A family of mimetic finite difference methods on polygonal and polyhedral meshes

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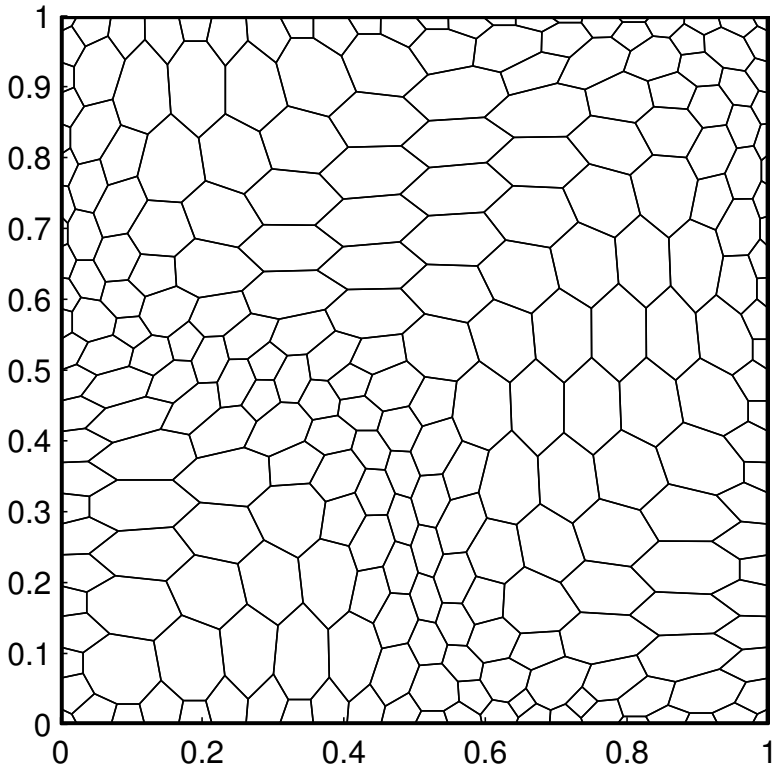
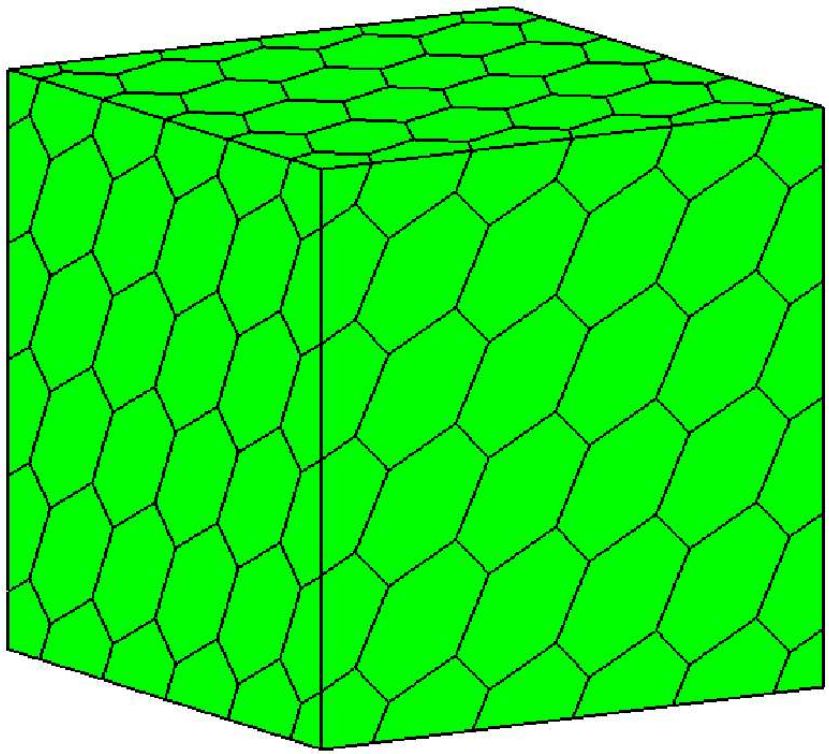
- R. Garimella for mesh generation support

Mimetic finite difference method

Continuum Problem	MFD method
$\begin{aligned} \operatorname{div} \vec{u} &= b \\ \vec{u} &= -\nabla p \end{aligned}$	$\begin{aligned} \mathcal{DIV}\, \boldsymbol{u}^h &= \boldsymbol{b}^h \\ \boldsymbol{u}^h &= -\mathcal{GRAD}\, \boldsymbol{p}^h \end{aligned}$
■ $\operatorname{div} = -\nabla^*$ ■ $\ker(\nabla) = constants$	■ $\mathcal{DIV} = -\mathcal{GRAD}^*$ ■ $\ker(\mathcal{GRAD}) = constants$

