Pergamon

Comp. Maths Math. Phys., Vol. 33, No. 11, pp. 1535–1541, 1993

0965–5425/93 $24.00 + 0.00

0965-5425(94)E0005-4

ESTIMATION OF TRACER MIGRATION TIME IN GROUND WATER FLOW†

A. R. Kasimov and D. M. Tartakovskii

Kazan

(*Received* 8 *December* 1992; *revised version received* 6 *May* 1993)

The problem of the spread of contamination by seepage is solved by optimizing the shape of the channel bed for the objective function — the particle travel time along a streamline with integral constraints, the flow rate and cross-sectional area of the reservoir. The problem can be reduced to the solution of Dirichlet's problem by means of an integral representation of the required analytic function, series expansion of the kernel of the Cauchy integral and the determination of the coefficients of the series from the extremum condition.

The travel time of a tracer particle along streamlines is an important parameter of the process of the spread of contamination by seepage [1–4]. Isoperimetric estimates of this time are obtained below by solving shape optimization problems [5, 6]. The objective functional is the travel time of the tracer along the depression curve, the free surface, and control is effected by the shape of the reservoir from which seepage of the contaminant occurs, and the integral constraint is the flow rate and cross-sectional area of the reservoir. The solution technique involves the integral representation of Dirichlet's boundary-value problem for an analytic function, series expansion of the kernel of Cauchy's integral, and determination of the coefficients of this series from the necessary conditions of an extremum either in explicit form, or from the solution of infinite systems of linear equations [7].

1. STATEMENT OF THE PROBLEM

Consider plane, steady filtration by Darcy's law from a reservoir BC with a vertical axis of symmetry, with free surfaces AB and DC (Fig.1a). The ground is assumed to be homogeneous and isotropic, and to have porosity m and filtration coefficient $\varkappa$. A contaminant has fallen into the reservoir and starts to enter the ground with the filtration flow. We shall assume that the process can be described by the "piston-like displacement" model ("neutral tracer" or "coloured liquid") [1, 2], that is, the contaminant moves along streamlines with a velocity determined from the solution of the hydrodynamic problem (for the limits of applicability of this model and a review of models which allow for dispersion, see [1, 8]).

From a practical point of view, the important questions are: how long does it take for the contaminant to permeate to a prescribed depth, what kind of time dependence has the concentration at a prescribed depth (breakthrough curve), and how does the geometry of the reservoir influence the dynamics of the contaminant? To answer these questions, we will examine the problem of

† *Zh. Vychisl. Mat. Mat. Fiz.* Vol. 33, No. 11, pp. 1751–1759, 1993.

The research was funded by the "Special Fund for the Award of Personal Scholarships and Grants to Gifted Young Academics".

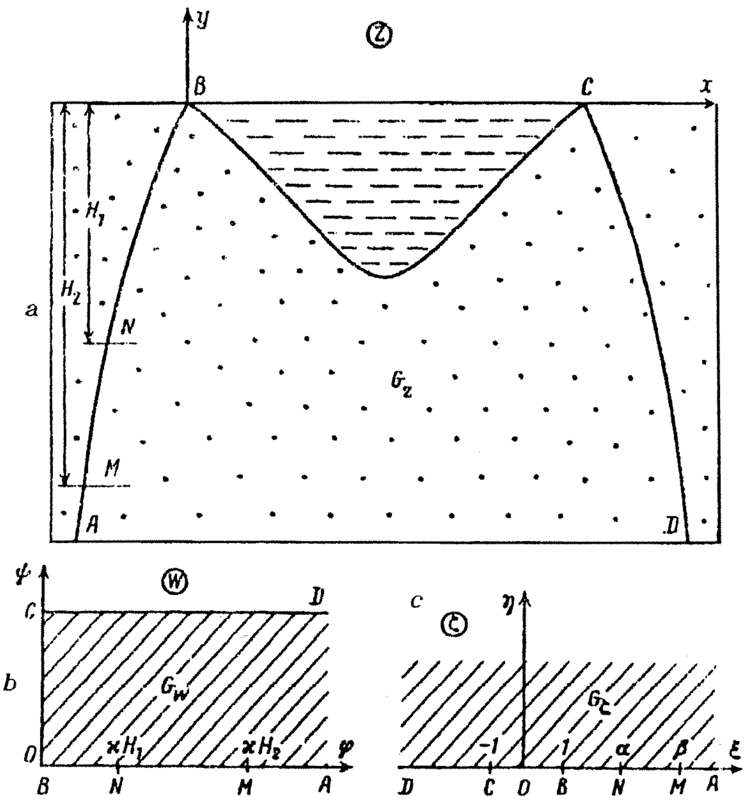

FIG. 1.

optimizing the shape BC, interpreting the solutions, obtained in the same way [9–13], as isoperimetric inequalities [14–16].

