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Correction to "Prediction of steady state flow in nonuniform geologic media by conditional moments: Exact nonlocal formalism, effective conductivities, and weak approximation" by Shlomo P. Neuman and Shlomo Orr

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In the paper "Prediction of steady state flow in nonuniform geologic media by conditional moments: Exact nonlocal formalism, effective conductivities, and weak approximation" by Shlomo P. Neuman and Shlomo Orr (*Water Resources Research*, *29*(2) 341–364, 1993) the authors developed a nonlocal formalism for steady state flow in randomly heterogeneous porous media by conditional moments. They considered the steady state flow equation and boundary conditions

$$\nabla \cdot [K(\mathbf{x})\nabla h(\mathbf{x})] + f(\mathbf{x}) = 0 \qquad \text{on } \Omega \tag{1}$$

$$h(\mathbf{x}) = H(\mathbf{x}) \qquad \text{on } \Gamma_D \tag{2}$$

$$K(\mathbf{x})\nabla h(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = Q(\mathbf{x}) \qquad \text{on } \Gamma_N \tag{3}$$

where $\mathbf{x}$ is the vector of space coordinates, ∇ is the grad operator with respect to $\mathbf{x}$, K is hydraulic conductivity, h is hydraulic head, f is a source term, Ω is the flow domain, H is prescribed head on the Dirichlet boundary segment Γ_D, $\mathbf{n}$ is a unit outward normal to the boundary Γ, and Q is prescribed flux across the Neumann boundary segment Γ_N. They took K, f, H, and Q to be statistically independent random fields such that $K = \langle K\rangle_\kappa + K'$, $f = \langle f\rangle + f'$, $H = \langle H\rangle + H'$, and $Q = \langle Q\rangle + Q'$, where angle brackets denote ensemble mean (statistical expectation), the subscript κ denotes conditioning on measured K values, and primed quantities represent zero (conditional) mean fluctuations. The authors showed that the corresponding conditional mean head satisfies the integrodifferential equation and boundary conditions

$$\nabla \cdot [\langle K(\mathbf{x})\rangle_\kappa \nabla\langle h(\mathbf{x})\rangle_\kappa - \mathbf{r}_\kappa(\mathbf{x})] + \langle f(\mathbf{x})\rangle = 0 \qquad \text{on } \Omega \tag{4}$$

$$\langle h(\mathbf{x})\rangle_\kappa = \langle H(\mathbf{x})\rangle \qquad \text{on } \Gamma_D \tag{5}$$

$$[\langle K(\mathbf{x})\rangle_\kappa \nabla\langle h(\mathbf{x})\rangle_\kappa - \mathbf{r}_\kappa(\mathbf{x})] \cdot \mathbf{n}(\mathbf{x}) = \langle Q(\mathbf{x})\rangle \qquad \text{on } \Gamma_N \tag{6}$$

where $\mathbf{r}_\kappa(\mathbf{x}) = -\langle K'(\mathbf{x})\nabla h'(\mathbf{x})\rangle_\kappa$ is a conditional residual flux given by integral expressions. From (F7), which is exact, of Neuman and Orr's paper it follows that

$$\mathbf{r}_\kappa(\mathbf{x}) = \int_\Omega \mathbf{a}_\kappa(\mathbf{x}, \mathbf{y})\nabla_y\langle h(\mathbf{y})\rangle_\kappa \, d\mathbf{y} + \int_\Omega \mathbf{b}_\kappa(\mathbf{x}, \mathbf{y})\mathbf{r}_\kappa(\mathbf{y}) \, d\mathbf{y} \tag{7}$$

where the kernels

$$\mathbf{a}_\kappa(\mathbf{x}, \mathbf{y}) = \langle K'(\mathbf{x})K'(\mathbf{y})\nabla\nabla_y^T \mathcal{G}(\mathbf{x}, \mathbf{y})\rangle_\kappa \tag{8}$$