Meshes

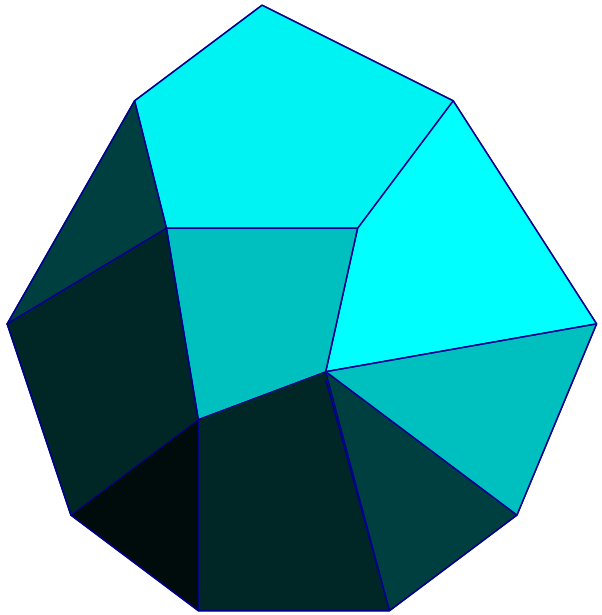




- moving mesh methods (Lagrangian, ALE)
- unlimited possibilities for mesh generation
- polyhedral meshes are preferable for some CFD applications

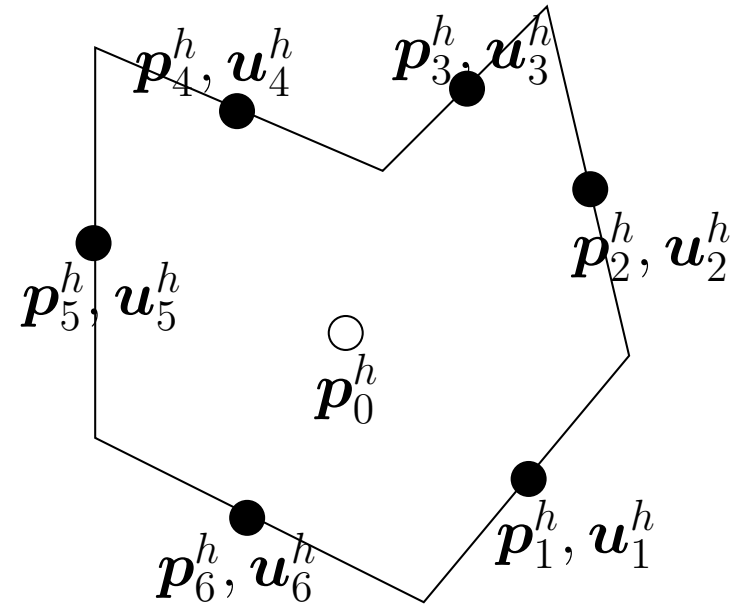
Degrees of freedom

- 1 pressure d.o.f. per mesh cell
- 1 velocity d.o.f. per mesh face
- 1 pressure d.o.f. per mesh face



Degrees of freedom

- 1 pressure d.o.f. per mesh cell
- 1 velocity d.o.f. per mesh edge
- 1 pressure d.o.f. per mesh edge

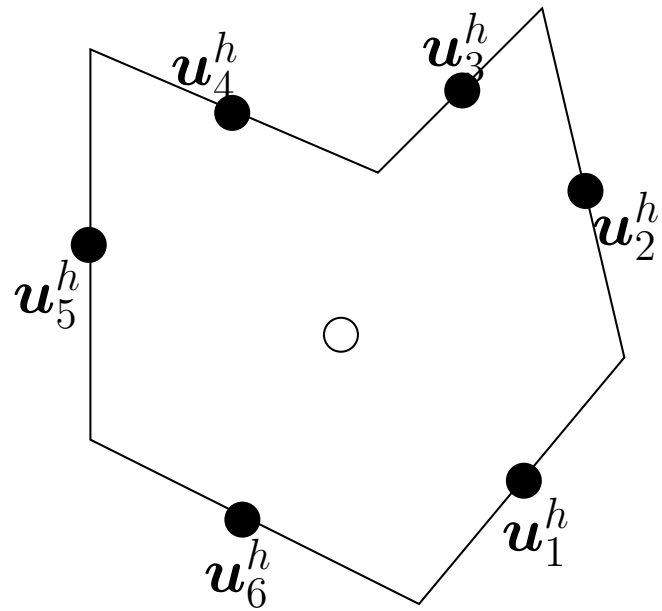


$$\boldsymbol{u}_i^h \approx \frac{1}{|e_i|} \int_{e_i} \vec{u} \cdot \vec{n}_i$$

