Moving grids for hyperbolic problems

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Objectives (1/4)

Let us consider the one-dimensional viscous Burgers equation:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, 1), \quad t \in (0, T),
\]

subject to the initial condition

\[
u(x, 0) = u_0(x).
\]

**The look-ahead strategy:**

- Assume that the data at \( t = t^n \) are given and exact.
- Derive exact and approximate error functionals at \( t = t^{n+1} \).
- Build an adapted grid minimizing the approximate error functional.
- Interpolate data to the adapted grid and perform one time step.
Objectives (2/4)

Let $t = t^n$. We consider a grid

$$0 = x^n_0 < x^n_1 < \ldots < x^n_{M+1} = 1$$

and define

$$x^n_{i+1/2} = (x^n_{i+1} + x^n_i)/2, \quad h^n_{i+1/2} = x^n_{i+1} - x^n_i.$$  

Let

$$\bar{u}^0_{i+1/2} = \frac{1}{h^n_{i+1/2}} \int_{x^n_i}^{x^n_{i+1}} u_0(x) \, dx.$$
Objectives (3/4)

Let us analyze an error functional associated with the donor scheme:

\[
\tilde{u}_{i+1/2}^{n+1} = L_{i+1/2}^n (\tilde{u}^n)
\]

\[
= \tilde{u}_{i+1/2}^n - \frac{\Delta t^n}{h_{i+1/2}} (f_{i+1}^n - f_i^n) + \frac{\varepsilon \Delta t^n}{h_{i+1/2}^n} \left( \left[ \frac{\delta u^n}{\delta x} \right]_{i+1} - \left[ \frac{\delta u^n}{\delta x} \right]_i \right)
\]

where \( f_{i}^n \) denotes the flux at point \( x_i^n \),

\[
f_{i}^n = \frac{1}{2} \begin{cases} 
(\tilde{u}_{i+1/2}^n)^2 & \text{if } \tilde{u}_{i+1/2} + \tilde{u}_{i-1/2} \geq 0, \\
(\tilde{u}_{i-1/2}^n)^2 & \text{otherwise,}
\end{cases}
\]

and

\[
\min_i \Delta t^n \left( \frac{\tilde{u}_{i+1/2}^n}{h_{i+1/2}^n} + \frac{2\varepsilon}{(h_{i+1/2}^n)^2} \right) < 1.
\]
Objectives (4/4)

Consider the following minimization problem:

$$\min_{x_1^n, \ldots, x_M^n} F^n_{ex}(\{x_i^n\})$$

where

$$F^n_{ex} = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x, t^{n+1}) - \bar{u}_{i+1/2}^{n+1}|^2 dx = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x, t^{n+1}) - \mathcal{L}_{i+1/2}^{n}(\bar{u}^n)|^2 dx$$

The stability condition and Taylor expansion give

$$\int_{x_i^n}^{x_{i+1}^n} \left(u(x, t^{n+1}) - \mathcal{L}_{i+1/2}^{n}(\{\bar{u}^n\})\right)^2 dx =$$

$$\left(\frac{\partial u}{\partial x}|_{x_i^{n+1/2}}\right)^2 \left[ \frac{(h_{i+1/2}^n)^3}{12} + \frac{(\Delta t^n u_{i+1/2}^n)^2}{4} \frac{(h_{i+1/2}^n - h_{i-1/2}^n)^2}{h_{i+1/2}^n} \right] + O(h_{i+1/2}^n)^4$$
Objectives (4/4)

Consider the following minimization problem:

$$
\min_{x_1^n, \ldots, x_M^n} F^n_{ex}(\{x_i^n\})
$$

where

$$
F^n_{ex} = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x, t^{n+1}) - \bar{u}_{i+1/2}^{n+1}|^2 \, dx = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x, t^{n+1}) - L^n_{i+1/2}(\bar{u}^n)|^2 \, dx
$$

The stability condition and Taylor expansion give

$$
\int_{x_i^n}^{x_{i+1}^n} \left( u(x, t^{n+1}) - L^n_{i+1/2}(\{\bar{u}^n\}) \right)^2 \, dx \approx \left( \frac{\partial u}{\partial x} \bigg|_{x_i^{n+1/2}} \right)^2 \left[ \frac{(h_{i+1/2}^n)^3}{12} \right].
$$
Consider a grid

\[ 0 = x_0 < x_1 < \ldots < x_{M+1} = 1 \]

and define

\[ x_{i+1/2} = (x_{i+1} + x_i)/2, \quad h_{i+1/2} = x_{i+1} - x_i. \]

