INTERFACE DYNAMICS IN RANDOM MEDIA

Yu.N. Lazarev[1], P.V. Petrov[1], Daniel M. Tartakovsky[2]

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Abstract. We consider the dynamics of fluid interface in heterogeneous porous media with uncertain hydraulic conductivities. Modeling hydraulic conductivity as a random field of given statistics allows us to predict the interface dynamics and to estimate the corresponding predictive uncertainty by means of statistical moments. The novelty of our approach to obtaining the interface statistics consists of dynamically mapping the Cartesian coordinate system onto a coordinate system associated with the moving front. This transforms a difficult problem of deriving closure relationships for highly nonlinear stochastic flows with free surfaces into a relatively simple problem of deriving stochastic closures for linear flows in domains with fixed boundaries. We derive a set of deterministic equations for the statistical moments of the interfacial dynamics, which hold in one and two spatial dimensions, and analyze their solutions for one-dimensional flow.

Key words: porous media, moving interface, random field, conductivity tensor, fluctuations, stochastic equations, moments

1. Introduction. Free surface (interface, moving front) problems arise in a variety of applications, such as wetting and drying of porous media, pumping in unconfined aquifers, secondary oil recovery, heat conduction and welding, diffusion-limited aggregation, crystal growth, semiconductor fabrication, and snowflake formation. Traditional, deterministic modeling of these and other similar phenomena assumes that the ambient environments are homogenous and/or that the relevant system

[1] Mathematical Division, Institute of Technical Physics, P.O.245, Snezhinsk, 456770, Russia, e-mail: *yulaz@snezhinsk.ru*, e-mail: *pvpetrov@snezhinsk.ru*.

[2] Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA, e-mail: *dmt@lanl.gov*

parameters, such as hydraulic conductivity and thermal diffusivity, are known with certainty in all of their relevant details. However, in most applications, interfaces propagate in heterogeneous environments, whose system parameters can only be sampled at selected locations in space and/or time. The need to assign parameter values to the points where measurements are not available, combined with measurement errors, introduces parameter uncertainty. This, in turn, leads to uncertainty in predictions of the interface dynamics. To make such predictions and to quantify the predictive uncertainty, it is common to treat the system parameters as random fields, and the corresponding governing equations as stochastic.

While flow and transport in randomly heterogeneous porous media with fixed boundaries have been studied extensively [1-4], analysis of the interfacial dynamics in random media is still in its infancy. A reason for the relative lack of progress in analyzing this important problem is its high degree of nonlinearity. Since the randomness of hydraulic conductivity of a porous medium causes the free surface dynamics to be stochastic, ensemble averaging of the flow equations involves calculating ensemble means of such quantities as integrals of random functions over random domains and random functionals. One approach to dealing with this problem is to employ simplifying physical assumptions, such as the Dupuit (shallow water) approximation [5] or a uniform flow approximation [6], which effectively eliminate moving boundaries (interfaces). Early attempts to analyze the interface dynamics [7,8] have relied on the expansions of integrals over the random domains into a Taylor series around the corresponding ensemble mean geometries. To make the analysis and numerical implementation of this procedure tractable, these studies linearize the problem by retaining only the leading term in such expansions. However, an analysis in [8] of the propagation of one-dimensional fronts demonstrates that such a linearization leads to approximations that are less than optimal, in that the subsequent perturbation expansions do not contain all the relevant terms.

The main goal of this study is to introduce an approach that does not require a linearization of the kind proposed in [7,8]. We formulate governing equations for the interface dynamics in random porous media in Section 2. The key part of our approach, a stochastic mapping of the random, time-varying flow domain onto a fixed domain, is presented in Section 3. Section 4 provides the corresponding mappings for the flow equations. This enables us to use standard perturbation techniques to derive, in Sections 5 and 6, closure approximations for the stochastic flow equations in two dimensions. Section 7 contains a brief outline of a numerical algorithm for solving the resulting

deterministic moment equations. In Section 8, we analyze the accuracy of our approximations in a one-dimensional setting, by comparing the analytical solutions of moment equations with their exact counterparts.

2. Problem Formulation. Consider the motion of a fluid-fluid interface in a randomly heterogeneous porous medium Ω_T that is bounded by the surface Γ_T. Following [8], we set gravity, capillary length, and the viscosity of one fluid to zero. In the inviscid fluid (air), the pressure is constant and may be set to zero. The viscous, incompressible fluid (water) occupies the flow domain Ω $(\Omega \subset \Omega_T)$, which is bounded either entirely by a free surface γ or by a combination of γ and some segments of Γ_T (Figure 1). Such flow is described by a combination of Darcy's law and mass conservation,

$$\mathbf{q}(\mathbf{r},t) = -K(\mathbf{r}) \cdot \nabla h(\mathbf{r},t), \qquad \nabla \cdot \mathbf{q} = f(\mathbf{r},t), \qquad \mathbf{r} \in \Omega(t), \tag{1}$$

subject to the boundary conditions

$$h(\mathbf{r},t) = H(\mathbf{r},t), \qquad \mathbf{r} \in \Gamma_D, \tag{2a}$$

$$\mathbf{n}(\mathbf{r}) \cdot \mathbf{q}(\mathbf{r},t) = Q(\mathbf{r},t), \qquad \mathbf{r} \in \Gamma_N, \tag{2b}$$

$$h(\mathbf{r},t) = 0, \qquad \mathbf{r} \in \gamma(t). \tag{2c}$$

where $\mathbf{q}$ is the Darcy flux, K is the hydraulic conductivity of a porous medium, h is the hydraulic head, f is the source function, and $\mathbf{n}$ is unit normal to the surface $\Gamma = \Gamma_D \cup \Gamma_N \cup \gamma$ consisting of Dirichlet segments Γ_D, Neumann segments Γ_N, and a moving front γ. The functions H and Q are the prescribed hydraulic head and flux on the Dirichlet and Neumann boundary segments, respectively. The dynamics of the free surface $\gamma(t)$ is described by

$$\frac{d\mathbf{R}}{dt} = \frac{V_n(\mathbf{R},t)\mathbf{n}}{n_e} = \frac{\mathbf{V}(\mathbf{R},t)}{n_e}, \qquad \mathbf{R} \in \gamma(t), \tag{2d}$$

where n_e is the medium's porosity, and mass conservation requires that the normal velocity of the front satisfies $V_n(\mathbf{R},t) = \mathbf{q}(\mathbf{R},t) \cdot \mathbf{n}(\mathbf{R},t)$. Equations (1) and (2) constitute the widely used Green and Ampt [9] model for the propagation of wetting fronts in porous media.

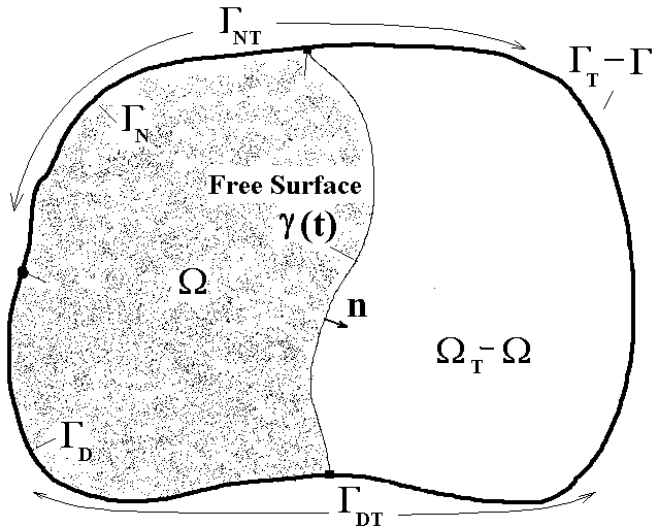

Figure.1. A schematic representation of the dynamics of free surfaces in porous media.

Uncertainty in the hydraulic conductivity of a porous medium is captured by representing $K = K(\mathbf{r})$ as a scalar random field with given mean $\bar{K}$, variance σ_K^2, and a two point correlation function $\rho_K(\mathbf{r}_1, \mathbf{r}_2)$. Other possible sources of randomness, which we do not consider here, are the driving forces f, H, and Q, and porosity n_e. Our goal is to develop a set of deterministic equations for the mean and variance of the system states. The former estimates the interfacial dynamics, while the latter quantifies the uncertainty associated with such an estimate.