Local divergence operator

The divergence theorem

$$\operatorname{div} \vec{u} = \lim_{V_E \to 0} \frac{1}{V_E} \oint_{\partial E} \vec{u} \cdot \vec{n}$$



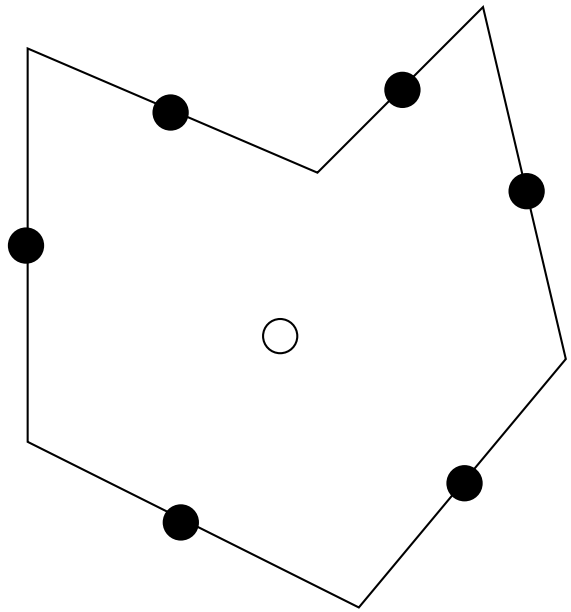
implies

$$\left(\mathcal{DIV}\ \boldsymbol{u}^h\right)_E = \frac{1}{V_E} \sum_i \boldsymbol{u}_i^h \, |e_i|$$

Local gradient operator

$$\boldsymbol{u}^h = -\mathcal{GRAD}\,\boldsymbol{p}^h$$

$$\begin{bmatrix} \boldsymbol{u}_1^h \\ \boldsymbol{u}_2^h \\ \vdots \\ \boldsymbol{u}_6^h \end{bmatrix} = \mathbb{M}_{6\times 6} \begin{bmatrix} |e_1|\,(\boldsymbol{p}_1^h - \boldsymbol{p}_0^h) \\ |e_2|\,(\boldsymbol{p}_2^h - \boldsymbol{p}_0^h) \\ \vdots \\ |e_6|\,(\boldsymbol{p}_6^h - \boldsymbol{p}_0^h) \end{bmatrix}$$



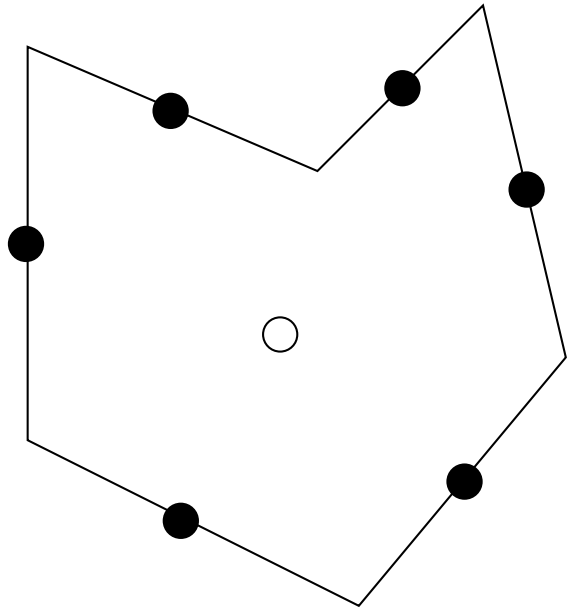
- $\mathbb{M} = \mathbb{M}^T > 0\ (symmetry \quad \& \quad positivity)$
- $\ker(\mathcal{GRAD}) = constants$

Patch test

Exact gradient for linear p:

$$p = x \quad \rightarrow \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{u}_i^h = \frac{1}{|e_i|} \int_{e_i} \vec{u} \cdot \vec{n}_i = n_{i,x}$$

$$\begin{bmatrix} n_{1,x} \\ n_{2,x} \\ \vdots \\ n_{6,x} \end{bmatrix} = \mathbb{M}_{6\times 6} \begin{bmatrix} |e_1| \, (x_1 - x_0) \\ |e_2| \, (x_2 - x_0) \\ \vdots \\ |e_6| \, (x_6 - x_0) \end{bmatrix}$$



Patch test (cont.)

