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A new error-minimizing strategy for the solution of the acoustic wave equation

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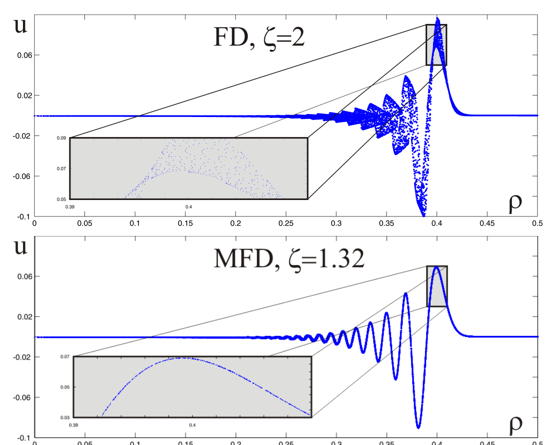
Many important physical problems require numerical solutions of the wave equations for long time intervals, e.g. radio, seismic and acoustic waves. Achieving the desired integration times requires balancing the efficiency and accuracy of the numerical scheme. The leading sources for long time integration error are numerical *anisotropy* and *dispersion* – numerical artifacts (absent in the physical problem) predicting different speed of propagation for waves depending on their direction and their wavelength. These artifacts are major challenge for existing methods, as even for moderate integration times this requires increasing the size of the problem significantly to resolve the smallest wavelength. We have developed a new computational technique, called *m-adaptation*, that significantly reduces the effect of the numerical dispersion and anisotropy for the acoustic wave equation, and may eliminate them completely.

The m-adaptation technique is based on Mimetic Finite Difference (MFD) discretizations [1] that were developed in the last 15 years by the team of M. Shashkov. One important feature that makes MFD discretizations exceptionally flexible is that they work on general polygonal and polyhedral meshes. Another feature, unique to MFD discretizations, is that unlike classical discretizations it produces not one but a parameterized family of methods [2] with equivalent properties – convergence rate, stability and computational complexity. Many classical methods, such as Finite Difference (FD) and Finite Element (FE) methods belong to this family. There are infinitely many more methods that do not correspond to any of the classical methods, but can be optimal for the problem of interest.

For the acoustic wave equation, $u_{tt} = c\triangle u$, written in a semi-discrete form $Mu_{tt} = Au$, the efficiency is achieved by selecting the mass matrix M to be diagonal. The stiffness matrix A is assembled from local matrices A_E element-by-element, like in the classical FE discretizations. On each element E the local matrix A_E has a form:

$$A_E = A_E^{\text{cons}} + A_E^{\text{stab}}(\zeta_1, \ldots, \zeta_k).$$

The symmetric consistency matrix A_E^{cons} is unique for all second-order accurate methods. The stability matrix A_E^{stab} is determined by the parameters $\zeta_1, \ldots, \zeta_k$, whose number k depends on the number of vertices in the element E. For the acoustic equation on squares $k = 1$; on cubes $k = 10$. The m-adaptation allows one to find the optimal parameters for a specified criteria, which for the wave equation are minimization of numerical anisotropy and numerical dispersion. The optimal parameter ζ_{opt} depends on the value κh, characterizing the resolution of the wavelength $\lambda = 2\pi/\kappa$ by the mesh of size h.



Radial slices of the numerical solutions of the wave equation with radially symmetric (around the origin) Gaussian initial displacement, zero initial velocity and zero Dirichlet boundary conditions. The solution obtained using FD (top) and FE (not shown) has about 30 times wider spread of displacement for any fixed distance ρ from the origin as compared with solution obtained using m-adaptation (bottom).