Let \( f^h(x) \) be a piecewise constant approximation of \( f(x) \). Then, the minimum of the functional

\[
\Phi(\{x_i\}, \{\bar{f}_{i+1/2}\}) = \int_0^1 (f(x) - f^h(x))^2 \, dx = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} (f(x) - \bar{f}_{i+1/2})^2 \, dx
\]

is achieved when

\[
\bar{f}_{i+1/2} = \frac{1}{h_{i+1/2}} \int_{x_i}^{x_{i+1}} f(x) \, dx.
\]
Exact error functional (2/2)

Thus, the problem

\[
\min_{x_1, \ldots, x_M, \bar{f}_{1/2}, \ldots, \bar{f}_{M+1/2}} \Phi(\{x_i\}, \{\bar{f}_{i+1/2}\})
\]

is reduced to

\[
\min_{x_1, \ldots, x_M} F_{ex}(\{x_i\}), \quad F_{ex}(\{x_i\}) = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} (f(x) - \bar{f}_{i+1/2})^2 \, dx.
\]

The Taylor expansion with the Lagrange remainder gives

\[
\min_{x_1, \ldots, x_M} F_{ex}(\{x_i\}), \quad F_{ex}(\{x_i\}) = \frac{1}{12} \sum_{i=0}^{M} \left( \left. \frac{\partial f}{\partial x} \right|_{x_i+1/2} \right)^2 h_i^{3+1/2}
\]

where \(x_{i+1/2}^*\) is a point from interval \((x_i, x_{i+1})\).
Lemma. Let \( e_{i+1/2}(\cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a set of functions defined by

\[
e_{i+1/2}(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} g(x) \, dx, \quad 0 \leq x_i \leq x_{i+1} \leq 1,
\]

where \( g(x) \geq 0 \) is an arbitrary bounded function. Then

\[
\min_{x_1, \ldots, x_M} \sum_{i=0}^{M} e_i^{p} = \frac{\mathcal{E}^p}{(M + 1)^{p-1}}
\]

where \( p \) is a positive integer and

\[
\mathcal{E} = \sum_{i=0}^{M} e_{i+1/2}(x_i, x_{i+1}) = \int_{0}^{1} g(x) \, dx.
\]

Moreover, the minimum is achieved when \( e_{i+1/2}(x_i, x_{i+1}) = \mathcal{E} / (M + 1) \).
Lemma. Let \( e_{i+1/2}(\cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a set of functions defined by

\[
e_{i+1/2}(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} g(x) \, dx, \quad 0 \leq x_i \leq x_{i+1} \leq 1,
\]

where \( g(x) \geq 0 \) is an arbitrary bounded function. Then

\[
\min_{x_1, \ldots, x_M} \sum_{i=0}^{M} e_{i+1/2}^p = \frac{\mathcal{E}^p}{(M+1)^{p-1}}
\]

where \( p \) is a positive integer and

\[
\mathcal{E} = \sum_{i=0}^{M} e_{i+1/2}(x_i, x_{i+1}) = \int_{0}^{1} g(x) \, dx.
\]

Moreover, the minimum is achieved when \( e_{i+1/2}(x_i, x_{i+1}) = \mathcal{E}/(M+1) \).
Minimization & equidistribution (2/4)

We introduce additional notations:

\[
\hat{\omega}_{i+1/2} = \left( \frac{1}{12} \left. \frac{\partial f}{\partial x} \right|_{x_i^{*}+1/2} \right)^{2/3}
\]

and

\[
\hat{e}_{i+1/2} = \hat{\omega}_{i+1/2} h_{i+1/2}.
\]

Then, we can rewrite the functional \( F_{ex} \) as follows:

\[
F_{ex}(\{x_i\}) = \frac{1}{12} \sum_{i=0}^{M} \left( \left. \frac{\partial f}{\partial x} \right|_{x_i^{*}+1/2} \right)^2 h_{i+1/2}^3 = \sum_{i=0}^{M} \hat{e}_{i+1/2}^3 = \sum_{i=0}^{M} \hat{\omega}_{i+1/2}^3 h_{i+1/2}^3.
\]

It is obvious that

\[
\hat{e}_{i+1/2} \rightarrow \int_{x_i}^{x_{i+1}} \left( \left. \frac{\partial f}{\partial x} \right| \right)^{2/3} dx
\]

and

\[
\sum_{i=0}^{M} \hat{e}_{i+1/2} \rightarrow \int_{0}^{1} \left( \left. \frac{\partial f}{\partial x} \right| \right)^{2/3} dx.
\]
Minimization & equidistribution (2/4)