Exact gradient for linear p:

$$p = y \quad \rightarrow \quad \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{u}_i^h = \frac{1}{|e_i|} \int_{e_i} \vec{u} \cdot \vec{n}_i = n_{i,y}$$

$$\begin{bmatrix} n_{1,x} & n_{1,y} \\ n_{2,x} & n_{2,y} \\ \vdots & \\ n_{6,x} & n_{6,y} \end{bmatrix} = \mathbb{M}_{6\times 6} \begin{bmatrix} |e_1|\,(x_1 - x_0) & |e_1|\,(y_1 - y_0) \\ |e_2|\,(x_2 - x_0) & |e_1|\,(y_2 - y_0) \\ \vdots & \vdots \\ |e_6|\,(x_6 - x_0) & |e_1|\,(y_6 - y_0) \end{bmatrix}$$

$$\mathbb{N}_{6\times 2} = \mathbb{M}_{6\times 6}\, \mathbb{R}_{6\times 2}$$

Properties of matrices $\mathbb{N}$ and $\mathbb{R}$

Lemma.

$$\mathbb{N}^T \mathbb{R} = \mathbb{R}^T \mathbb{N} = \mathbb{I}_{2\times 2}$$

Proof.

$$\begin{aligned} V_E = \int_E \nabla x \cdot \nabla (x - x_0) &= \int_{\partial E} (\nabla x \cdot \vec{n})(x - x_0) = \sum_i n_{i,x} \int_{e_i} (x - x_0) \\ &= \sum_i n_{i,x} |e_i| (x_i - x_0) = \sum_i \mathbb{N}_{i,1} \mathbb{R}_{i,1} \\ &= (\mathbb{N}^T \mathbb{R})_{1,1} \end{aligned}$$

Similarly,

$$0 = \int_E \nabla y \cdot \nabla (x - x_0) = \int_{\partial E} (\nabla y \cdot \vec{n})(x - x_0) = (\mathbb{N}^T \mathbb{R})_{2,1}$$

Solution method

A solution to

$$\mathbb{N} = \mathbb{M}\,\mathbb{R}$$

is

$$\mathbb{M}_0 = \mathbb{N}\,\mathbb{N}^T$$

- check: $\mathbb{M}_0\,\mathbb{R} = \mathbb{N}\,\mathbb{N}^T\,\mathbb{R} = \mathbb{N}\,\mathbb{I} = \mathbb{N}$
- valid for *any* polygon and *any* polyhedron with planar faces
- $\mathbb{M}_0 = \mathbb{M}_0^T \geq 0$
- general form for the solution is $\mathbb{M}_0 + \mathbb{M}_1$ where $\mathbb{M}_1\,\mathbb{R} = 0$

Family of MFD methods

Theorem. Let columns of $\mathbb{D}$ span $\ker(\mathbb{R}^T)$, i.e.

$$\mathbb{R}^T\,\mathbb{D} = 0 \qquad \text{and} \qquad \mathbb{R}\,\mathbb{D}^T = 0.$$

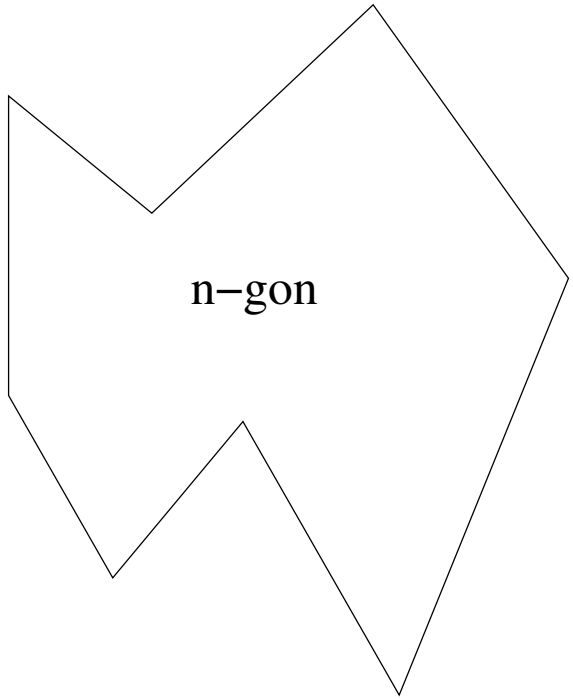
Then,

$$\mathbb{M} = \mathbb{N}\,\mathbb{N}^T + \mathbb{D}\,\mathbb{U}\,\mathbb{D}^T$$

is the SPD matrix for any $\mathbb{U} = \mathbb{U}^T > 0$.