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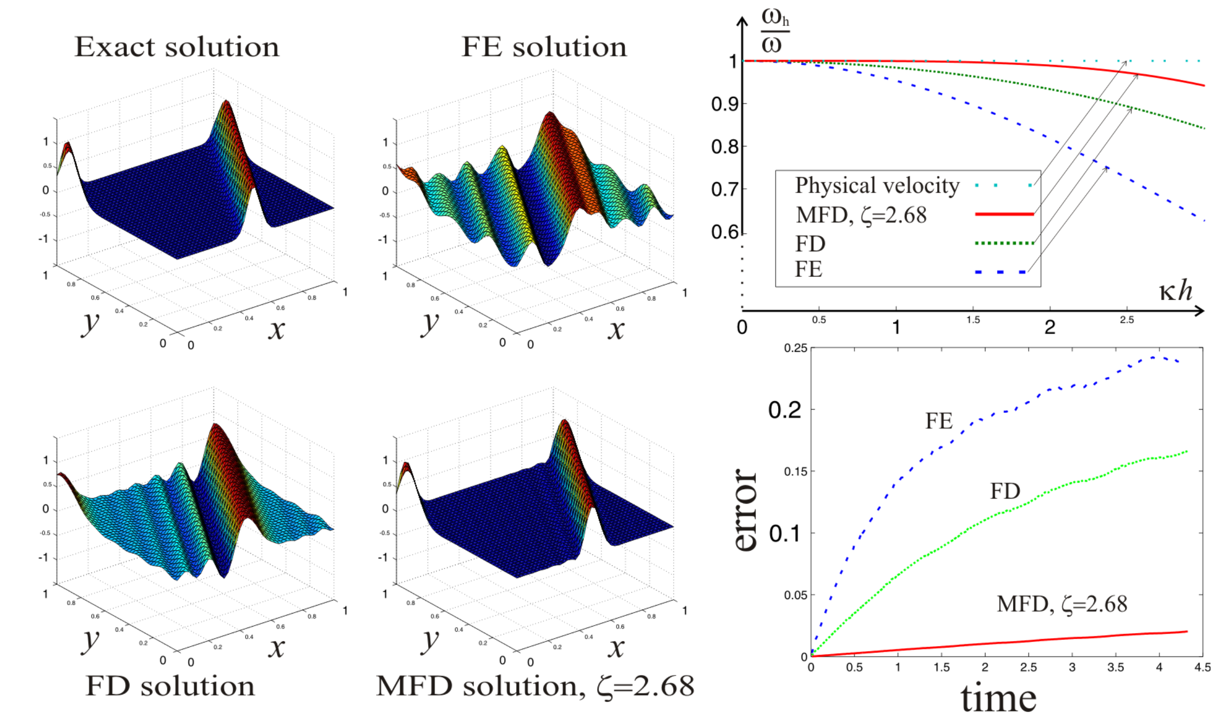

A packet of plain waves moving a distance of 15 mean wavelengths at 30° *to a square mesh. Error in* L^2*-norm as a function of time for FE, FD, and MFD with* $\zeta_{opt} = 2.68$*, optimized for* $\kappa h = 0.5$*. The dispersion curves are shown (top right) for all three methods. Note a wide region,* $\kappa h < 1.5$*, where the dispersion curve for the optimized MFD matches almost perfectly with the physical velocity (an order of magnitude improvement over FD and FE). This region corresponds to waves with 4 or more points per wavelength, which is a dramatic improvement over the common practice of using at least 12-20 points per wavelength.*

We performed a number of numerical experiments that demonstrate at least an order of magnitude reduction of error for m-adaptation compared with the classical FD and FE methods.

For the minimization of numerical anisotropy, we considered a test problem with a radially symmetric physical solution. Due to mesh anisotropy, the numerical solutions produce radially asymmetric solutions, leaving a mesh imprint on the problem. The deviation of numerical solution from a radially symmetric one can be measured through the spread of values $u(\rho)$ on circles of radius ρ, depicted on the first page. The smaller the spread is, the closer the solution is to a radially symmetric one, with zero spread corresponding to a perfectly radially symmetric solution. The spread of the values obtained using the m-adaptation, is a factor of 30 smaller compared with the classical FD and FE methods, virtually eliminating mesh imprint on the problem.

The results of comparison for the numerical dispersion are similar, with typical reduction of L^2-error by an order of magnitude.

Acknowledgements

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[2] L. Beirão da Veiga, V. Gyrya, K. Lipnikov and M. Manzini, Mimetic finite difference method for the Stokes problem on polygonal meshes. *J. Comput. Phys.*, (2009), **228**, 7215-7232.