We introduce additional notations:

\[ \hat{\omega}_{i+1/2} = \left( \frac{1}{12} \frac{\partial f}{\partial x} \right)^{2/3} \]

and

\[ \hat{\varepsilon}_{i+1/2} = \hat{\omega}_{i+1/2} h_{i+1/2}. \]

Then, we can rewrite the functional $F_{ex}$ as follows:

\[ F_{ex}(\{x_i\}) = \frac{1}{12} \sum_{i=0}^{M} \left( \frac{\partial f}{\partial x} \right)_{x_{i+1/2}}^2 h_{i+1/2}^3 = \sum_{i=0}^{M} \hat{\varepsilon}_{i+1/2}^3 = \sum_{i=0}^{M} \hat{\omega}_{i+1/2}^3 h_{i+1/2}^3. \]

In other words, taking $g(x) = |\partial f / \partial x|^{2/3}$, we get

\[ \hat{\varepsilon}_{i+1/2} \rightarrow \int_{x_i}^{x_{i+1}} g(x) \, dx \quad \text{and} \quad \sum_{i=0}^{M} \hat{\varepsilon}_{i+1/2} \rightarrow \int_{0}^{1} g(x) \, dx. \]
The equidistribution principle,

\[ \hat{e}_{i+1/2} = \hat{e}_{i-1/2}, \quad i = 1, \ldots, M, \]

may be rewritten as follows:

\[ \hat{\omega}_{i+1/2}(x_{i+1} - x_i) - \hat{\omega}_{i-1/2}(x_i - x_{i-1}) = 0. \]

It is a discretization of the non-linear elliptic equation

\[ \frac{\partial}{\partial \xi} \left( \omega(x) \frac{\partial x}{\partial \xi} \right) = 0, \quad x(0) = 0, \quad x(1) = 1, \]

on a uniform grid with the coefficient \( \omega(x) \) given by

\[ \omega(x) = \left| \frac{\partial f}{\partial x} \right|^{2/3}. \]
A discrete analog of the nonlinear elliptic equation can be directly derived from

$$\nabla F_{ex} = 0.$$ 

Recall that

$$F_{ex}(\{x_i\}) = \sum_{i=0}^{M} \int_{x_i}^{x_{i+1}} (f(x) - \bar{f}_{i+1/2})^2 \, dx.$$ 

Then

$$\frac{\partial F_{ex}}{\partial x_i} = 2f(x_i) - \bar{f}_{i-1/2} - \bar{f}_{i+1/2} = 0.$$ 

The Taylor expansion at point $x_i$ results in

$$\left. \frac{\partial f}{\partial x} \right|_{x_i} (h_{i+1/2} - h_{i-1/2}) + \frac{1}{3} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_i} \left( h_{i+1/2}^2 + h_{i-1/2}^2 \right) = 0.$$
Approximate error functional (1/7)

Let $R_h$ be an interpolation operator from grid $\{x_i^0\}$ to grid $\{x_i\}$. Consider the following minimization problem:

$$\min_{x_1, \ldots, x_M} \int_0^1 |f(x) - [R_h(f_h^0)](x)|^2 \, dx.$$  

We assume that

- $R_h$ is exact for linear functions;
- $R_h$ is conservative.

$$\int_0^1 |f(x) - [R_h(f_h^0)](x)|^2 \, dx = \sum_{i=0}^M \int_{x_i}^{x_{i+1}} \left| f(x) - \bar{f}_{i+1/2} + O(h_{i+1/2}^2) \right|^2 \, dx$$

$$= \sum_{i=0}^M \left[ \frac{1}{12} \left( \frac{\partial f}{\partial x} \bigg|_{x_i^{*+1/2}} \right)^2 h_{i+1/2}^3 + O(h_{i+1/2}^4) \right].$$
Recall that
\[ F_{ex} = \sum_{i=0}^{M} \frac{\hat{\omega}_{i+1/2}^3}{h_{i+1/2}^3}. \]