We shall assume the two horizontal levels H_1 and H_2 ($H_1 < H_2$), measured from the water level in the reservoir, as well as the filtration flow rate Q, are given.

In this model, the flow rate $v = \nabla\varphi(x, y)$, where φ is the potential, satisfying Laplace's equation in the flow region G_z. The time taken by the particle to move along the streamline $\psi = \text{const}$, $\psi \in [0, Q]$ between points N and M is given by the formula (see [1])

$$t = m \int_{\varphi_1}^{\varphi_2} v^{-2} (\varphi)_{\psi=\text{const}}\, d\varphi, \tag{1}$$

where φ_1 and φ_2 are the values of the potential at points N and M, and v is the velocity.

We shall consider the motion of a tracer along the depression contour AB, on which, as we know [1, 17], two conditions are satisfied: $\varphi + \varkappa y = 0$ and $\psi = 0$. Hence, in the optimization problem formulated below, the abscissae of points N and M are subject to definition (as is the whole boundary of the domain G_z), while their ordinates are fixed: $y_1 = -H_1$ and $y_2 = -H_2$.

Problem 1. For prescribed $\varkappa$, m, Q, H_1, H_2, it is required to determine the shape of the curve BC such that the travel time t of a particle along AB from level $y = -H_1$ to $y = -H_2$ is an extremum.

2. SHAPE OPTIMIZATION

We will introduce the complex coordinate $z = x + iy$ and complex potential $w = \varphi + i\psi$, in the plane of which the domain G_z corresponds to the half-strip G_w (Fig. 1*b*). We map G_w conformally on to the half-plane $\operatorname{Im} \zeta > 0$ of the variable $\zeta = \xi + i\eta$ (Fig. 1*c*) by the function

$$\zeta = \cos(\pi w / iQ), \tag{2a}$$

from which it follows, for real ζ in particular, that

$$\psi = (Q/\pi) \arccos \xi, \quad |\xi| \le 1, \quad \varphi = (Q/\pi)\,\mathrm{arcch}\,\xi, \quad |\xi| > 1, \tag{2b}$$

the coordinates of the points N and M on the $O\xi$ axis being equal to $\alpha = \mathrm{ch}\,(\varkappa\pi H_1/Q)$ and $\beta = \mathrm{ch}\,(\varkappa\pi H_2/Q)$ respectively.

To solve Problem I, we introduce control of the shape of BC in the form

$$y = y\,[\psi(\xi)], \tag{3}$$

where $\psi(\xi)$ is found from (2), and $y[\psi]$ is assumed to belong to the Hölder class.

For the function $z(\zeta)$ with boundary conditions (3) and $y = -\varphi(\xi)/\varkappa$ for $|\xi| \ge 1$, the solution of Dirichlet's boundary-value problem has the form (see [11, 17])

$$\varkappa z = -iw + \frac{\varkappa}{\pi} \int_{-1}^{1} \frac{y(\psi(\tau))}{\tau - \zeta}\, d\tau. \tag{4}$$

Expanding (3) in series in Chebyshev polynomials and taking the limit in (4) as $\zeta \to \xi \in [-1, 1]$, we obtain the equations of the equipotential BC:

$$x = \frac{\psi(\xi)}{\varkappa} - \frac{Q}{\varkappa}\left[\sum b_{2n-1} T_{2n-1}(\xi) + b_0\right], \quad y = \frac{Q}{\varkappa} \sum b_{2n-1} U_{2n-1}(\xi), \tag{5a}$$

$$T_n(\xi) = \cos(n \arccos \xi), \quad U_n(\xi) = \sin(n \arccos \xi), \quad n = 1, 2, \ldots, \infty \tag{5b}$$

Letting $\zeta \to \xi > 1$, we obtain the equations of the curve BA from (4):

$$x = \frac{Q}{\varkappa} \sum b_{2n-1} (\xi - \sqrt{\xi^2 - 1})^{2n-1}, \quad y = -\varphi(\xi)/\varkappa. \tag{6}$$