3. Mapping of the Flow Domain. Consider a curvilinear coordinate system (ξ, η), which is tied to the moving interface $\gamma(t)$. An advantage of using such a coordinate system is that the random, time-varying flow domain Ω in the (x, y) Cartesian coordinate system becomes a fixed regular-shaped domain (e.g., a square or a rectangle) W in the (ξ, η) coordinate system (see Figure 2).

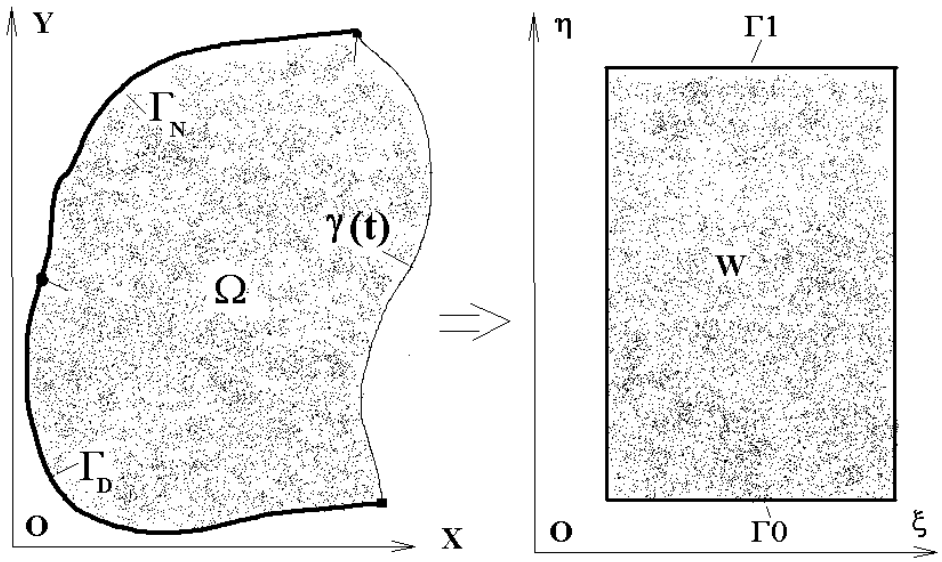

Figure 2. A mapping of the flow domain.

Following [10], we define a stochastic mapping $\Omega \to W$ as a solution of the Laplace equations

$$\frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \eta^2} = 0, \qquad \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} = 0, \tag{3}$$

subject to the boundary conditions

$$\begin{aligned} x(1,\eta) &= x_\gamma(\eta), \quad & y(1,\eta) &= y_\gamma(\eta) \\ x(0,\eta) &= x_{\Gamma D}(\eta), \quad & y(0,\eta) &= y_{\Gamma D}(\eta) \end{aligned} \tag{4a}$$

and

$$\begin{aligned} x(\xi,0) &= x_{\Gamma 0}(\xi), \quad & y(\xi,0) &= y_{\Gamma 0}(\xi) \\ x(\xi,1) &= x_{\Gamma 1}(\xi), \quad & y(\xi,1) &= y_{\Gamma 1}(\xi) \end{aligned} . \tag{4b}$$

For the mapping (3) - (4) to exist, it is necessary that the boundary Γ of the flow domain Ω be piecewise smooth [10]. This condition holds for most physical applications, such as (unstable) front propagation in porous media. Moreover, as will become clear below, it is sufficient for (3) – (4) to exist in a weak sense, which further smoothes the boundary Γ through its ensemble averaging.

We use Reynolds decomposition $A = \bar{A} + \tilde{A}$ to represent a random field A as the sum of its mean $\bar{A}$ and a zero-mean random fluctuation $\tilde{A}$. (In the following, we use $\bar{A}$ and $\langle A \rangle$ interchangeably to indicate the ensemble mean of *A*.) Then stochastic averaging of (3) - (4) yields the ensemble mean component of the stochastic mapping as a solution of

$$\frac{\partial^2 \overline{x}}{\partial \xi^2} + \frac{\partial^2 \overline{x}}{\partial \eta^2} = 0, \tag{5}$$

subject to

$$\begin{aligned} \overline{x}(1,\eta) &= \overline{x}_\gamma(\eta), \quad \overline{x}(\xi,0) = x_{\Gamma 0}(\xi), \\ \overline{x}(0,\eta) &= x_{\Gamma D}(\eta), \quad \overline{x}(\xi,1) = x_{\Gamma 1}(\xi). \end{aligned} \tag{6}$$

Here $\overline{x}_\gamma(0) = x_{\Gamma 0}(1)$, $\overline{x}_\gamma(1) = x_{\Gamma 1}(1)$. Introduction of the Green's function $L(\xi,\eta \mid \xi_1,\eta_1)$ as a solution of the Poisson equation,

$$\frac{\partial^2 L}{\partial \xi_1^2} + \frac{\partial^2 L}{\partial \eta_1^2} = -\delta(\xi_1 - \xi)\delta(\eta_1 - \eta), \tag{7}$$

subject to the homogeneous boundary conditions,

$$L(\xi,\eta \mid \xi_1 = 0,\eta_1) = L(\xi,\eta \mid \xi_1 = 1,\eta_1) = L(\xi,\eta \mid \xi_1,\eta_1 = 0) = L(\xi,\eta \mid \xi_1,\eta_1 = 1) = 0, \tag{8}$$

allows us to rewrite (5) – (6) as

$$\overline{x}(\xi,\eta) = -\int_0^1 d\eta_1 \left(\overline{x}_\gamma(\eta_1) \frac{\partial L}{\partial \xi_1}\bigg|_{\xi_1=1} - x_{\Gamma D}(\eta_1) \frac{\partial L}{\partial \xi_1}\bigg|_{\xi_1=0} \right) - \int_0^1 d\xi_1 \left(x_{\Gamma 1}(\xi_1) \frac{\partial L}{\partial \eta_1}\bigg|_{\eta_1=1} - x_{\Gamma 0}(\xi_1) \frac{\partial L}{\partial \eta_1}\bigg|_{\eta_1=0} \right). \tag{9}$$

An expression for $\overline{y}$ is obtained in a similar fashion.

An equation for the random fluctuations $\tilde{x}$ is obtained by subtracting (5) – (6) from (3) – (4),

$$\frac{\partial^2 \tilde{x}}{\partial \xi^2} + \frac{\partial^2 \tilde{x}}{\partial \eta^2} = 0, \tag{10}$$

subject to

$$\begin{aligned} \tilde{x}(1,\eta) &= \tilde{x}_\gamma(\eta), \quad \tilde{x}(\xi,0) = \tilde{x}_{\Gamma 0}(\xi), \\ \tilde{x}(0,\eta) &= 0, \quad \tilde{x}(\xi,1) = \tilde{x}_{\Gamma 1}(\xi). \end{aligned} \tag{11}$$

To find the boundary functions $\tilde{x}_{\Gamma 0}(\xi)$ and $\tilde{x}_{\Gamma 1}(\xi)$, we note that both $\overline{x}$ and $\overline{x} + \tilde{x}$ belong to $\Gamma 0$ and $\Gamma 1$, and that the following equalities hold,

$$\tilde{x}_{\Gamma 0}(1) = \tilde{x}_\gamma(0), \quad \tilde{x}_{\Gamma 0}(0) = 0, \quad \tilde{x}_{\Gamma 1}(1) = \tilde{x}_\gamma(1), \quad \tilde{x}_{\Gamma 1}(0) = 0. \tag{12}$$

Expanding $\tilde{x}_{\Gamma 0}(\xi)$ and $\tilde{x}_{\Gamma 1}(\xi)$ in a Taylor series, and retaining the leading terms in these expansions, yields

$$\tilde{x}_{\Gamma 0}(\xi) = \tilde{x}_{\gamma}(0) \cdot \xi \frac{\left(\dfrac{dx_{\Gamma 0}}{d\xi}\right)_{\xi}}{\left(\dfrac{dx_{\Gamma 0}}{d\xi}\right)_{\xi=1}}, \qquad \tilde{x}_{\Gamma 1}(\xi) = \tilde{x}_{\gamma}(1) \cdot \xi \frac{\left(\dfrac{dx_{\Gamma 1}}{d\xi}\right)_{\xi}}{\left(\dfrac{dx_{\Gamma 1}}{d\xi}\right)_{\xi=1}}. \tag{13}$$