Since the precise computation of coefficients \( \hat{\omega}_{i+1/2} \) is impossible, they are replaced by computable coefficients \( \omega_{i+1/2} \) such that \( \omega_{i+1/2} \approx \hat{\omega}_{i+1/2} \),

\[ \omega_{i+1/2} = \frac{1}{h_{i+1/2}^3} \sum_{k=0}^{M} \frac{\hat{x}_{ik}}{\bar{x}_{ik}} \int_{\hat{x}_{ik}}^{\bar{x}_{ik}} \left( f_{k+1/2}^0 + \left[ \frac{\delta f_{h,0}}{\delta x} \right]_{k+1/2} (x - x_{k+1/2}) - [R_h(f^0,h)](x) \right)^2 \, dx \]

where \( [\hat{x}_{ik}, \bar{x}_{ik}] = [x_i, x_{i+1}] \cap [x_k^0, x_{k+1}^0] \). This results in an approximate minimization problem:

\[ \min_{x_1, \ldots, x_M} F_{ap}(\{x_i\}), \quad F_{ap}(\{x_i\}) = \sum_{i=0}^{M} \omega_{i+1/2}^3 h_{i+1/2}^3. \]
Approximate error functional (3/7)

\[
\frac{\partial}{\partial \xi} \left( \omega(x) \frac{\partial x}{\partial \xi} \right) = 0, \quad x(0) = 0, \quad x(1) = 1,
\]

**Algorithm** (equidistribution principle)

For \( k = 1, \ldots, K_{max} \) do

1. For the given grid \( \{x_i^k\} \) compute values \( \omega_{i+1/2}^k, i = 0, \ldots, M \).
2. Perform one Gauss-Seidel sweep

   \[
   \omega_{i+1/2}^k (x_{i+1}^k - x_i^{k+1}) - \omega_{i-1/2}^k (x_i^{k+1} - x_{i-1}^{k+1}) = 0, \quad i = 1, \ldots, M.
   \]

3. Stop iterations if \( \max_i |x_i^k - x_i^{k+1}| \leq TOL \) where \( TOL \) is the user given tolerance.
Approximate error functional (4/7)

\[
\min_{x_1, \ldots, x_M} F_{ap}(\{x_i\}), \quad F_{ap}(\{x_i\}) = \sum_{i=0}^{M} \omega_i^{3} h_{i+1/2}^{3}
\]

**Algorithm** (direct minimization)

For \( k = 1, \ldots, K_{\text{max}} \) do

1. For the given grid \( \{x_i^k\} \) compute values \( \omega_{i+1/2}^k, i = 0, \ldots, M \).

2. Perform one Gauss-Seidel sweep

\[
\min_{x_i^{k+1}} \left\{ \left[ \hat{\omega}_{i+1/2}^{k+1} (x_{i+1}^k - x_i^{k+1}) \right]^3 + \left[ \hat{\omega}_{i-1/2}^{k+1} (x_i^{k+1} - x_{i-1}^k) \right]^3 \right\},
\]

where \( i = 1, \ldots, M \), \( R_h \) is the interpolation operator from grid \( \{x_i^k\} \) to grid \( \{x_i^{k+1}\} \), and \( \hat{\omega}_h^{k+1} = R_h(\omega_h^k) \).

3. Stop iterations if \( \max_i |x_i^k - x_i^{k+1}| \leq TOL \).
Approximate error functional (4/7)

\[
\sum_{i=0}^{M} \omega_{i+1/2}^{k} h_{i+1/2}^{k} = \sum_{i=0}^{M} \hat{\omega}_{i+1/2}^{k+1} h_{i+1/2}^{k+1}.
\]

**Algorithm** (direct minimization)

For \( k = 1, \ldots, K_{\text{max}} \) do

1. For the given grid \( \{x_{i}^{k}\} \) compute values \( \omega_{i+1/2}^{k} \), \( i = 0, \ldots, M \).

2. Perform one Gauss-Seidel sweep

\[
\min_{x_{i+1/2}^{k+1}} \left\{ \left[ \hat{\omega}_{i+1/2}^{k+1} (x_{i+1}^{k+1} - x_{i}^{k+1}) \right]^3 + \left[ \hat{\omega}_{i-1/2}^{k+1} (x_{i}^{k+1} - x_{i-1}^{k+1}) \right]^3 \right\},
\]

where \( i = 1, \ldots, M \), \( R_{h} \) is the interpolation operator from grid \( \{x_{i}^{k}\} \) to grid \( \{x_{i}^{k+1}\} \), and \( \hat{\omega}_{h,k+1} = R_{h}(\omega_{h,k}) \).