The value of the velocity along a streamline is given in the form

$$v^2 = (\partial\varphi/\partial s)^2 = (\partial\varphi/\partial\xi)^2 [(\partial x/\partial\xi)^2 + (\partial y/\partial\xi)^2]^{-1}. \tag{7}$$

Substituting $\partial\varphi/\partial s$ from (2), and $\partial x/\partial\xi$ and $\partial y/\partial\xi$ from (6), for the objective functional (1) we obtain

$$t = -\frac{m}{\varkappa} \int_{\alpha}^{\beta} \frac{\dot{x}^2 + \dot{y}^2}{\dot{y}}\, d\xi = \frac{\pi Q m}{\varkappa^2} \int_{\alpha}^{\beta} \left[\sum (2n-1)\, b_{2n-1} (\xi - \sqrt{\xi^2 - 1})^{2n-1}\right]^2 \times$$
$$\times (\sqrt{\xi^2 - 1})^{-1}\, d\xi + t_h, \quad t_h = m\,(H_2 - H_1)/\varkappa, \tag{8}$$

where the dots above x and y denote differentiation.

Making the replacement of variables $\xi = \mathrm{ch}\,\gamma$ and changing to dimensionless time $t^* = t\varkappa^2/(mQ)$, we obtain from (8):

$$t^* = \pi \int_{\pi H_1^*}^{\pi H_2^*} \left[\sum (2n-1)\, b_{2n-1} e^{-(2n-1)\gamma}\right]^2 d\gamma + t_h^*, \tag{9}$$

where $H_1^* = \varkappa H_1/Q$, $H_2^* = \varkappa H_2/Q$, and the quantity t_h^* gives the tracer travel time in one-dimensional flow in the case where BC is the horizontal segment $y = 0$, and the free surfaces AB and CD are the rays $x = 0$ and $x = L$ respectively.

Since the integral in (9) is positive, it follows at once that Problem 1 has a unique global minimum $t^* = t_h^*$, which is reached when all $b_{2n-1} = 0$. Thus, a solution (a plane horizontal channel) has been obtained in the class of arbitrary profiles for one, an extreme, streamline. We will now find the extremum of the functional of time along an arbitrary streamline, restricting ourselves to the first term in the expansion (5), corresponding to Coseni channels (see [17]). Restriction to one or more terms of the series or a similar restriction to a definite class of optimized shapes in the physical plane in this way often permits a quite close approximation to the optimum [18, 19].

The parametric equations of a Coseni contour (the channel shown in Fig. 1a is for $H^* = 0.4$) have the form

$$x^* = \psi^* - H^* \sin(\pi\psi^*), \quad y^* = -H^* \cos(\pi\psi^*), \quad -1/2 \le \psi^* \le 1/2, \tag{10}$$

where x^*, y^*, φ^*, ψ^* are values relative to Q, and H^* is the dimensionless depth of the channel, that is, we have taken $b_1 = -H^*$ and $b_n = 0$, $n = 2, 3, \ldots$, in expansion (5). Note that, for the profiles described by (19),

$$z^* = -iw^* - iH^* \exp(-\pi w^*), \tag{11}$$

where z^* and w^* are the dimensionless (that is, relative to half the flow rate $q = Q/2$) physical coordinate and complex potential [17].

As the objective functional, we will take the particle travel time along any streamline from a point on BC to the prescribed horizontal H_2. Substituting the expression $v(\varphi, \psi) = |dw/dz|$ found from (11) into (1) and letting $\varphi_1 = 0$, corresponding to the channel floor, we have

$$t^*(\varphi_2^*, \psi^*, H^*) = \varphi_2^* - (\pi H^{*2}/2)[\exp(-2\pi\varphi_2^*) - 1] + \tag{12}$$

$$+2H^* \cos(\pi\psi^*)[\exp(-\pi\varphi_2^*) - 1], \quad -1/2 \le \psi^* \le 1/2.$$

Problem II. For prescribed $\varkappa$, m, Q, H_1, H_2, it is required to find the depth H^* of a Coseni channel for which the functional $t(\psi^*)$ reaches an extremum during the motion of a tracer along an arbitrary streamline $\psi^* \in [-1/2, 1/2]$ from the bottom of the reservoir to the level $y = -H_2^*$.