$$\mathbf{b}_\kappa(\mathbf{x}, \mathbf{y}) = \langle K'(\mathbf{x})\nabla\nabla_y^T \mathcal{G}(\mathbf{x}, \mathbf{y})\rangle_\kappa \tag{9}$$

form second-rank positive semidefinite symmetric and nonsymmetric tensors, respectively, $\mathcal{G}$ being the random Green's function associated with (1)–(3), and ∇_y the grad operator with respect to $\mathbf{y}$. Equation (7) is implicit in $\mathbf{r}_\kappa$; we include it here because it was not presented in this form by Neuman and Orr. Instead, they presented an explicit alternative to (7) which, according to their (12), has the form

$$\mathbf{r}_\kappa(\mathbf{x}) = \int_\Omega \mathbf{a}_\kappa(\mathbf{x}, \mathbf{y})\nabla_y h_\kappa(\mathbf{y}) \, d\mathbf{y} + \text{boundary integrals} \tag{10}$$

where $h_\kappa(\mathbf{x})$ is the solution of (4)–(6) corresponding to $\mathbf{r}_\kappa \equiv 0$. The boundary integrals, as defined in (12), (19b), and (19c) of Neuman and Orr, require correction. We demonstrate in Appendix A that the correct form of (10) is

$$\mathbf{r}_\kappa(\mathbf{x}) = \int_\Omega \mathbf{a}_\kappa(\mathbf{x}, \mathbf{y})\nabla_y h_\kappa(\mathbf{y}) \, d\mathbf{y} + \int_{\Gamma_N} \mathbf{c}_\kappa(\mathbf{x}, \mathbf{y})\langle Q(\mathbf{y})\rangle \, d\mathbf{y} + \int_\Omega \int_{\Gamma_N} \mathbf{d}_\kappa(\mathbf{x}, \mathbf{y}, \mathbf{z})\langle Q(\mathbf{z})\rangle \, d\mathbf{z} \, d\mathbf{y} \tag{11}$$

where the kernels $\mathbf{c}_\kappa(\mathbf{x}, \mathbf{y})$ and $\mathbf{d}_\kappa(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are conditional vectors independent of the source and boundary terms f, H, Q, and given by

$$\mathbf{c}_\kappa(\mathbf{x}, \mathbf{y}) = -\langle K'(\mathbf{x})K'(\mathbf{y})\nabla\mathcal{G}(\mathbf{x}, \mathbf{y})\rangle_\kappa \langle K(\mathbf{y})\rangle_\kappa^{-1} - \langle K'(\mathbf{x})K(\mathbf{y})^{-1}\rangle_\kappa \langle K(\mathbf{y})\rangle_\kappa \nabla G(\mathbf{x}, \mathbf{y}) \tag{12}$$

$$\mathbf{d}_\kappa(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle K'(\mathbf{x})\nabla\mathcal{G}(\mathbf{x}, \mathbf{y})\nabla_y^T K'(\mathbf{y})\nabla_y G(\mathbf{y}, \mathbf{z}) \cdot [\langle K(\mathbf{z})\rangle_\kappa K(\mathbf{z})^{-1} - 1]\rangle_\kappa \tag{13}$$

where G is the deterministic Green's function associated with (4)–(6) upon setting $\mathbf{r}_\kappa \equiv 0$. Equations (11)–(13) supersede (12), (19b), and (19c) of Neuman and Orr.

The last two terms in (11) drop out when all Neumann boundary conditions are of the mean no-flow type, $\langle Q(\mathbf{x})\rangle \equiv 0$. This can be achieved in practice by moving Γ_N a small distance ε outward, defining it as a mean no-flow boundary, then formally absorbing Q along the original Neumann boundary (just inside the newly defined mean no-flow boundary) into the interior source term f. The last two terms in (11) also drop out

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in the special case where $K(\mathbf{x})$ along Γ_N is deterministic (known with certainty) so that $K'(\mathbf{x}) \equiv 0$.