3. Stop iterations if \( \max_{i} |x_{i}^{k} - x_{i}^{k+1}| \leq TOL \).
Approximate error functional (5/7)

Let us consider a test function \( f(x) \) given by

\[
f(x) = 1 - \frac{9r_1 + 5r_1^5}{10(r_1 + r_1^5 + r_2)}, \quad r_1 = \exp \frac{1/2 - x}{20\varepsilon}, \quad r_2 = \exp \frac{3/8 - x}{2\varepsilon},
\]

with \( \varepsilon = 0.005 \). Let

\[
E(\{x_i\}) = \sqrt{F_{ex}(\{x_i\})}
\]

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<th>( E({x_i^{ap,st}}) )</th>
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Approximate error functional (6/7)
Approximate error functional (7/7)
Grid smoothing (1/4)

Let the mesh steps satisfy the following condition:

\[
\frac{\alpha}{\alpha + 1} \leq \frac{h_{i-1/2}}{h_{i+1/2}} \leq \frac{\alpha + 1}{\alpha}, \quad i = 1, \ldots, M.
\]

**Lemma.** Let \( \omega_{i+1/2}, i = 0, \ldots, M \), be given values of a monitor function. The values \( \tilde{\omega}_{i+1/2}, i = 0, \ldots, M \), of a smoothed monitor function satisfying

\[
\frac{\alpha}{\alpha + 1} \leq \frac{\tilde{\omega}_{i+1/2}}{\tilde{\omega}_{i-1/2}} \leq \frac{\alpha + 1}{\alpha}, \quad i = 1, \ldots, M,
\]

can be computed by solving the system of \( M + 1 \) linear equations:

\[
\tilde{\omega}_{i+1/2} - \alpha(\alpha + 1)(\tilde{\omega}_{i+3/2} - 2\tilde{\omega}_{i+1/2} + \tilde{\omega}_{i-1/2}) = \omega_{i+1/2},
\]

where \( \tilde{\omega}_{-1/2} = \omega_{1/2} \) and \( \tilde{\omega}_{M+3/2} = \omega_{M+1/2} \).
Grid smoothing (1/4)

Let the mesh steps satisfy the following condition:

\[
\frac{\alpha}{\alpha + 1} \leq \frac{\omega_{i+1/2}}{\omega_{i-1/2}} \leq \frac{\alpha + 1}{\alpha}, \quad i = 1, \ldots, M.
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\]

can be computed by solving the system of \( M + 1 \) linear equations:

\[
\tilde{\omega}_{i+1/2} - \alpha(\alpha + 1)(\tilde{\omega}_{i+3/2} - 2\tilde{\omega}_{i+1/2} + \tilde{\omega}_{i-1/2}) = \omega_{i+1/2},
\]

where \( \tilde{\omega}_{-1/2} = \omega_{1/2} \) and \( \tilde{\omega}_{M+3/2} = \omega_{M+1/2} \).
Grid smoothing (2/4)

**Algorithm** (direct minimization with smoothing)

For $k = 1, \ldots, K_{max}$ do

1. For the given grid $\{x_i^k\}$ compute values $\omega_{i+1/2}^k, i = 0, \ldots, M$.

2. Compute the smoothed values $\tilde{\omega}_{i+1/2}^k, i = 0, \ldots, M$, by solving the tridiagonal system.

3. Perform one Gauss-Seidel sweep

   \[ \min_{x_{i+1}^{k+1}} \left\{ \left[ \tilde{\omega}_{i+1/2}^{k+1} (x_{i+1}^{k+1} - x_{i}^{k+1}) \right]^3 + \left[ \tilde{\omega}_{i-1/2}^{k+1} (x_{i}^{k+1} - x_{i-1}^{k+1}) \right]^3 \right\} , \]

   where $i = 1, \ldots, M$, $R_h$ is the interpolation operator from grid $\{x_i^k\}$ to grid $\{x_{i}^{k+1}\}$, and $\tilde{\omega}^{h,k+1} = R_h(\tilde{\omega}^{h,k})$.

4. Stop iterations if $\max_i |x_i^k - x_i^{k+1}| \leq TOL$ where $TOL$ is the user given tolerance.
Grid smoothing (3/4)

Recall that

\[ E(\{x_i\}) = \sqrt{F_{ex}(\{x_i\})}. \]

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Grid smoothing (4/4)
Burgers equation (1/2)

Let \( T = 0.9, \varepsilon = 0.005 \) and

\[
E(\{x_i^N\}) = \left[ \sum_{i=0}^{M} \int_{x_i^N}^{x_{i+1}^N} (u(x, T) - \bar{u}_{i+1/2}^N)^2 \, dx \right]^{1/2}.
\]
Burgers equation (2/2)

Let $T = 0.5$, $\varepsilon = 0$ and the initial condition be the periodic function

$$u_0(x) = 0.5 + \sin(2\pi x).$$
Conclusion

- The error introduced by the numerical scheme can be ignored even for lower order time integration schemes.
- The error introduced by the numerical interpolation can be ignored when the interpolation operator is more accurate than the discretization.
- Necessity of a grid smoothing has been observed in many numerical experiments.
- The algorithms have shown a robust behavior for 1D Burgers equation.