Notice that in this formulation the level $y = -H_2^*$ is not exponential, and according to (11), along this line

$$H_2^* = \varphi_2^* + H^* \exp(-\pi\varphi_2^*/2) \cos(\pi\psi^*). \tag{13}$$

Substituting H^* from (13) into (12), we reduce Problem II to the search for an extremum of the function $t^*(H^*)$, where the control H^* must satisfy the inequality $0 \le H^* \le 2/\pi$. The constraint $H^* \le 2/\pi$ is not a consequence of physical considerations or of the optimization technique, but of the character of the chosen control. Thus, for $H > 2/\pi$, the conformal mapping $z(w)$ of (11) becomes non-one-sheeted, and the profile BC becomes selfintersecting [17].

Figure 2 shows graphs of $t^*(H^*)$ for $H_2^* = 1.0$ and $\psi^* = 0.0$, 0.1, 0.2, 0.3, 0.4 (curves *1–5* respectively).

It is clear from the graphs that the function $t^*(H^*)$ can have either an internal minimum or a minimum for $H^* = 2/\pi$, depending on the particular streamline chosen.

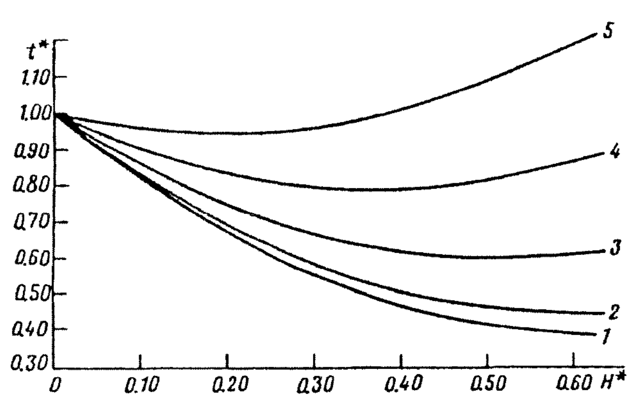

FIG. 2.

One isoperimetric constraint has been used in Problems I and II: the filtrational flow rate, the extremal of Problem I being a degenerate profile, a channel with zero "living" cross-sectional area and, therefore, a zero hydraulic flow rate. We will add to the formulation of Problem I a different isoperimetric constraint, the cross-sectional area of the reservoir S.

Problem III. For prescribed $\varkappa$, m, Q, H_1, H_2 and S, it is required to determine the shape of the curve BC such that the particle travel time t along AB from level $y=-H_1$ to level $y=-H_2$ is an extremum.

For the flow model under consideration here, the expression for S has the form

$$S^* = -\frac{\pi}{2}\sum (2n-1)\, b_{2n-1}^2 + \frac{2}{\pi}\sum \frac{b_{2n-1}}{2n-1}, \text{ where } S^* = \varkappa^2 S/Q. \tag{14}$$

We introduce the Lagrange functional $L^* = t^* + \mu S^*$, where μ is a Lagrange multiplier, to be determined. Substituting into the necessary condition for an extremum

$$\frac{\partial L^*}{\partial b_{2k-1}} = \frac{\partial t^*}{\partial b_{2k-1}} + \frac{\mu \partial S^*}{\partial b_{2k-1}} = 0, \quad k = 1, 2, \ldots, \infty, \tag{15}$$

the expressions obtained from (14):

$$\partial S^*/\partial b_{2k-1} = -\pi (2k-1)\, b_{2k-1} + 2/[\pi (2k-1)],$$

and from (8):

$$\frac{\partial t^*}{\partial b_{2k-1}} = -2\pi(2k-1) \int_{\pi H_1^*}^{\pi H_2^*} \sum (2n-1) b_{2n-1} \exp[-2(n+k-1)\gamma]\, d\gamma =$$

$$= -2\pi (2k-1) \sum (2n-1)\, b_{2n-1} \frac{\beta_1^{-2(n+k-1)} - \alpha_1^{-2(n+k-1)}}{2(n+k-1)},$$

where $\alpha_1 = \alpha + \sqrt{\alpha^2 - 1}$, $\beta_1 = \beta + \sqrt{\beta^2 - 1}$, we obtain an infinite system of linear algebraic equations for the coefficients b_{2n-1}:

$$b_{2k-1} = \sum c_{2k-1,2n-1} b_{2n-1} + a_{2k-1}, \quad k = 1, 2, \ldots, \infty, \tag{16}$$