In terms of the Green's function (8) – (9), the solution of (10) – (11) is given by

$$\tilde{x}(\xi,\eta) = \int_0^1 d\xi_1 \tilde{x}_{\Gamma 0}(\xi_1) \frac{\partial L}{\partial \eta_1}\bigg|_{\eta_1=0} - \int_0^1 d\xi_1 \tilde{x}_{\Gamma 1}(\xi_1) \frac{\partial L}{\partial \eta_1}\bigg|_{\eta_1=1} - \int_0^1 d\eta_1 \tilde{x}_{\gamma}(\eta_1) \frac{\partial L}{\partial \xi_1}\bigg|_{\xi_1=1}. \tag{14}$$

An expression for $\tilde{y}$ is obtained in a similar fashion. Equation (14) and the corresponding equation for $\tilde{y}$ define linear integral operators $\hat{X}$ and $\hat{Y}$ that relate the mapping fluctuations inside the flow domain, $\tilde{x}(\xi,\eta)$ and $\tilde{y}(\xi,\eta)$, to their counterparts on the moving interface, $\tilde{x}_{\gamma}$ and $\tilde{y}_{\gamma}$,

$$\tilde{x}(\xi,\eta) = \hat{X} \cdot \tilde{x}_{\gamma}, \qquad \tilde{y}(\xi,\eta) = \hat{Y} \cdot \tilde{y}_{\gamma}. \tag{15}$$

4. Transformed Flow Equations. Let the subscripts ξ and η denote the partial derivatives with respect to ξ and η, respectively, and

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \tag{16}$$

denote the mapping Jacobian. Then

$$\frac{\partial}{\partial x} = \frac{y_{\eta}}{J}\frac{\partial}{\partial \xi} - \frac{y_{\xi}}{J}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial y} = -\frac{x_{\eta}}{J}\frac{\partial}{\partial \xi} + \frac{x_{\xi}}{J}\frac{\partial}{\partial \eta}, \tag{17}$$

and (1) becomes

$$\frac{\partial y_{\eta} q_1}{\partial \xi} - \frac{\partial y_{\xi} q_1}{\partial \eta} - \frac{\partial x_{\eta} q_2}{\partial \xi} + \frac{\partial x_{\xi} q_2}{\partial \eta} = J \cdot f(x(\xi,\eta), y(\xi,\eta)). \tag{18}$$

The Darcy flux components q_1 and q_2 are given by

$$q_1 = -K(x(\xi,\eta), y(\xi,\eta)) \cdot \left[\frac{y_{\eta}}{J}\frac{\partial h}{\partial \xi} - \frac{y_{\xi}}{J}\frac{\partial h}{\partial \eta}\right], \qquad q_2 = -K(x(\xi,\eta), y(\xi,\eta)) \cdot \left[-\frac{x_{\eta}}{J}\frac{\partial h}{\partial \xi} + \frac{x_{\xi}}{J}\frac{\partial h}{\partial \eta}\right]. \tag{19}$$

Substituting (19) into (18) and denoting $F = J \cdot f$ yields

$$-\frac{\partial}{\partial \xi} K^{11} \frac{\partial h}{\partial \xi} - \frac{\partial}{\partial \eta} K^{22} \frac{\partial h}{\partial \xi} + \frac{\partial}{\partial \xi} K^{12} \frac{\partial h}{\partial \eta} + \frac{\partial}{\partial \eta} K^{21} \frac{\partial h}{\partial \xi} = F, \tag{20}$$

where the components of the hydraulic conductivity tensor are given by

$$K^{11} = K\frac{x_\eta^2 + y_\eta^2}{J}, \qquad K^{22} = K\frac{x_\xi^2 + y_\xi^2}{J}, \qquad K^{12} = K^{21} = K\frac{x_\xi x_\eta + y_\xi y_\eta}{J}. \tag{21}$$

Since the mapping $\Omega \to W$ is not orthogonal in general, $K^{12} = K^{21} \neq 0$. A unifying form of (21) is

$$K^{\alpha\beta} = \left(\Re^{-1T} \cdot \Re^{-1}\right)^{\alpha\beta}, \qquad \alpha, \beta = 1,2, \qquad \Re = \frac{\begin{Vmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{Vmatrix}}{\sqrt{K \cdot J}}. \tag{22}$$

While hydraulic conductivity in the fixed coordinate system (x, y) was taken to be a scalar, hydraulic conductivity in the moving coordinate system (ξ, η) becomes a second rank tensor. Of course, (20) remains valid even if hydraulic conductivity in the (x, y) coordinate system were a tensor.

Since the transformed flow equations involve first derivatives x_ξ, x_γ, y_ξ, and y_γ, the boundary Γ (or, more precisely, its ensemble mean) must be at least once differentiable. This explains the existence condition for the mapping $\Omega \to W$ in Section 3.

Boundary conditions for (20) are derived by rewriting (2) in the moving coordinate system. This gives, for Dirichlet and Neumann boundaries,

$$h(\xi = 0, \eta) = H(x_{\Gamma D}(\eta), y_{\Gamma D}(\eta)), \qquad h(\xi = 1, \eta) = 0, \tag{23a}$$

$$-q_\eta\big|_{\xi=0} = K^{22}\left(x_{\Gamma 0}(\xi), y_{\Gamma 0}(\xi)\right)\frac{\partial h}{\partial \eta} - K^{21}(x_{\Gamma 0}(\xi), y_{\Gamma 0}(\xi))\frac{\partial h}{\partial \xi} = \sqrt{x_\xi^2 + y_\xi^2} \cdot Q(x_{\Gamma 0}(\xi), y_{\Gamma 0}(\xi)), \tag{23b}$$

$$-q_\eta\big|_{\xi=1} = K^{22}\left(x_{\Gamma 1}(\xi), y_{\Gamma 1}(\xi)\right)\frac{\partial h}{\partial \eta} - K^{21}(x_{\Gamma 1}(\xi), y_{\Gamma 1}(\xi))\frac{\partial h}{\partial \xi} = -\sqrt{x_\xi^2 + y_\xi^2} \cdot Q(x_{\Gamma 1}(\xi), y_{\Gamma 1}(\xi)). \tag{23c}$$

The equations for the interface dynamics become (Appendix B)

$$\frac{\partial x_\gamma}{\partial t} = \chi^x \frac{\partial h}{\partial \xi}, \qquad \frac{\partial y_\gamma}{\partial t} = \chi^y \frac{\partial h}{\partial \xi}, \tag{24a}$$

where

$$\chi^x = -\frac{K}{n_e J}\frac{\partial y_\gamma}{\partial \eta}, \qquad \chi^y = \frac{K}{n_e J}\frac{\partial x_\gamma}{\partial \eta}. \tag{24b}$$

Introducing a new notation for the coordinates $\xi^1 = \xi$ and $\xi^2 = \eta$, and for the components of the Darcy flux $q^1 = q_\xi$ and $q^2 = q_\eta$, and using the Einstein summation convention allows us to rewrite (18) – (19) in a compact form

$$\frac{\partial q^{\alpha}}{\partial \xi^{\alpha}} = Jf\,, \qquad q^{\alpha} = -K^{\alpha\beta}\frac{\partial h}{\partial \xi^{\beta}}\,. \tag{25}$$

Introducing hydraulic resistivity,

$$Z^{\alpha\beta} = \left(K^{-1}\right)^{\alpha\beta} = \left(\Re \cdot \Re^{T}\right)^{\alpha\beta}, \tag{26}$$

recasts the flow equation (25) as

$$\frac{\partial q^{\alpha}}{\partial \xi^{\alpha}} = Jf\,, \qquad \frac{\partial h}{\partial \xi^{\alpha}} = -Z^{\alpha\beta}q^{\beta}\,, \qquad \left(\xi^{1},\xi^{2}\right)\in W. \tag{27}$$

This is the form we use below to derive moment equations for the hydraulic head and the interface dynamics.

5. Statistical Moments of Head. Stochastic averaging of the flow equation (27), defined on the fixed domain W and subject to the Dirichlet and Neumann boundary conditions (23a) – (23c), has received considerable attention. Most of such studies have assumed that hydraulic conductivity (resistivity) is a scalar, while in (27) it is a tensor.

