, 8 (1996), pp. 353–369.

[4] G. Christakos and D. T. Hristopulos, *Stochastic Radon operators in porous media hydrodynamics*, Quart. Appl. Math., 55 (1997), pp. 89–112.

[5] R. L. Chuoke, P. van Meurs, and C. van der Poel, *The instability of slow, immiscible, viscous liquid-liquid displacements in permeable media*, Trans. AIME, 216 (1959), pp. 188–194.

[6] J. H. Cushman, *Development of stochastic partial differential equations for subsurface hydrology*, Stochastic Hydrol. Hydraul., 1 (1987), pp. 241–262.

[7] G. Dagan, *Flow and Transport in Porous Formations*, Springer-Verlag, New York, 1989.

[8] G. Dagan and D. G. Zeitoun, *Free-surface flow toward a well and interface upconning in stratified aquifers of random conductivity*, Water Resour. Res., 34 (1998), pp. 3191–3196.

[9] E. R. G. Eckeret and R. M. Drake, *Analysis of Heat and Mass Transfer*, McGraw-Hill, New York, 1972.

[10] J. P. van der Eerden and H. Hoche, *The Green function approach to several problems in crystal growth*, J. Cryst. Growth, 65 (1983), pp. 60–67.

[11] G. Fennemore and J. X. Xin, *Wetting fronts in one-dimensional periodically layered soils*, SIAM J. Appl. Math., 58 (1998), pp. 387–427.

[12] G. A. Fenton and D. V. Griffith, *Statistics of free surface flow through a stochastic earth dam*, J. Geotech. Engrg., ASCE, 122 (1996), pp. 422–436.

[13] W. H. Green and G. A. Ampt, *Studies in soil physics,* I. *The flow of air and water through soil*, J. Agr. Sci., 4 (1911), pp. 1–24.

[14] P. Indelman, *Averaging of unsteady flows in heterogeneous media of stationary conductivity*, J. Fluid Mech., 310 (1996), pp. 39–60.

[15] D. A. Kessler, J. Koplik, and H. Levine, *Numerical simulation of two-dimensional snowflake growth*, Phys. Rev. A, 30 (1984), pp. 2820–2823.

[16] P. R. King, *The use of field theoretic methods for the study of flow in heterogeneous porous medium*, J. Phys. A, 20 (1987), pp. 3935-3947.

[17] R. Lenormand, E. Touboul, and C. Zarcone, *Numerical models and experiments on immiscible displacement in porous media*, J. Fluid Mech., 189 (1988), pp. 165–187.

[18] C. C. Mei and J.-L. Auriault, *Mechanics of heterogeneous porous media with several spatial scales*, Proc. Roy. Soc. London Ser. A, 426 (1989), pp. 391–423.

[19] R. C. Morgan and I. Babuška, *An approach for constructing families of homogenized equations for periodic media.* I. *An integral representation and its consequences*, SIAM J. Math. Anal., 22 (1991), pp. 1–15.

[20] S. P. Neuman and S. Orr, *Prediction of steady state flow in nonuniform geologic media by conditional moments: Exact nonlocal formalism, effective conductivities, and weak approximation*, Water Resour. Res., 29 (1993), pp. 341–364.

[21] L. Paterson, *Diffusion limited aggregation and two-fluid displacements in porous media*, Phys. Rev. Lett., 52 (1984), pp. 1621–1624.

[22] J. R. Philip, *Stability analysis of infiltration*, Soil. Sci. Soc. Am. Proc., 39 (1975), pp. 1042–1049.

[23] P. A. C. Raats, *Unstable wetting fronts in uniform and nonuniform soils*, Soil. Sci. Soc. Am. Proc., 37 (1973), pp. 681–685.

[24] G. F. Roach, *Green's Functions*, Cambridge University Press, Cambridge, UK, 1982, p. 150.

[25] P. G. Saffman and G. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous fluid*, Proc. Roy. Soc. London Ser. A, 245 (1958), pp. 312–329.

[26] L. M. Sander, P. Ramanlal, and E. Ben-Jacob, *Diffusion-limited aggregation as a deterministic growth process*, Phys. Rev. A, 32 (1985), pp. 3160–3163.

[27] M. I. Shvidler, *Filtration Flows in Heterogeneous Media (A Statistical Approach)*, Consultants Bureau, New York, 1964.

[28] D. M. Tartakovsky and S. P. Neuman, *Transient flow in bounded randomly heterogeneous domains:* 1. *Exact conditional moment equations and recursive approximations*, Water Resour. Res., 34 (1998), pp. 1–12.

[29] D. M. Tartakovsky and S. P. Neuman, *Transient effective hydraulic conductivities under*

slowly and rapidly varying mean gradients in bounded three-dimensional random media, Water Resour. Res., 34 (1998), pp. 21–32.

[30] D. M. Tartakovsky and S. P. Neuman, *Extension of "Transient flow in bounded randomly heterogeneous domains:* 1. *Exact conditional moment equations and recursive approximations,"* Water Resour. Res., 35 (1999), pp. 1921–1925.

[31] C. L. Winter, C. M. Newman, and S. P. Neuman, *A perturbation expansion for diffusion in a random velocity field*, SIAM J. Appl. Math., 44 (1984), pp. 411–424.

[32] Y. C. Yortsos and F. J. Hickernell, *Linear stability of immiscible displacement in porous media*, SIAM J. Appl. Math., 49 (1989), pp. 730–748.

[33] D. Zhand and C. L. Winter, *Nonstationary stochastic analysis of steady state flow through variably saturated, heterogeneous media*, Water Resour. Res., 34 (1998), pp. 1091–1100.