Family of MFD methods (cont.)

$$\mathbb{M} = \mathbb{N}\,\mathbb{N}^T + \mathbb{D}\,\mathbb{U}\,\mathbb{D}^T$$



- size of $\mathbb{U}$ is $n-2$
- $(n-1)(n-2)/2$ free coefficients
- If both term are balanced, then
  - 2nd order convergence for pressure
  - 1st order convergence for velocity
- the same formula holds for polyhedral meshes
- straightforward generalization to full tensor $\mathbb{K}$

Numerical experiments

$$\mathbb{M} = \mathbb{N}\,\mathbb{N}^T + \mathbb{D}\,\mathbb{U}\,\mathbb{D}^T$$

Let $\mathbb{D}$ be symmetric orthogonal projector onto $\ker(\mathbb{R}^T)$ and

$$\mathbb{U} = u\mathbb{I}, \qquad u = \frac{1}{V_E}.$$

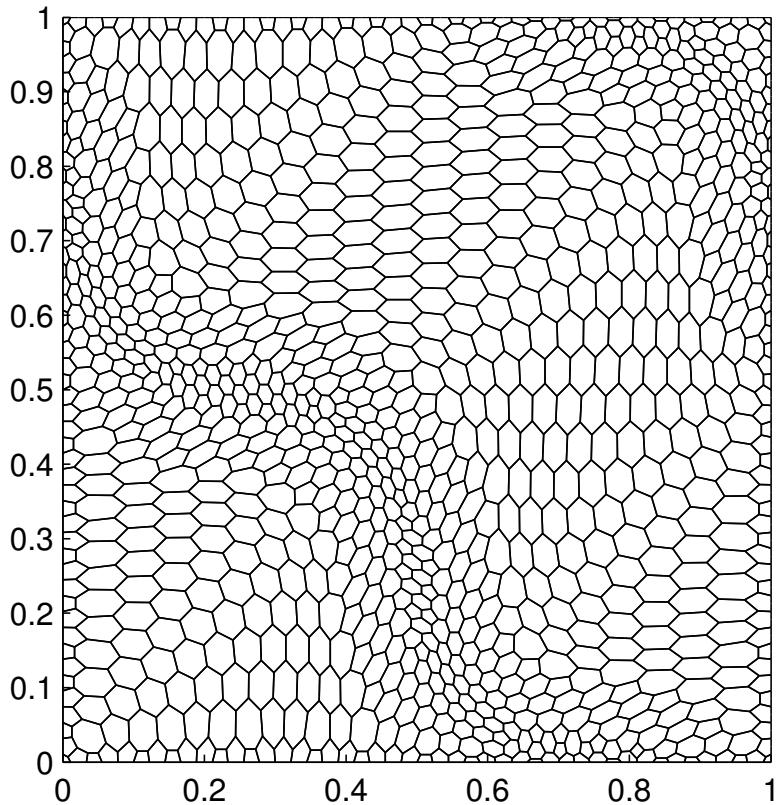
Then

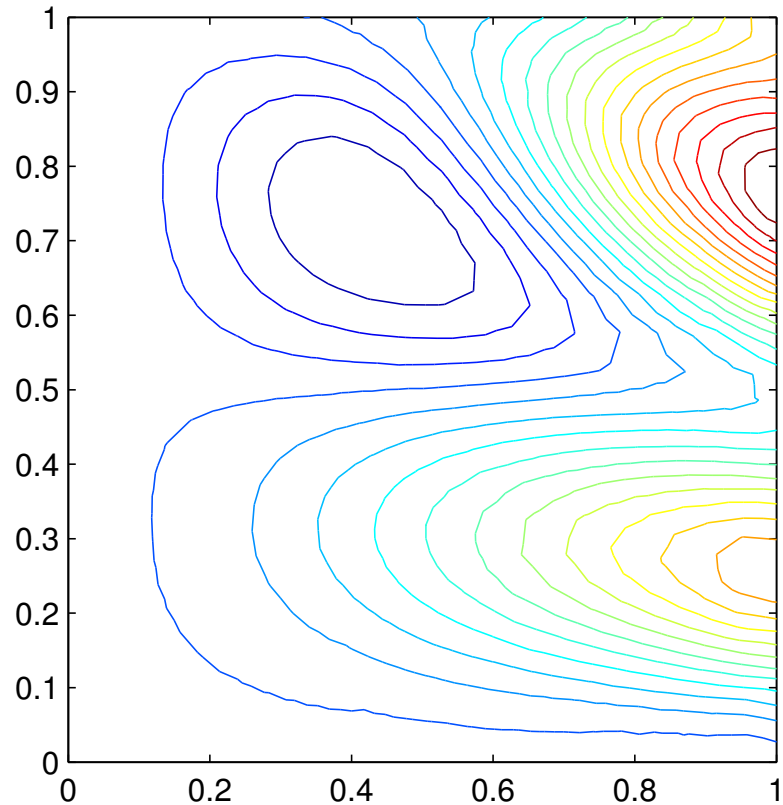
$$\mathbb{M} = \mathbb{N}\,\mathbb{N}^T + u\mathbb{D}$$