To simplify presentation, we assume deterministic initial conditions and set the source function and boundary fluxes to zero, i.e. $f \equiv 0$ and $Q \equiv 0$. Taking the ensemble average of (27) and (23) yields equations for the mean hydraulic head,

$$\frac{\partial}{\partial \xi^{\alpha}}\overline{q}^{\alpha} = 0\,, \qquad -\frac{\partial \overline{h}}{\partial \xi^{\alpha}} = \overline{Z}^{\alpha\beta}\overline{q}^{\beta} + \left\langle \tilde{Z}^{\alpha\beta}\tilde{q}^{\beta}\right\rangle. \tag{28}$$

This equation is subject to the boundary conditions

$$\overline{h}\left(\xi^{1}=0,\xi^{2}\right) = H_{0}\left(\xi^{2},t\right), \quad \overline{h}\left(\xi^{1}=1,\xi^{2}\right)=0\,, \quad \overline{q}^{2}\left(\xi^{1},\xi^{2}=0\right)=0\,, \quad \overline{q}^{2}\left(\xi^{1},\xi^{2}=1\right)=0\,. \tag{29}$$

The presence of an unknown second moment $\left\langle \tilde{Z}^{\alpha\beta}\tilde{q}^{\beta}\right\rangle$ in the equation for the first moment $\overline{h}$ is known as a closure problem. One of the most widely used closures is obtained through perturbation expansions of the relevant random fields, such as hydraulic resistivity, hydraulic head, and their moments in the powers of the conductivity fluctuations. The second-order approximation of $\left\langle \tilde{Z}^{\alpha\beta}\tilde{q}^{\beta}\right\rangle$ gives rise to the non-local mean Darcy equation (see Appendix A),

$$-\frac{\partial \overline{h}}{\partial \xi^{\alpha}} = \left[\overline{Z}^{\alpha\beta} - \left\langle \tilde{Z}^{\alpha\alpha_1}\left(\overline{Z}^{-1}\right)^{\alpha_1\beta_1}\tilde{Z}^{\beta_1\beta}\right\rangle\right]\overline{q}^{\beta}(\boldsymbol{\xi}) + \int_0^1\int_0^1 \left\langle \tilde{Z}^{\alpha\alpha_1}T^{\alpha_1\beta_1}\tilde{Z}_1^{\beta_1\beta}\right\rangle \overline{q}^{\beta}(\boldsymbol{\xi}_1)\,d\xi_1^1 d\xi_1^2\,. \tag{30}$$

(Appendix A also outlines a procedure for calculating the head covariance.) Here $Z = Z(\boldsymbol{\xi})$, $Z_1 = Z(\boldsymbol{\xi}_1)$, and

$$T^{\alpha_1\beta_1} = \left(\overline{Z}^{-1}\right)^{\alpha_1\alpha_2} \frac{\partial^2 E(\boldsymbol{\xi},\boldsymbol{\xi}_1)}{\partial \xi^{\alpha_2}\partial \xi_1^{\beta_2}} \left(\overline{Z}_1^{-1}\right)^{\beta_2\beta_1}, \tag{31}$$

where $E(\boldsymbol{\xi}_1 \mid \boldsymbol{\xi}_2)$ is the Green's function defined as a solution of

$$\frac{\partial}{\xi_2^\alpha}\left(\overline{Z}^{-1}\right)^{\alpha\beta} \frac{\partial}{\partial \xi_2^\beta} E = -\delta\left(\xi_2^1 - \xi_1^1\right)\delta\left(\xi_2^2 - \xi_1^2\right), \tag{32a}$$

subject to the boundary conditions

$$E(\boldsymbol{\xi}_1 \mid \xi_2^1 = 0, \xi_2^2) = E(\boldsymbol{\xi}_1 \mid \xi_2^1 = 1, \xi_2^2) = 0, \qquad \left(\overline{Z}^{-1}\right)^{2\alpha} \frac{\partial G}{\partial \xi_2^\alpha}\bigg|_{\xi_2^2=0} = 0, \qquad \left(\overline{Z}^{-1}\right)^{2\alpha} \frac{\partial G}{\partial \xi_2^\alpha}\bigg|_{\xi_2^2=1} = 0. \tag{32b}$$

To obtain the second-order approximations of the correlation matrices of $\tilde{Z}^{\alpha\beta}$ in (30), we linearize the random fluctuations $\tilde{Z}^{\alpha\beta}$ about the corresponding means,

$$\tilde{Z}^{\alpha\beta}(\tilde{x},\tilde{y},\tilde{K};\boldsymbol{\xi}) \approx Z_x^{\alpha\beta}\cdot\tilde{x}(\boldsymbol{\xi}) + Z_y^{\alpha\beta}\cdot\tilde{y}(\boldsymbol{\xi}) + Z_K^{\alpha\beta}\cdot\tilde{K}\left(\overline{x}(\boldsymbol{\xi}),\overline{y}(\boldsymbol{\xi})\right), \tag{33a}$$

where $Z_x^{\alpha\beta}$, $Z_y^{\alpha\beta}$, and $Z_K^{\alpha\beta}$ are the linear deterministic operators,

$$Z_x^{\alpha\beta} = \frac{1}{\overline{x}_\xi\overline{y}_\eta - \overline{x}_\eta\overline{y}_\xi}\left(\frac{1}{\overline{K}}\begin{Vmatrix} \overline{x}_\xi\dfrac{\partial}{\partial\xi} + \overline{y}_\xi\dfrac{\partial}{\partial\eta} & 0 \\ \overline{x}_\eta\dfrac{\partial}{\partial\xi} + \overline{y}_\eta\dfrac{\partial}{\partial\eta} & 0 \end{Vmatrix} - \begin{Vmatrix} \overline{x}_\xi^2 + \overline{y}_\xi^2 & \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta \\ \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta & \overline{x}_\eta^2 + \overline{y}_\eta^2 \end{Vmatrix}\left(\overline{y}_\eta\frac{\partial}{\partial\xi} - \overline{y}_\xi\frac{\partial}{\partial\eta}\right)\right), \tag{33b}$$

$$Z_y^{\alpha\beta} = \frac{1}{\overline{x}_\xi\overline{y}_\eta - \overline{x}_\eta\overline{y}_\xi}\left(\frac{1}{\overline{K}}\begin{Vmatrix} 0 & \overline{x}_\xi\dfrac{\partial}{\partial\xi} + \overline{y}_\xi\dfrac{\partial}{\partial\eta} \\ 0 & \overline{x}_\eta\dfrac{\partial}{\partial\xi} + \overline{y}_\eta\dfrac{\partial}{\partial\eta} \end{Vmatrix} - \begin{Vmatrix} \overline{x}_\xi^2 + \overline{y}_\xi^2 & \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta \\ \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta & \overline{x}_\eta^2 + \overline{y}_\eta^2 \end{Vmatrix}\left(\overline{x}_\xi\frac{\partial}{\partial\eta} - \overline{x}_\eta\frac{\partial}{\partial\xi}\right)\right), \tag{33c}$$

and

$$Z_K^{\alpha\beta} = -\frac{1}{\overline{K}}\begin{Vmatrix} \overline{x}_\xi^2 + \overline{y}_\xi^2 & \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta \\ \overline{x}_\xi\overline{x}_\eta + \overline{y}_\xi\overline{y}_\eta & \overline{x}_\eta^2 + \overline{y}_\eta^2 \end{Vmatrix}. \tag{33d}$$

Hence, up to second order, the correlation matrices in (30) can be expressed in terms of the cross-correlations between $\tilde{x}$, $\tilde{y}$, and $\tilde{K}$. In turn, (15) relates the statistics of $\tilde{x}$, and $\tilde{y}$ to the statistics of the interface fluctuations $\tilde{x}_\gamma$ and $\tilde{y}_\gamma$.