Equation (12) of Neuman and Orr contains a Dirichlet boundary integral which no longer appears in our correct expression (11). Hence all Dirichlet boundary integrals which stem from Neuman and Orr's (12) must be eliminated from their paper. Likewise, all Neumann boundary integrals which stem from (12) in their paper must be replaced by the sum of two integrals compatible with the last two terms in our (11).

Based on the presence of a Dirichlet boundary integral in their (30), Neuman and Orr concluded that when an effective hydraulic conductivity tensor exists in the presence of Dirichlet boundaries separated by a finite distance, this tensor is nonsymmetric. Upon dropping the Dirichlet boundary integral from their (30) there no longer remains a basis for this conclusion.

We end by correcting a typographical error in (38) of Neuman and Orr. This equation should have f_i^2 rather than f_i in the numerator:

$$F_i = \frac{2\sigma_Y^2}{\pi\lambda_i^2}\int_o^{\pi/2}\int_o^{\pi/2}\frac{f_i^2}{\mathbf{f}^T\boldsymbol{\lambda}^{-2}\mathbf{f}}\sin\phi\, d\phi\, d\theta \tag{14}$$

Appendix A

Equations (A6) and (A10) of Neuman and Orr show correctly that

$$\begin{aligned} h(\mathbf{x}) = &\int_\Omega \mathcal{G}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \\ &- \int_{\Gamma_D} K(\mathbf{y})\nabla_y \mathcal{G}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) H(\mathbf{y})\, d\mathbf{y} \\ &+ \int_{\Gamma_N} \mathcal{G}(\mathbf{x}, \mathbf{y}) Q(\mathbf{y})\, d\mathbf{y} \end{aligned} \tag{A1}$$

$$\begin{aligned} h_\kappa(\mathbf{x}) = &\int_\Omega G(\mathbf{x}, \mathbf{y})\langle f(\mathbf{y})\rangle\, d\mathbf{y} \\ &- \int_{\Gamma_D} \langle K(\mathbf{y})\rangle_\kappa \nabla_y G(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\langle H(\mathbf{y})\rangle\, d\mathbf{y} \\ &+ \int_{\Gamma_N} G(\mathbf{x}, \mathbf{y})\langle Q(\mathbf{y})\rangle\, d\mathbf{y} \end{aligned} \tag{A2}$$

In accord with their (A11) and (A12), we rewrite (1) as

$$\mathcal{L}h(\mathbf{x}) + f(\mathbf{x}) = (L + \Re)h(\mathbf{x}) + f(\mathbf{x}) = 0 \tag{A3a}$$

where

$$\mathcal{L} = \nabla \cdot [K(\mathbf{x})\nabla] \qquad L = \nabla \cdot [\langle K(\mathbf{x})\rangle_c \nabla] \qquad \Re = \nabla \cdot [K'(\mathbf{x})\nabla] \tag{A3b}$$

$$\langle \mathcal{L}\rangle_c = L = \nabla \cdot [\langle K(\mathbf{x})\rangle_c \nabla] \qquad \langle \Re \rangle_c \equiv 0. \tag{A3c}$$

Rewriting (A3a) as $Lh(\mathbf{x}) = -f(\mathbf{x}) - \Re h(\mathbf{x})$; premultiplying by G; applying Green's identity twice; and recognizing that G satisfies $LG + \delta = 0$ where δ is the Dirac delta function, subject to homogeneous boundary conditions, yields

$$h(\mathbf{x}) = -L^{-1}f(\mathbf{x}) - L^{-1}\Re h(\mathbf{x}) + B[H(\mathbf{x}), Q(\mathbf{x})]. \tag{A4}$$