- complexity of computing $\mathbb{M}_{n\times n}$ is $(2d+1)n^2 + 4d^2n$

Numerical experiments

$$p(x,\, y) = x^3 y^2 + x \sin(2\pi x y) \sin(2\pi y), \qquad \mathbb{K} = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}$$





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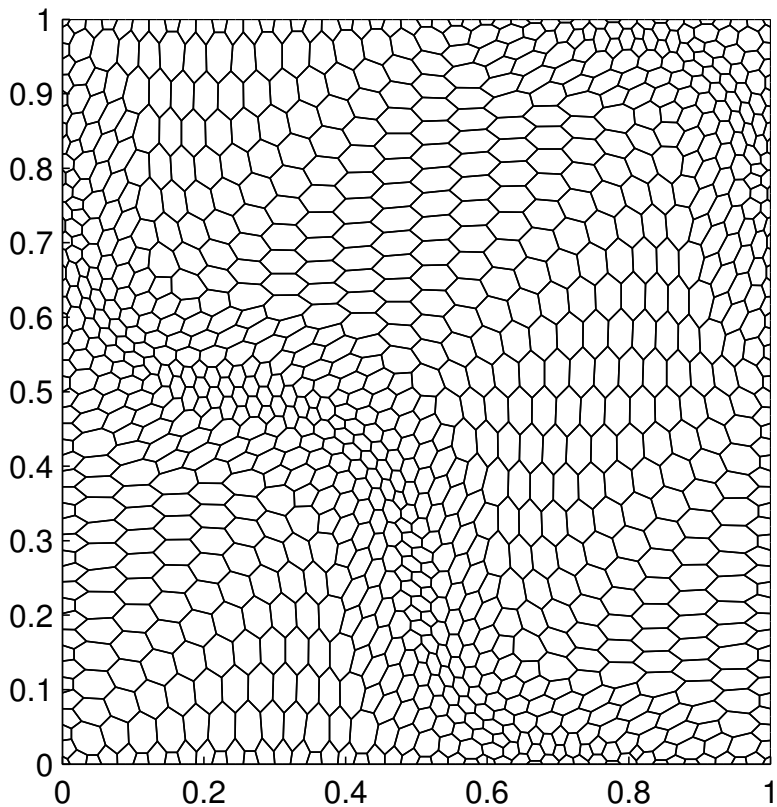
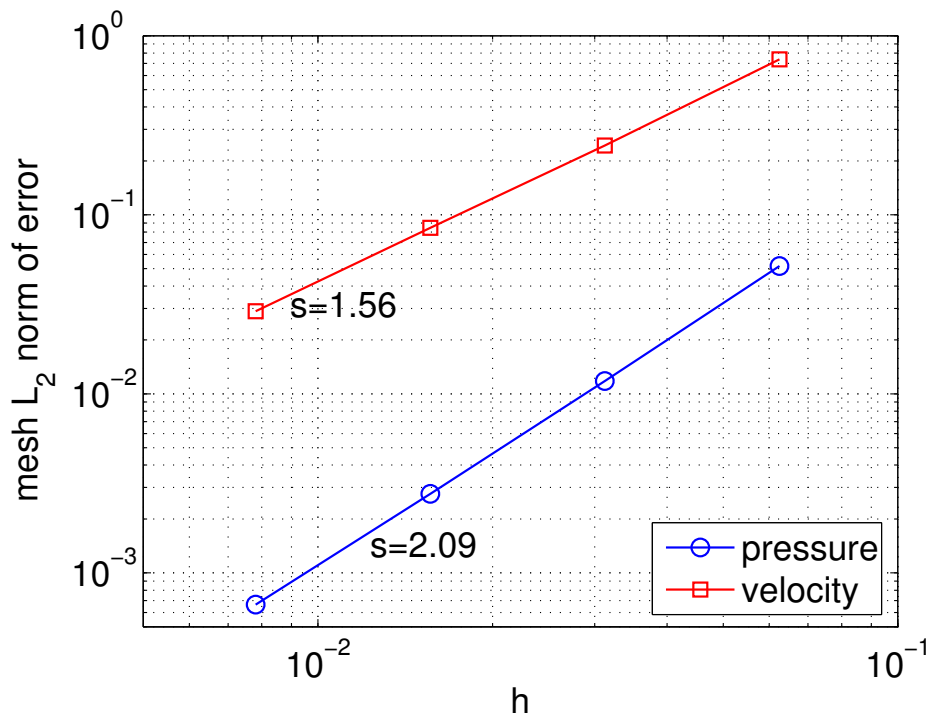
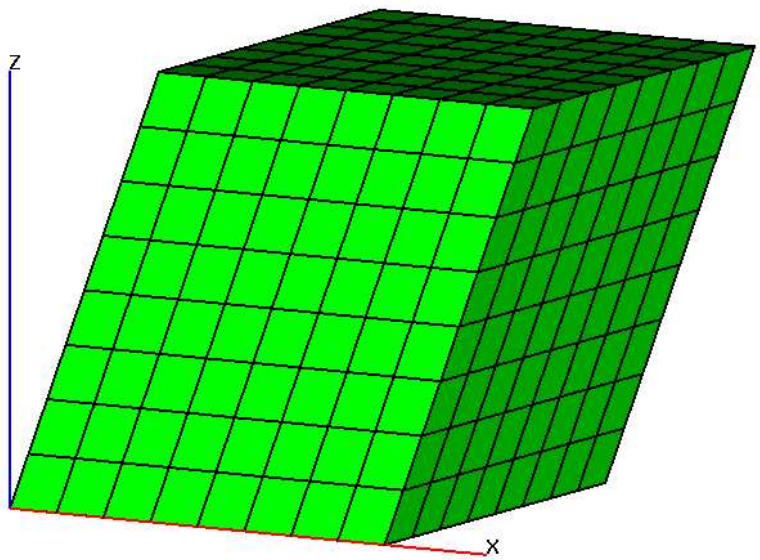
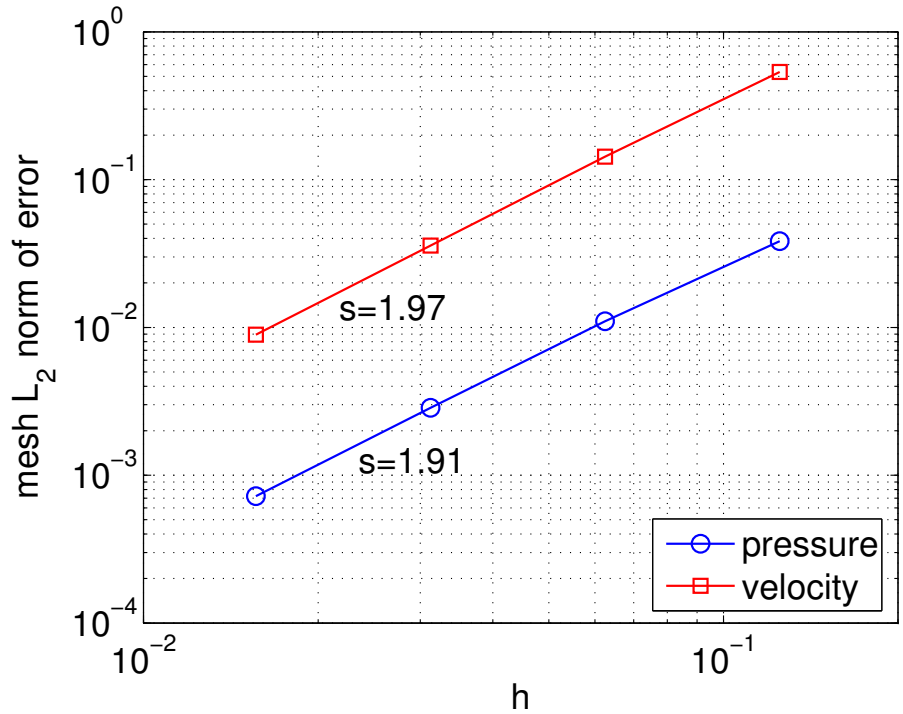