6. Statistical Moments of the Interface Dynamics. The ensemble averaging of (24) yields

$$\frac{\partial \overline{x}_\gamma}{\partial t} = \overline{\chi}^x \frac{\partial \overline{h}}{\partial \xi} + \left\langle \tilde{\chi}^x \frac{\partial \tilde{h}}{\partial \xi} \right\rangle . \tag{34}$$

To derive an approximate expression for the mixed moment in (34), we note that, up to second order in the conductivity fluctuations, $\tilde{\chi}^x$ $\left(\chi^x = \overline{\chi}^x + \tilde{\chi}^x\right)$ can be written as

$$\tilde{\chi}^x(\tilde{x}_\gamma, \tilde{y}_\gamma, \tilde{K}) \approx \chi_k^x \tilde{K} + \chi_x^x \tilde{x}_\gamma + \chi_y^x \tilde{y}_\gamma , \tag{35a}$$

where $\chi_k^x, \chi_x^x, \chi_y^x$ are deterministic coefficients,

$$\chi_x^x = \frac{\overline{K}\overline{y}_{\gamma\eta}}{n_e\left(\overline{x}_{\gamma\xi}\overline{y}_{\gamma\eta} - \overline{x}_{\gamma\eta}\overline{y}_{\gamma\xi}\right)^2}\left(\overline{y}_{\gamma\xi}\frac{\partial}{\partial \eta} - \overline{y}_{\gamma\eta}\frac{\partial}{\partial \xi}\right), \tag{35b}$$

$$\chi_y^x = \frac{\overline{K}\overline{y}_{\gamma\eta}}{n_e\left(\overline{x}_{\gamma\xi}\overline{y}_{\gamma\eta} - \overline{x}_{\gamma\eta}\overline{y}_{\gamma\xi}\right)^2}\left(\frac{\overline{x}_{\gamma\xi}\overline{y}_{\gamma\eta} - \overline{x}_{\gamma\eta}\overline{y}_{\gamma\xi}}{\overline{y}_{\gamma\eta}}\frac{\partial}{\partial \eta} - \overline{x}_{\gamma\xi}\frac{\partial}{\partial \eta} + \overline{x}_{\gamma\eta}\frac{\partial}{\partial \xi}\right), \tag{35c}$$

and

$$\chi_K^x = \frac{\overline{y}_{\gamma\eta}}{n_e\left(\overline{x}_{\gamma\xi}\overline{y}_{\gamma\eta} - \overline{x}_{\gamma\eta}\overline{y}_{\gamma\xi}\right)} . \tag{35d}$$

Subtracting (34) from (24) and omitting the terms of orders higher than second gives

$$\frac{\partial \tilde{x}_\gamma}{\partial t} = \overline{\chi}^x \frac{\partial \tilde{h}}{\partial \xi} + \tilde{\chi}^x \frac{\partial \overline{h}}{\partial \xi} . \tag{36}$$

Substituting (35) and a similar expansion for the hydraulic head fluctuations $\tilde{h}$ into (36) leads to

$$\frac{\partial \tilde{x}_\gamma(\eta_1)}{\partial t} = X_k(\eta_1, \mathbf{v})\tilde{K}(\mathbf{v}) + X_x(\eta_1, \eta_3)\tilde{x}_\gamma(\eta_3) + X_y(\eta_1, \eta_3)\tilde{y}_\gamma(\eta_3) , \tag{37a}$$

where the deterministic coefficients X_k, X_x, X_y are given by

$$X_\delta = \chi_\delta^x \frac{\partial \overline{h}}{\partial \xi^1} - \overline{\chi}^x \int_0^1\int_0^1 \frac{\partial^2 E(\boldsymbol{\xi} \mid \boldsymbol{\xi}_1)}{\partial \xi^1 \partial \xi_1^\alpha}\left(\overline{Z}^{-1}\right)^{\alpha\alpha_1} Z_\delta^{\alpha_1\beta}\overline{q}^\beta d\xi_1^1 d\xi_1^2 , \qquad \delta = k, x, y \tag{37b}$$

Similarly, an equation for $\tilde{y}_\gamma$ is

$$\frac{\partial \tilde{y}_\gamma(\eta_1)}{\partial t} = Y_k(\eta_1, \mathbf{v})\tilde{K}(\mathbf{v}) + Y_x(\eta_1, \eta_3)\tilde{x}_\gamma(\eta_3) + Y_y(\eta_1, \eta_3)\tilde{y}_\gamma(\eta_3) . \tag{38}$$

Equations for the covariances $C_{xx}^{\gamma}(\eta_1,\eta_2)=\left\langle \tilde{x}_{\gamma}(\eta_1)\tilde{x}_{\gamma}(\eta_2)\right\rangle$ and $C_{yy}^{\gamma}(\eta_1,\eta_2)=\left\langle \tilde{y}_{\gamma}(\eta_1)\tilde{y}_{\gamma}(\eta_2)\right\rangle$, and the cross-covariance $C_{xy}^{\gamma}(\eta_1,\eta_2)=\left\langle \tilde{x}_{\gamma}(\eta_1)\tilde{y}_{\gamma}(\eta_2)\right\rangle$ are derived from (37) and (38) by noting that

$$\frac{\partial C_{xx}^{\gamma}(\eta_1,\eta_2)}{\partial t}=\left\langle \frac{\partial \tilde{x}_{\gamma}(\eta_1)}{\partial t}\tilde{x}_{\gamma}(\eta_2)\right\rangle+\left\langle \frac{\partial \tilde{x}_{\gamma}(\eta_2)}{\partial t}\tilde{x}_{\gamma}(\eta_1)\right\rangle, \tag{39a}$$

and

$$\frac{\partial C_{xy}^{\gamma}(\eta_1,\eta_2)}{\partial t}=\left\langle \frac{\partial \tilde{x}_{\gamma}(\eta_1)}{\partial t}\tilde{y}_{\gamma}(\eta_2)\right\rangle+\left\langle \frac{\partial \tilde{y}_{\gamma}(\eta_2)}{\partial t}\tilde{x}_{\gamma}(\eta_1)\right\rangle. \tag{39b}$$

This gives

$$\begin{aligned}\frac{\partial C_{xx}^{\gamma}(\eta_1,\eta_2)}{\partial t}&=X_k(\eta_1,\mathbf{v})C_{Kx}^{\gamma}(\mathbf{v},\eta_2)+X_x(\eta_1,\eta_3)C_{xx}^{\gamma}(\eta_2,\eta_3)+X_y(\eta_1,\eta_3)C_{yy}^{\gamma}(\eta_2,\eta_3)\\&+X_k(\eta_2,\mathbf{v})C_{Kx}^{\gamma}(\mathbf{v},\eta_1)+X_x(\eta_2,\eta_3)C_{xx}^{\gamma}(\eta_1,\eta_3)+X_y(\eta_2,\eta_3)C_{yy}^{\gamma}(\eta_1,\eta_3),\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\partial C_{xy}^{\gamma}(\eta_1,\eta_2)}{\partial t}&=X_k(\eta_1,\mathbf{v})C_{Ky}^{\gamma}(\mathbf{v},\eta_2)+X_x(\eta_1,\eta_3)C_{xy}^{\gamma}(\eta_3,\eta_2)+X_y(\eta_1,\eta_3)C_{yy}^{\gamma}(\eta_2,\eta_3)\\&+X_k(\eta_2,\mathbf{v})C_{Kx}^{\gamma}(\mathbf{v},\eta_1)+X_x(\eta_2,\eta_3)C_{xx}^{\gamma}(\eta_1,\eta_3)+X_y(\eta_2,\eta_3)C_{xy}^{\gamma}(\eta_1,\eta_3),\end{aligned} \tag{41}$$

and an equation for $C_{yy}^{\gamma}(\eta_1,\eta_2)$, which is analogous to (40).

We start the derivation of approximate solutions for the cross-covariances $C_{Kx}^{\gamma}(\mathbf{v},\eta)=\left\langle \tilde{K}(\mathbf{v})\tilde{x}_{\gamma}(\eta)\right\rangle$ and $C_{Ky}^{\gamma}(\mathbf{v},\eta)=\left\langle \tilde{K}(\mathbf{v})\tilde{y}_{\gamma}(\eta)\right\rangle$ by noting that second-order approximations require $\tilde{K}=\tilde{K}\left(\overline{x},\overline{y}\right)$, while the dependence of conductivity fluctuations, $\tilde{K}$, on $\tilde{x}$ and $\tilde{y}$ manifests itself in the third- and higher-order terms. Then

$$\frac{\partial \tilde{K}}{\partial t}=\frac{\partial \overline{x}}{\partial t}\frac{\partial \tilde{K}}{\partial \overline{x}}+\frac{\partial \overline{y}}{\partial t}\frac{\partial \tilde{K}}{\partial \overline{y}} \tag{42}$$

and (42) can be written in the following form

$$\frac{\partial \tilde{K}}{\partial t}=\hat{C}\tilde{K}. \tag{43a}$$

It follows from (9) that the deterministic linear operator $\hat{C}$ has the form