where $c_{2k-1,2n-1} = -(2n-1)[\beta_1^{-2(n+k-1)} - \alpha_1^{-2(n+k-1)}][\mu(k+n-1)]^{-1}$ and $a_{2k-1} = 2[\pi(2k-1)]^{-2}$, which can be solved by the technique described in [7]. The situation is complicated in this case by the fact that the coefficients of the system contain the constant μ, determined from the prescribed constraint of Problem III. We know [7] that the existence and uniqueness of the solution of systems of the form (16) follow from the condition that it should be regular:

$$\sum |c_{k,n}| < 1, \quad k = 1, 2, \ldots, \infty. \tag{17}$$

To prove (17), we note that $(2k-1)/(2n-1+2k-1) \le 1$ for (16). Then for (16)

$$\sum |c_{k,n}| \le 2 \sum \left| \frac{\beta_1^{-2(k+n-1)} - \alpha_1^{-2(k+n-1)}}{\mu} \right|,$$

or, since $\alpha_1 \le \beta_1$,

$$|c_{k,n}| \le \frac{2}{|\mu|} \left[\beta_1^{-(2k-1)} \sum \beta_1^{-(2n-1)} - \alpha_1^{-(2k-1)} \sum \alpha_1^{-(2n-1)} \right].$$

Thus, condition (17) is satisfied if

$$|\mu| > 2 \{ [\alpha_1^{2k-1}(\alpha_1 - 1)]^{-1} - [\beta_1^{2k-1}(\beta_1 - 1)]^{-1} \}. \tag{18}$$

It is easy to show that the right-hand side of (18) reaches its maximum at $k=1$. Hence, the system of inequalities (18) holds if

$$|\mu| > 2\{ [\alpha_1(\alpha_1 - 1)]^{-1} - [\beta_1(\beta_1 - 1)]^{-1}\}. \quad (19)$$

Thus, if condition (19) holds, system (16) has a unique solution, which is found below by two methods: by the reduction method, and by an iterational method [7].

The first step in the reduction method is to choose an initial value of the Lagrange multiplier $\mu^{(1)}$, satisfying (19), and the number of equations $n = N$. Then the solutions $b_{2n-1}^{(1)}$ of the finite system found by Gauss's method are substituted into (14). If it is not satisfied to the prescribed accuracy ε_1, $\mu^{(2)}$ is found by solving Eq. (14). The iterations are continued until $b_{2n-1}^{(i)}$ and $\mu^{(i)}$ satisfy (16) and (14). The number of equations and unknowns is then increased ($n = N + 1$), and the process is repeated until the condition $|b_{2n-1}^{(i,N+1)} - b_{2n-1}^{(i,N)}| < \varepsilon$, $n = 1, 2, \ldots$, is satisfied.

As a first approximation of the iterational method, we chose $b_{2k-1}^{(0)} = 0$. Then

$$b_{2k-1}^{(1)} = 2\,[\pi(2k-1)]^{-2}, \quad b_{2k-1}^{(2)} = b_{2k-1}^{(1)} - (1/\mu^{(1)}) \sum c_{2k-1,2n-1} b_{2n-1}^{(1)}. \quad (20)$$

Substituting (20) into (14) we have

$$\mu^{(1)} = \pm \frac{\pi}{2} \sqrt{\sum_k (2k-1) \Big(\sum_n c_{2k-1,2n-1} b_{2n-1}^{(1)}\Big)}\ (\sqrt{S_0 - S^*})^{-1}, \quad (21)$$

$$S_0 = \frac{2}{\pi^3} \sum (2k-1)^{-3}.$$

A similar iterational procedure to that described in the reduction method is then followed. Note that (21) gives the well-known condition for Problem I to be solvable: $S^* \leq S_0$.

Graphs of t^* against S^* are shown in Fig. 3.

Unfortunately we have not succeeded in obtaining a sufficient condition for an extremum of Problem III.