Here we have defined the inverse operator L^{-1} as

$$L^{-1}f(\mathbf{x}) = -\int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \tag{A5}$$

so that $h_1 = -L^{-1}f$ is the solution of $Lh_1 + f = 0$, subject to homogeneous boundary conditions $H \equiv Q \equiv 0$, and defined the nonhomogeneous boundary integral B as

$$\begin{aligned} B[H(\mathbf{x}), Q(\mathbf{x})] = &-\int_{\Gamma_D} \langle K(\mathbf{y})\rangle_\kappa \nabla_y G(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) H(\mathbf{y})\, d\mathbf{y} \\ &+ \int_{\Gamma_N} G(\mathbf{x}, \mathbf{y})\langle K(\mathbf{y})\rangle_\kappa K(\mathbf{y})^{-1} Q(\mathbf{y})\, d\mathbf{y} \end{aligned} \tag{A6}$$

Analogous to (A5), we define the inverse operator $\mathcal{L}^{-1}$ as

$$\mathcal{L}^{-1}f(\mathbf{x}) = -\int_\Omega \mathcal{G}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \tag{A7}$$

so that $h_2 = \mathcal{L}^{-1}f$ is the solution of $\mathcal{L}h_2 + f = 0$ subject to homogeneous boundary conditions $H \equiv Q \equiv 0$. Note that our definitions of L^{-1} and $\mathcal{L}^{-1}$ here differ from those in Appendix A of Neuman and Orr in that the latter include nonhomogeneous boundary integrals while our (A5) and (A7) do not. Only if L^{-1} and $\mathcal{L}^{-1}$ are defined as we have done here are the following algebraic manipulations valid, hence the need for this correction.

Let us rewrite (A4) as

$$(1 + L^{-1}\Re)h = -L^{-1}f + B. \tag{A8}$$

It is easy to verify that $I = LL^{-1} = (\mathcal{L} - \Re)L^{-1}$ where I is the identity operator. Premultiplying by $\mathcal{L}^{-1}$, and recognizing that $\mathcal{L}^{-1}\mathcal{L}L^{-1} = L^{-1}$ (because L^{-1} generates a function that satisfies homogeneous boundary conditions, and $\mathcal{L}^{-1}\mathcal{L} = I$ when operating on such a function), yields $\mathcal{L}^{-1} = L^{-1} - \mathcal{L}^{-1}\Re L^{-1}$. With this it is easy to check that

$$(1 - \mathcal{L}^{-1}\Re)(1 + L^{-1}\Re) = I. \tag{A9}$$

Operating on (A8) with $(1 - \mathcal{L}^{-1}\Re)$ and using (A9) yields

$$h = (1 - \mathcal{L}^{-1}\Re)(-L^{-1}f + B). \tag{A10}$$

We rewrite (A10) as

$$h = (1 - \mathcal{L}^{-1}\Re)(-L^{-1}f + B_o + N) \tag{A11}$$

where

$$\begin{aligned} B_o[H(\mathbf{x}), Q(\mathbf{x})] = &-\int_{\Gamma_D} \langle K(\mathbf{y})\rangle_\kappa \nabla_y G(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) H(\mathbf{y})\, d\mathbf{y} \\ &+ \int_{\Gamma_N} G(\mathbf{x}, \mathbf{y})\, Q(\mathbf{y})\, d\mathbf{y} \end{aligned} \tag{A12}$$

and

$$N[H(\mathbf{x}), Q(\mathbf{x})] = \int_{\Gamma_N} G(\mathbf{x}, \mathbf{y})[\langle K(\mathbf{y})\rangle_\kappa K(\mathbf{y})^{-1} - 1]Q(\mathbf{y})\, d\mathbf{y}. \tag{A13}$$

Operating on (A11) with $K'\nabla$; taking conditional mean; noting that $\langle -L^{-1}f + B_o\rangle_\kappa = -L^{-1}\langle f\rangle + B_o[\langle H\rangle, \langle Q\rangle] = h_\kappa$ by virtue of (A2); substituting the definitions of $\mathcal{L}^{-1}$, $\Re$, and N; applying Green's identity; and considering that $\mathbf{r}_\kappa = -\langle K'\nabla h'\rangle_\kappa = -\langle K'\nabla h\rangle_\kappa$ leads directly to (11)–(13).

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