$$\hat{C}=\mathbf{U}\frac{\partial}{\partial \boldsymbol{\xi}}, \tag{43b}$$

where $\mathbf{U}=(U_1,U_2)^T$,

$$U_1 = \frac{\bar{y}_{\xi^2}\dfrac{\partial \bar{x}}{\partial t} - \bar{x}_{\xi^2}\dfrac{\partial \bar{y}}{\partial t}}{\bar{x}_{\xi^1}\bar{y}_{\xi^2} - \bar{x}_{\xi^2}\bar{y}_{\xi^1}}, \qquad U_2 = \frac{-\bar{y}_{\xi^1}\dfrac{\partial \bar{x}}{\partial t} + \bar{x}_{\xi^1}\dfrac{\partial \bar{y}}{\partial t}}{\bar{x}_{\xi^1}\bar{y}_{\xi^2} - \bar{x}_{\xi^2}\bar{y}_{\xi^1}}, \tag{43c}$$

and

$$\frac{\partial}{\partial t}\begin{pmatrix}\bar{x}(\boldsymbol{\xi},t)\\ \bar{y}(\boldsymbol{\xi},t)\end{pmatrix} = -\int_0^1 \frac{\partial L(\boldsymbol{\xi} \mid \xi_1 = 1, \eta_1)}{\partial \xi_1}\frac{\partial}{\partial t}\begin{pmatrix}\bar{x}_\gamma(\eta_1,t)\\ \bar{y}_\gamma(\eta_1,t)\end{pmatrix} d\eta_1 . \tag{43d}$$

Combining (43) and (37) gives

$$\frac{\partial C^\gamma_{Kx}(\mathbf{v},\eta)}{\partial t} = \widehat{C}(\mathbf{v},\mathbf{v}_1)C^\gamma_{Kx}(\mathbf{v}_1,\eta) + X_k(\eta,\mathbf{v}_1)\rho_K(\mathbf{v},\mathbf{v}_1) + X_x(\eta,\eta_1)C^\gamma_{Kx}(\mathbf{v},\eta_1) + X_y(\eta,\eta_1)C^\gamma_{Ky}(\mathbf{v},\eta_1) . \tag{44}$$

Similarly, combining (43) and (38) leads to an equation for $C^\gamma_{Ky}(\mathbf{v},\eta)$.

Equations (40), (41), and (44) are subject to the homogeneous initial conditions. The conductivity correlation function in the moving coordinate system $\rho_K(\mathbf{v},\mathbf{v}_1)$ at time *t* is computed, to second order in the conductivity fluctuations, at the mean coordinates.

7. Numerical Algorithm. We present a detailed description of our numerical algorithm and its implementation in a companion paper. A brief outline is as follows.

- Given the mean location of the interface at time *t*, solve (3) – (4) to construct the dynamic mapping $\Omega \to W$.
- Compute the correlation matrices in the mean Darcy law (30).
- Solve the mean flow equation and the second moment equation (Appendix A) to obtain the mean hydraulic head $\bar{h}$ and the hydraulic head variance σ_h^2.
- Calculate the mean velocity of the interface from (34) and compute the mean position of the interface at time $t + \Delta t$.
- Solve equations (40), (41), and (44) to obtain cross-covariances C^γ_{xx}, C^γ_{yy}, C^γ_{xy}, and C^γ_{Ky} at time $t + \Delta t$.
- Repeat calculations.

8. A Computational Example. While in general our moment equations have to be solved numerically, some flow scenarios are amenable to analytical treatment. Consider the one-dimensional front propagation in a randomly heterogeneous porous medium of log-normal hydraulic

conductivity, $K = \ln Y$, with the geometric mean $K_g = \exp(\overline{Y})$, variance σ_Y^2, correlation function ρ_Y, and correlation length λ. The front is driven by hydraulic head $h(0) = H_0$ imposed at the boundary $x = 0$.

We recast the problem in a dimensionless form by introducing

$$x_d = \frac{x}{\lambda}, \qquad t_d = 2\frac{tK_g H_0}{n_e \lambda^2}, \qquad K_d = \frac{K}{K_g}, \qquad \text{and} \qquad h_d = \frac{h}{H_0}. \tag{45}$$

In the following, we drop the subscript *d*.

8.1. Mapping. Solving the one-dimensional version of (3) yields a mapping

$$x = x_\gamma \xi, \tag{46a}$$

whose Jacobian is

$$J = \frac{\partial x}{\partial \xi} = x_\gamma. \tag{46b}$$

8.2. Transformed flow equations. The mapping (46) transforms the flow equations into the one-dimensional version of (27),

$$\frac{\partial q}{\partial \xi} = 0, \qquad \frac{\partial h}{\partial \xi} = -Zq, \qquad h(0) = 1, \qquad h(1) = 0 \tag{47a}$$

Hydraulic resistivity $Z \equiv Z^{11}$ is obtained from (22) and (26) as

$$Z = x_\gamma K^{-1}. \tag{47b}$$

8.3. Statistical moments of head. The one-dimensional version of (28) gives

$$\frac{\partial \overline{q}}{\partial \xi} = 0, \qquad \frac{\partial \overline{h}}{\partial \xi} = -\overline{Z}\overline{q} - \left\langle \tilde{Z}\tilde{q} \right\rangle, \qquad \overline{h}(0) = 1, \qquad \overline{h}(1) = 0. \tag{48}$$

We seek the first-order approximation of the mean head, i.e., $\overline{h} = \overline{h}^{(0)} + \overline{h}^{(1)} + O(\sigma_Y^4)$.

It follows from (47b), $\overline{K} = K_g(1 + \sigma_Y^2/2) + O(\sigma_Y^4)$, and $\left\langle \tilde{K}^2 \right\rangle = \sigma_Y^2 + O(\sigma_Y^4)$, that

$$\overline{Z} = \overline{x}_\gamma \overline{z} + h.o.t., \qquad \text{and} \qquad \tilde{Z} = \overline{x}_\gamma \tilde{z} + h.o.t., \tag{49a}$$

where

$$\bar{z} = 1 + \frac{\sigma_Y^2}{2} - \Phi - \xi \frac{\partial \Phi}{\partial \xi}, \qquad \tilde{z} = \frac{\tilde{x}_\gamma}{\bar{x}_\gamma} - \tilde{K}, \tag{49b}$$

and $\Phi = \left\langle \tilde{K}\tilde{x}_\gamma \right\rangle / \bar{x}_\gamma$ and $\tilde{K} = \tilde{K}\left(\bar{x}(\xi)\right)$. Within the first-order framework, the random field $z = \bar{z} + \tilde{z}$ represents a normalized dimensionless hydraulic resistivity, i.e., $z = Z / \bar{x}_\gamma$.

The one-dimensional Green's function in (32) has the form

$$E\left(\xi \mid \xi_1\right) = \left(\xi_1 - \xi\right)\theta\left(\xi - \xi_1\right) + \left(1 - \xi_1\right)\xi, \tag{50}$$

where $\theta\left(\xi\right)$ is the Heaviside function defined as $\theta = 1$ for $\xi \geq 0$ and $\theta = 0$ otherwise. Substituting (50) into the one-dimensional versions of (30) and (31), and introducing $Q = \bar{x}_\gamma q$, yields a solution for the mean hydraulic head,

$$-\frac{\partial \bar{h}}{\partial \xi} = \bar{Q}\left(\bar{z}(\xi) - \int_0^1 C_z(\xi, \nu) d\nu \right), \qquad \frac{\partial \bar{Q}}{\partial \xi} = 0. \tag{51}$$

Integrating the first equation in (51), while taking into account the boundary conditions (47a), yields

$$\bar{Q} = \left(\int_0^1 \bar{z}(\nu) d\nu - \int_0^1 \int_0^1 C_z(\mu, \nu) d\mu d\nu \right)^{-1}. \tag{52}$$

It follows from (49b) that the covariance function $C_z(\mu, \nu) = \left\langle \tilde{z}(\mu)\tilde{z}(\nu) \right\rangle$ in (51) and (52) is given by

$$C_z(\mu, \nu) = \rho_Y\left(\bar{x}_\gamma(\mu - \nu)\right) - \Phi(\mu) - \Phi(\nu) + r_\gamma, \tag{53}$$

where $r_\gamma = \sigma_\gamma^2 / \bar{x}_\gamma^2$ is the square of the coefficient of variation of the interface, $\sigma_\gamma^2 = \left\langle \tilde{x}_\gamma^2 \right\rangle$ being the variance of the interface position.