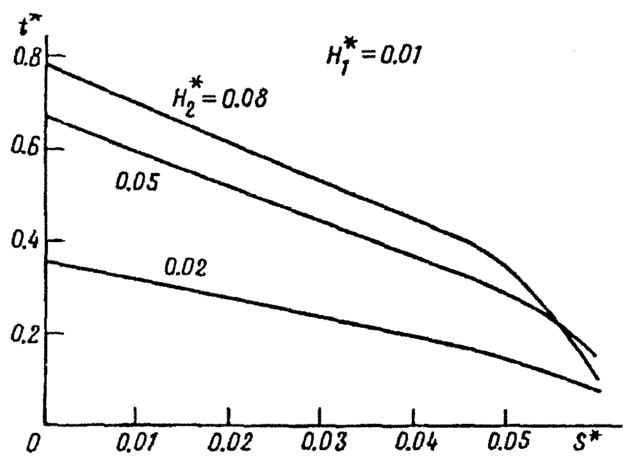

FIG. 3.

CONCLUSION

A unique explicit global extremum in the class of arbitrary curves, extrema in a one-parameter class of curves and curves satisfying a necessary condition have been constructed for three kinds of isoperimetric problem with the travel time along unspecified streamlines as the criterion. The technique described for finding the extremum of the functional of particle travel time in a flow can be extended to other models of both profile and plane filtration. It is not difficult to show that the problem of estimating the effective breakthrough time for the ground flow to the subterranean contour of a concrete dam [17] can be solved in the same way.

REFERENCES

1. BEAR J., *Dynamics of Fluids in Porous Media*. Elsevier, New York, 1972.
2. GUVANASEN V. and GUVANASEN V. M., An approximate semianalytical solution for tracer injection tests in a confined aquifer with a radially converging flow field and finite volume of tracer and chase fluid. *Water Resources Res.* **23**, 1607–1619, 1987.
3. PHILIP J. R., Travel times from buried and surface infiltration point sources. *Water Resources Res.* **20**, 990–994, 1984.
4. STRACK O. D., Three-dimensional streamlines in Dupuit–Forchheimer models. *Water Resources Res.* **20**, 812–822, 1984.
5. PIRONNEAU O., *Optimal Shape Design for Elliptic Systems*. Springer, New York, 1984.
6. HASLINGER J. and NEITTAANMÄKI P., *Finite Element Approximation for Optimal Shape Design: Theory and Applications.* Wiley, Chichester, 1988.
7. KANTOROVICH L. V. and KRYLOV V. I., *Approximate Methods of Higher Analysis*. Fizmatgiz, Moscow, 1962.
8. SAMSONOV B. G. and SAMSONOVA L. M., *Travel of a Substance and the Solution of Hydrogeological Problems*. Nedra, Moscow, 1987.
9. CARLEMAN T., Über ein Minimalproblem der mathematischen Physik. *Math. Z.* **1**, 208–212, 1918.
10. ILYINSKY N. B. and KASIMOV A. R., The estimation of integral seepage characteristics of hydraulic structures in terms of the theory of inverse boundary-value problems. *Z. Angew. Math. Mech.* **72**, 2, 103–112, 1992.
11. IL'INSKII N. B. and KASIMOV A. R., Estimates of backwater and draining levels by hydrodynamic modelling. *Izv. Akad. Nauk SSSR. Mekhan.Zhidkosti i Gaza*, No. 2, 82–90, 1991.
12. KASIMOV A. R., Optimization of irrigation in a hydrodynamic model. *Prikl. Mat. Mekh.* **55**, 2, 338–341, 1991.
13. ILYINSKY N. B. and KASIMOV A. R., Problems of seepage to empty ditch and drain. *Water Resources Res.* **28**, 3, 871–877, 1992.
14. POLYA G. and SZEGO, G., *Isoperimetric Inequalities in Mathematical Physics*. Fizmatgiz, Moscow, 1962.
15. GOL'DSHTEIN R. V. and YENTOV V. M., *Qualitative Methods in the Mechanics of Continuous Media*. Nauka, Moscow, 1989.
16. ILYINSKY N. B. and KASIMOV A. R., Analytical estimation of ground water flow around cutoff walls and into interceptor trenches. *J. Ground Water* **30**, 6, 901–907, 1992.
17. POLUBARINOVA-KOCHINA P. Ya., *Theory of the Movement of Ground Water*. Nauka, Moscow, 1977.
18. VILLAGGIO P., A mathematical theory of the guillotine. *Arch. Ration. Mech. Analys.* **109**, 2, 93–101, 1990.
19. KASIMOV A. R., Seepage optimization for trapezoidal channel. *J. Irrig. Drainage Div. Proc. ASCE.* **118**, 4, 520–526, 1992.