Mathematical Modeling and Analysis

Mimetic Finite Differences for Modeling Stokes Flow on Polygonal Meshes

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Stokes flow is fluid flow where advective inertial forces are negligibly small compared to viscous forces. This is a typical situation on a microscale or when the fluid velocity is very small. Stokes flow is a good and important approximation for a number of physical problems such as sedimentation, modeling of bio-suspensions, construction of efficient fibrous filters, developing energy efficient micro-fluidic devices (e.g. mixers), etc. Efficient numerical solution of Stokes flow requires unstructured meshes adapted to geometry and solution as well as accurate discretization methods capable of treating such meshes. We developed a new mimetic finite difference (MFD) method that remains accurate on general polygonal meshes that may include non-convex and degenerate elements [1].

Triangular meshes allow one to model complex geometric objects. However, compared to quadrilateral and more general polygonal meshes, the triangular meshes with the same resolution do not provide optimal cover of the space, which result in larger algebraic problems. The MFD method was designed to provide accurate approximation of differential operators on general meshes. These meshes may include degenerate elements, as in adaptive mesh refinement methods, non-convex elements, as in moving mesh methods, and even elements with curved edges near curvilinear boundaries.

The incompressible Stokes equations are

$$-\mathrm{div}\left(\mu(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right) = \mathbf{F}-\nabla p$$
$$\mathrm{div}\,\mathbf{u} = 0$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, $\mathbf{F}$ is the given external force, and μ is the fourth-order symmetric positive definite *tensor* viscosity.

Since μ is a tensor, the developed MFD method can be applied to problems of linear elasticity that can be written in a similar form.

The MFD method has many similarities with a low-order finite element (FE) method. Both methods try to preserve fundamental properties of physical and mathematical models. Various approaches to extend the FE method to non-simplicial elements have been developed over the last decade. Construction of basis functions for such elements is a challenging task and may require extensive analysis of geometry. Contrary to the FE method, the *MFD method uses only boundary representation of discrete unknowns* to build stiffness and mass matrices. Since no extension inside the mesh element is required, practical implementation of the MFD method is simple for general polygonal meshes.
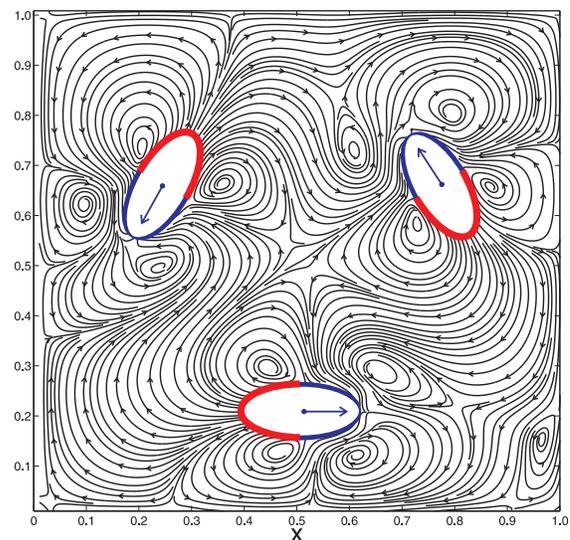
Streamlines for the flow generated by three self-propelled bacteria (colored ellipses) moving counterclockwise in a closed box with no-slip conditions on the walls. On blue parts of the ellipses fluid sticks to bacteria, on red parts fluid is pushed back to generate propulsion. Calculations were performed with the mimetic finite difference method on a polygonal mesh obtained by intersection of a square 50×50 mesh with ellipses.

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The MFD method is flexible in selecting discrete unknowns. In [1], the velocity is approximated at mesh vertices and the velocity flux is approximated at mesh edges. The pressure is approximated by one constant (e.g. average) on each mesh element. This set of discrete velocity unknowns is abundant and will be reduced in the future. On triangular meshes, the MFD method coincides with the FE method that uses the same set of discrete unknowns. The numerical experiments in [1] have shown the *second-order* convergence for the velocity variable and the *first-order* for the pressure on unstructured polygonal meshes. The convergence rates have remained the same in experiments with anisotropic tensor μ.
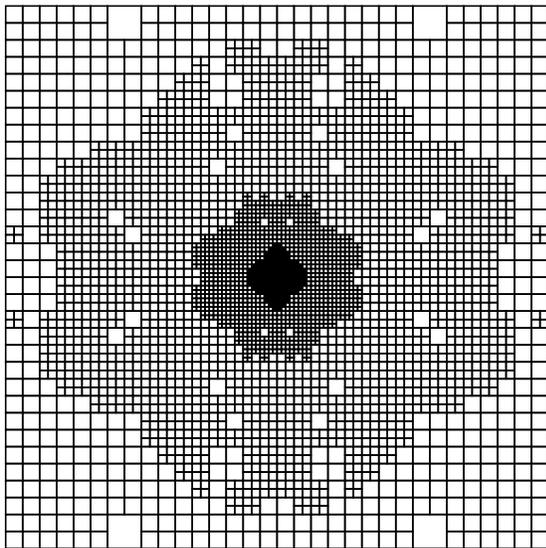
Example of an adapted mesh in Stokes flow with a singular point force in the middle of the domain. The mesh consists of regular quadrilateral elements and degenerate elements with 5, 6 and 8 edges. The MFD method uses the same construction for all these elements.

Like the MFD method for the diffusion problem [2, 3], the novel MFD method [1] is again a parametric family of methods with equivalent properties. In numerical experiments, we have used a particular member of this family. Analysis of this family is an open question. The answer to this question may result in new adaptive methods. In addition to traditional mesh refinement (h-adaptation) and enrichment of discretization space (p-adaptation), the MFD method provides a basis for selecting an optimal discretization method.

The novel MFD method has been developed for elements with straight edges. Applying ideas from [2], it will be possible to extend it to meshes with curved edges. The ideas described in the last two paragraphs will be the topics of future research.

A similar MFD method has been developed independently by Lourenco Beirão da Veiga and Marco Manzini [1].

Acknowledgements

References

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