By the same token, substituting (50) into the one-dimensional versions of (A3) and (A4) yields an expression for the random fluctuations of hydraulic head,

$$\tilde{h}(\xi) = \bar{Q}\left(\xi \int_0^1 \tilde{z}(\nu) d\nu - \int_0^\xi \tilde{z}(\nu) d\nu \right), \qquad \tilde{Q} = -\bar{Q} \int_0^1 \tilde{z}(\nu) d\nu. \tag{54}$$

Hence, the variances of hydraulic head and flux are given by

$$\sigma_h^2 = \bar{Q}^2 \left(\xi^2 \int_0^1 \int_0^1 C_z(\mu, \nu) d\mu d\nu - 2\xi \int_0^1 \int_0^\xi C_z(\mu, \nu) d\mu d\nu + \int_0^\xi \int_0^\xi C_z(\mu, \nu) d\mu d\nu \right) \tag{55}$$

and

$$\sigma_Q^2 = \overline{Q}^2 \int_0^1 \int_0^1 C_z(\mu,\nu)\, d\mu\, d\nu\,, \tag{56}$$

respectively.

8.4. Statistical moments of the interface dynamics. The equations of motion can now be written as

$$\frac{\partial \overline{x}_\gamma}{\partial t} = \frac{\overline{Q}(\xi=1)}{2\overline{x}_\gamma}\,, \qquad \frac{\partial \widetilde{x}_\gamma}{\partial t} = \frac{\widetilde{Q}(\xi=1)}{2\overline{x}_\gamma}\,. \tag{57}$$

Recalling the definition of $\tilde{z}$ in (49b), it follows from (57) that

$$\frac{\partial}{\partial t}\left(\frac{\widetilde{x}_\gamma}{\overline{x}_\gamma}\right) = -\frac{\overline{Q}}{2\overline{x}_\gamma^2}\left(\int_0^1 \widetilde{z}(\nu)\, d\nu + \frac{\widetilde{x}_\gamma}{\overline{x}_\gamma}\right) = \frac{\overline{Q}}{2\overline{x}_\gamma^2}\left(\int_0^1 \widetilde{K}[\overline{x}(\nu)]\, d\nu - 2\frac{\widetilde{x}_\gamma}{\overline{x}_\gamma}\right). \tag{58}$$

The one-dimensional version of (43) gives an equation for the conductivity fluctuations,

$$\frac{\partial \widetilde{K}}{\partial t} = \frac{\xi}{\overline{x}_\gamma}\frac{\partial \overline{x}_\gamma}{\partial t}\frac{\partial \widetilde{K}}{\partial \xi} = \frac{\overline{Q}\,\xi}{2\overline{x}_\gamma^2}\frac{\partial \widetilde{K}}{\partial \xi}\,. \tag{59}$$

Combining (58) and (59) leads to equations for the covariances r_γ and Φ,

$$\frac{\partial r_\gamma}{\partial t} = \frac{\overline{Q}}{\overline{x}_\gamma^2}\left[\int_0^1 \Phi(\nu)\, d\nu - 2r_\gamma\right] \tag{60}$$

and

$$\frac{\partial \Phi}{\partial t} - \frac{\overline{Q}\,\xi}{2\overline{x}_\gamma^2}\frac{\partial \Phi}{\partial \xi} = \frac{\overline{Q}}{2\overline{x}_\gamma^2}\left[\int_0^1 \rho_Y\big(\overline{x}(\xi), \overline{x}(\nu)\big)\, d\nu - 2\Phi(\xi)\right]. \tag{61}$$

For the homogeneous initial conditions, the solutions of (60) and (61) are

$$r_\gamma = \int_0^1 \int_0^1 (1-\xi)(1-\eta)\rho_Y\big(\overline{x}_\gamma(\xi-\eta)\big)\, d\xi\, d\eta \tag{62}$$

and

$$\Phi(\xi) = \int_0^1 (1-\eta)\rho_Y\big(\overline{x}_\gamma(\xi-\eta)\big)\, d\eta\,, \tag{63}$$

respectively. Equations (51) - (53), (55) - (57), (62), and (63) form a closed set of deterministic equations for the statistics of the interface dynamics and related state variables.

8.5. Comparison with the exact solution. The one-dimensional flow equations (47) can be integrated exactly to obtain

$$h = 1 - Q^{-1}\int_0^{\xi} z(\nu)d\nu\,, \qquad Q^{-1} = \int_0^1 z(\nu)d\nu\,. \tag{64}$$

The statistics for the interface dynamics can be computed exactly for perfectly correlated porous media ($\lambda \to \infty$), so that hydraulic conductivity K and the normalized hydraulic resistivity z become random constants [8]. Then (64) yields [8]

$$h = 1 - \frac{x}{x_\gamma}, \qquad q = \frac{K}{x_\gamma}, \qquad x_\gamma^2 = Kt\,, \tag{65}$$

so that the mean and variance of the interface position are given by

$$\left[\bar{x}_\gamma\right]_{exact} = \mathrm{e}^{\sigma_Y^2/8}\sqrt{t} \qquad \text{and} \qquad \left[\sigma_\gamma^2\right]_{exact} = \left(\mathrm{e}^{\sigma_Y^2/2} - \mathrm{e}^{\sigma_Y^2/4}\right)t\,, \tag{66}$$

respectively. Thus the mean position of the interface scales as $\sqrt{t}$, while its variance is linearly proportional to t. Additionally, the normalized cross-covariance $\Phi = \left\langle \tilde{K}\tilde{x}_\gamma \right\rangle / \bar{x}_\gamma$ has the form

$$\left[\Phi\right]_{exact} = \mathrm{e}^{9\sigma_Y^2/8} - \mathrm{e}^{5\sigma_Y^2/8} \tag{67}$$

and is time invariant.

We now proceed to compare our first-order perturbation solutions derived in the previous section with their exact counterparts. Since for perfectly correlated media $C_Y = \sigma_Y^2$, (60) and (61) yield

$$r_\gamma = \frac{\sigma_Y^2}{4} \qquad \text{and} \qquad \Phi = \frac{\sigma_Y^2}{2}\,, \tag{69}$$

respectively. Then it follows from (49b) that $\bar{z} = 1$ and $\sigma_z^2 \equiv \rho_Y - 2\Phi + r_\gamma = \sigma_Y^2/4$, so that (52) gives

$$\overline{Q} = \left(1 - \frac{\sigma_Y^2}{4}\right)^{-1}. \tag{70}$$

Combining (57) and (70) results in

$$\bar{x}_\gamma = \sqrt{\overline{Q}t} = \sqrt{\frac{t}{1-\sigma_Y^2/4}} = \left(1 + \frac{\sigma_Y^2}{8} + O(\sigma_Y^4)\right)\sqrt{t} \tag{71}$$

and

$$\sigma_\gamma^2 \equiv \bar{x}_\gamma^2 r_\gamma = \frac{\sigma_Y^2}{4}\left(1 + \frac{\sigma_Y^2}{8}\right)^2 t = \left(\frac{\sigma_Y^2}{4} + O(\sigma_Y^4)\right)t\,. \tag{72}$$

It is easy to see that the perturbation solutions (71) and (72) for the statistics of the interface dynamics coincide, up to second order in the log conductivity variance σ_Y^2, with their exact counterparts (66). While these perturbation solutions are formally valid for $\sigma_Y^2 < 1$, this comparison shows that they remain accurate for σ_Y^2 as large as 2. Additionally, up to any order in σ_Y^2 they have the same scaling laws.

Another advantage of the proposed approach is that it involves relative fluctuations of the dependent and independent random fields, e.g., $\tilde{x}_\gamma / \bar{x}_\gamma$ and $\tilde{Q}/\overline{Q}$, rather than their absolute counterparts, e.g., $\tilde{x}_\gamma$ and $\tilde{Q}$. The former can be small even when the latter are large, which is important for the accuracy of our perturbative solutions. In particular, the coefficient of variation $\rho_\gamma \equiv \left\langle \tilde{x}_\gamma^2 / \bar{x}_\gamma^2 \right\rangle = \sigma_Y^2 / 4$ remains smaller than unity as long as $\sigma_Y^2 < 4$, while the corresponding variance $\left\langle \tilde{x}_\gamma^2 \right\rangle = t\sigma_Y^2 / 4$ increases with time t and, hence, can be arbitrary large.