Numerical experiments

Problem similar to the above 2D problem

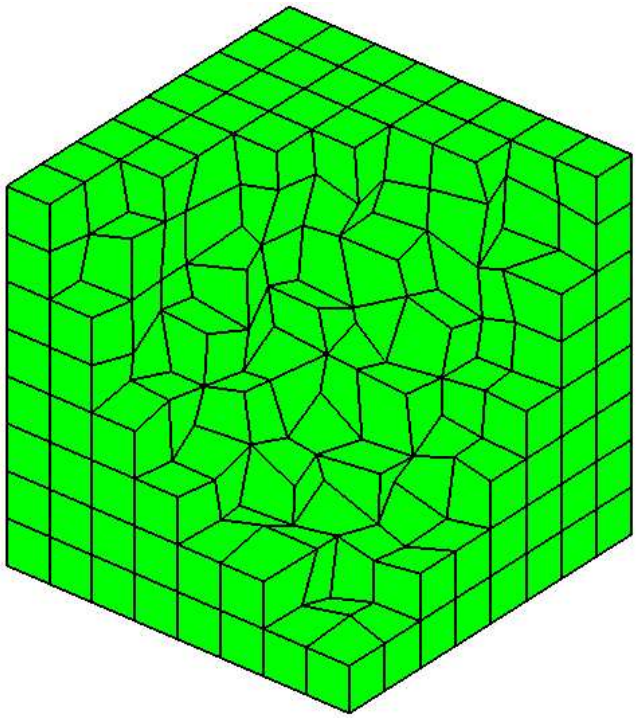


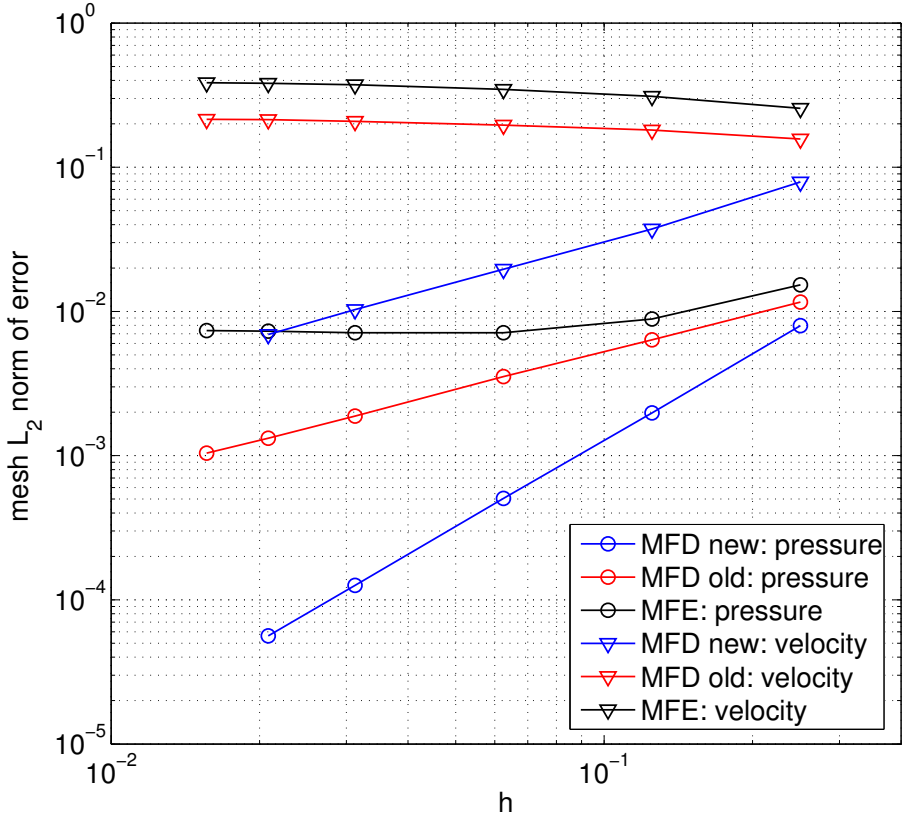


Conclusion

- For diffusion problems on unstructured polygonal and polyhedral meshes, we developed a *family of mimetic finite difference* methods with the following properties:
  - methods have optimal convergence rates
  - they result in SPD matrices
- the methodology can be extended to cells with strongly curved faces

Conclusion (cont.)





- the mixed FE method does *not* converge on randomly perturbed meshes