High-order Mimetic Finite Difference Methods on Arbitrary Meshes

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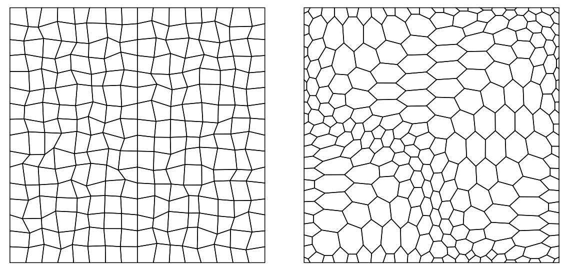
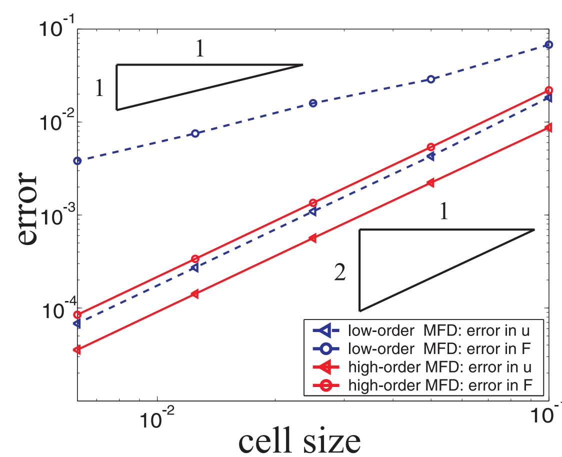
The mimetic finite difference (MFD) methods mimic important properties of physical and mathematical models. As the result, conservation laws, solution symmetries, and the fundamental identities of the vector and tensor calculus are held for discrete models. The existing MFD methods for solving diffusion-type problems on arbitrary meshes are *second-order* accurate for the conservative variable (temperature, pressure, energy, etc.) and only *first-order* accurate for its flux. In many physical simulations such as reactive transport in porous media, compressible flows, etc., the flux accuracy makes significant impact on evolution of conservative quantities. We developed new high-order MFD methods which are *second-order* accurate for both conservative variable and its flux [1]. These methods are well suited for simulations on arbitrary polygonal meshes.

Modelling with arbitrary polygonal meshes has a number of advantages. Such meshes allow to describe accurately small, detailed structures such as tilted layers, irregular inclusions, rugged surfaces and interfaces, etc. The polygonal meshes cover the plane more efficiently than triangular meshes which eventually reduces the number of discrete unknowns without lose of accuracy. The locally refined meshes, used to improve resolution in region of interest, are particular examples of polygonal meshes with degenerate elements.

There are a few fundamentally different approaches to increase accuracy of discretization methods. Finite volume and finite difference methods increase stencils of discrete operators which impose severe restrictions on mesh smoothness. These methods are usually applied on smooth meshes and lose accuracy on rough ones. The finite element and spectral element methods increase the number of unknowns inside each element but impose severe restrictions on the shape of admissible mesh elements. To develop new high-order MFD methods, we blend ideas of the finite element [4] and the low-order MFD methods [2].



Sample meshes used in analysis. Both randomly perturbed (left) and polygonal (right) meshes are challenging tests for any discretization method. The new high-order MFD methods have similar approximation properties on both meshes.



Convergence rates for the low-order MFD (blue) and the new high-order MFD (red) methods on randomly perturbed quadrilateral meshes for a manufactured solution. Both methods are second-order accurate for the conservative variable u. The high-order MFD method is second-order accurate for the flux $\vec{F}$, while the low-order MFD method is only first-order accurate.

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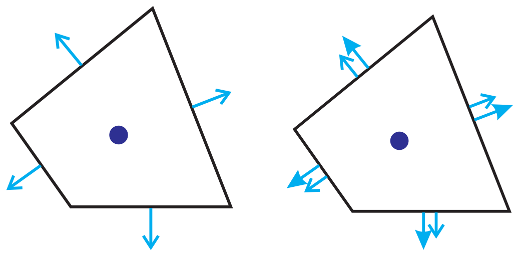

(a) (b)

Schematic illustration of degrees of freedom in the low-order (a) and high-order (b) MFD methods. One arrow represents the average normal flux through the edge. Two close arrows represent the average normal flux (0-th moment) through the edge and its first moment. The solid circles represent degrees of freedom for conservative variable.

In our analysis we consider a stationary diffusion problem for the conservative variable u and its flux $\vec{F}$:

$$\operatorname{div}\vec{F} = q, \qquad \vec{F} = -K\nabla u.$$

In the high-order MFD method, the scalar function u is represented by one unknown, its average value, in each mesh element. The flux $\vec{F}$ (the vector function) is represented by two unknowns on each mesh edge, the number which is twice more than in the low-order MFD method.

Similarly to the low-order MFD method, the key step in the high-order MFD method is the definition of the inner product in the space of discrete fluxes. This inner product can be also viewed as a quadrature rule for the integral of a dot-product of two continuous fluxes. Since the chosen degrees of freedom are normal fluxes on mesh edges, the construction of this inner product is a non-trivial task. Due to additivity of integration, this inner product can be defined independently on each mesh element. We developed two methods for building elemental inner products [1].

The first method extends further the ideas of the low-order Kuznetsov-Repin finite element method [3]. We divide *virtually* each polygonal element into triangles and use the existing formula for exact integration of linear fluxes on a triangle [4]. The virtual triangular partition introduces additional flux unknowns on interior edges. Half of these unknowns, 0-th moments of the normal flux, are eliminated using the Kuznetsov-Repin approach. The remaining unknowns (1-st moments) are eliminated by mimicking integral identities for particular spaces of vector functions. In the finite element community this technique is know as the static condensation. The method is useful for problems where the flux has to be recovered at some points inside a polygonal element.

The second method was inspired in part by the methods developed in [2]. Only boundary data (normals to polygon edges, length of edges, and quadrature rules for edge integrals) are used to build the elemental inner product. Since, no auxiliary triangular partition is required, the proposed method can be easily applied on meshes with degenerate polygons, which appear in adaptive mesh refinement (AMR) methods, and non-convex polygons, which appear in moving mesh methods.

Acknowledgements

Los Alamos Report LA-UR-07-5883. Funded by the Department of Energy at Los Alamos National Laboratory under contracts DE-AC52-06NA25396 and the DOE Office of Science Advanced Computing Research (ASCR) program in Applied Mathematical Sciences.

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