9. Summary. We considered interface dynamics in heterogeneous environments with uncertain parameters. While our approach is applicable to a wide range of applications, we formulate it in terms of free-surface flow in porous media. To predict the evolution of a fluid-fluid interface and to quantify the uncertainty associated with such a prediction, we treated the hydraulic conductivity (permeability) of a porous medium as random and the corresponding governing equations as stochastic. The previous attempts to address this problem involve mathematical objects -- such as integrals of random functions over random domains and random functionals -- that are not readily amenable to standard perturbation techniques. To overcome this difficulty, we introduced a dynamic stochastic mapping of the domain with moving boundaries onto a fixed domain. This allowed us to use the well-understood ensemble averaging approaches to derive deterministic differential equations for the statistical moments of hydraulic head, Darcian flux, and interface dynamics.

We used perturbation expansions in a small parameter σ_Y^2, the variance of log hydraulic conductivity, to derive closure approximations for these moment equations. This formally limits the applicability of our approach to mildly heterogeneous porous media ($\sigma_Y^2 < 1$). However, the comparison of analytical solutions of the one-dimensional moment equations with their exact solutions demonstrates that the perturbation approximations remain accurate for σ_Y^2 as large as 4.

We demonstrated that, for one-dimensional flow, the mean position of the interface scales as $\sqrt{t}$, while its variance is linearly proportional to t. These findings are in line with previous studies [8].

Appendix A. Mixed Moments. To derive the second-order approximation of $\left\langle \tilde{Z}^{\alpha\beta}\tilde{q}^{\beta} \right\rangle$ in (24), we consider an equation for the hydraulic head and flux fluctuations, which is obtained by subtracting (24) and (25) from (23) and (22) and retaining the terms up to $\tilde{Z}^2$-order,

$$\frac{\partial}{\partial \xi^{\alpha}}\tilde{q}^{\alpha} = 0, \qquad -\frac{\partial \tilde{h}}{\partial \xi^{\alpha}} = \overline{Z}^{\alpha\beta}\tilde{q}^{\beta} + \tilde{Z}^{\alpha\beta}\overline{q}^{\beta} \tag{A1}$$

subject to the boundary conditions

$$\tilde{h}\left(\xi^1 = 0, \xi^2\right) = 0, \ \tilde{h}\left(\xi^1 = 1, \xi^2\right) = 0, \ \tilde{q}^2\left(\xi^1, \xi^2 = 0\right) = 0, \ \tilde{q}^2\left(\xi^1, \xi^2 = 1\right) = 0. \tag{A2}$$

In terms of $E(\xi_1 \,|\, \xi_2)$, the Green's function defined by (28), the solution of (A1) – (A2) is

$$\tilde{h}(\xi) = -\int_0^1\int_0^1 \frac{\partial E(\xi \,|\, \xi_1)}{\partial \xi_1^{\alpha}}\left(\overline{Z}^{-1}\right)^{\alpha\alpha_1} \tilde{Z}^{\alpha_1\beta}\overline{q}^{\beta} d\xi_1^1 d\xi_1^2 . \tag{A3}$$

It follows from (21), (22), and (A3) that

$$\tilde{q}^{\alpha}(\xi) = -\left(\overline{Z}^{-1}\right)^{\alpha\alpha_1} \tilde{Z}^{\alpha_1\beta}\overline{q}^{\beta}(\xi) + \left(\overline{Z}^{-1}\right)^{\alpha\alpha_1} \int_0^1\int_0^1 \frac{\partial^2 E(\xi \,|\, \xi_1)}{\partial \xi^{\alpha_1}\partial \xi_1^{\alpha_2}}\left(\overline{Z}^{-1}\right)^{\alpha_2\beta_2} \tilde{Z}^{\beta_2\beta}\overline{q}^{\beta} d\xi_1^1 d\xi_1^2 . \tag{A4}$$

Substituting (A4) into (24) gives the non-local mean Darcy's law (30). (Co)variances of the hydraulic head and the Darcy flux are obtained by squaring (A3) and (A4) and taking the ensemble mean.

Appendix B. Equations for the Interface Dynamics. The mapping $\Omega \to W$ transforms the dynamics conditions on the interface (2) into

$$\mathbf{n}_{\gamma} \cdot \mathbf{q} = V_n . \tag{B1}$$

and

$$\frac{\partial x_{\gamma}}{\partial t} = \frac{V_x(x_{\gamma}, y_{\gamma})}{n_e(x_{\gamma}, y_{\gamma})}, \qquad \frac{\partial y_{\gamma}}{\partial t} = \frac{V_y(x_{\gamma}, y_{\gamma})}{n_e(x_{\gamma}, y_{\gamma})} . \tag{B2}$$

Here $x_{\gamma} = x_{\gamma}(\eta, t)$, $y_{\gamma} = y_{\gamma}(\eta, t)$, and the normal to the free surface is given by

$$\mathbf{n}_{\gamma} = \frac{\nabla\gamma}{|\nabla\gamma|} = \frac{(\partial y_{\gamma}/\partial\eta)\mathbf{e}_x - (\partial x_{\gamma}/\partial\eta)\mathbf{e}_y}{\sqrt{(\partial x_{\gamma}/\partial\eta)^2 + (\partial y_{\gamma}/\partial\eta)^2}}, \tag{B3}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ denote the unit vectors in the Cartesian (x, y) coordinate system.

Substituting (B3) and the Darcy flux (19) evaluated at the interface into (B1) yields

$$V_n = -\frac{K}{J\sqrt{(\partial x_\gamma / \partial \eta)^2 + (\partial y_\gamma / \partial \eta)^2}} \left\{ \left[\left(\frac{\partial x_\gamma}{\partial \eta} \right)^2 + \left(\frac{\partial y_\gamma}{\partial \eta} \right)^2 \right] \frac{\partial h}{\partial \xi} - \left(\frac{\partial x_\gamma}{\partial \xi} \frac{\partial x_\gamma}{\partial \eta} + \frac{\partial y_\gamma}{\partial \xi} \frac{\partial y_\gamma}{\partial \eta} \right) \frac{\partial h}{\partial \eta} \right\}. \tag{B4}$$

Since the interface is an equipotential, i.e. a surface of the constant hydraulic head, the tangential derivative of the hydraulic head $\partial h / \partial \eta \equiv 0$. Hence it follows from (B4) and (B3) that the components of the interface velocity vector $\mathbf{V} = V_n \mathbf{n}_\gamma$ are given by

$$V_x = -\frac{K}{J} \frac{\partial y_\gamma}{\partial \eta} \frac{\partial h}{\partial \xi} \quad \text{and} \quad V_y = \frac{K}{J} \frac{\partial x_\gamma}{\partial \eta} \frac{\partial h}{\partial \xi}. \tag{B5}$$

Substituting (B5) into (B2 gives (24).

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References

[1] Dagan, G., Flow and Transport in Porous Formations, 465 pp., Springer-Verlag, 1989.

[2] Gelhar, L.W., Stochastic Subsurface Hydrology, 390 pp., Prentice Hall, 1994.

[3] Cushman, J. H, The Physics of Fluids in Hierarchical Porous Media: Angstroms to Miles, Kluwer, 1997.

[4] Dagan, G. & Neuman, S. P. (ed.), Subsurface Flow and Transport: a Stochastic Approach, Cambridge University Press, 1997.

[5] Dagan, G. and D. G. Zeitoun, Seawater-freshwater interface in a stratified aquifer of random permeability distribution, J. Contam. Hydrol., **29**, 185-203, 1998.

[6] Amir O. and G. Dagan, Uniform steady free-surface flow in heterogeneous porous formations. J. Fluid Mech., **454** (2002), 327-343.

[7] Tartakovsky, D. M., Stochastic analysis of heterogeneous phreatic aquifers, Water Resour. Res., **35**(12), 3941-3946, 1999.

[8] D. M. Tartakovsky, C. L. Winter, Dynamics of free surfaces in random porous media, SIAM J. Appl. Math., **61** (2001), 1857-1876.

[9] W. H. Green and G. A. Ampt, Studies in soil physics, I. The flow of air and water through soil, J. Agr. Sci., **4** (1991), pp.1-24.

[10] V. M. Goloviznin, O. G. Simacheva, On one method of constructing computational grids in domains with curved boundaries, J. of Computational Mathematics and Mathematical Physics., **5** (1983), pp. 1245-1249 [in Russian].