Mimetic discretizations for diffusion equation on polygonal meshes in Cartesian and Cylindrical geometries

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Objectives

- mimetic ideas in nature

Mimic octopus

Objectives

- mimetic ideas in numerical analysis
- mimetic finite elements
- mimetic finite differences

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- mimetic ideas in numerical analysis
  `(e.g., variational methods)`
- mimetic finite elements
  `(e.g., Raviart-Thomas and Nedelec elements)`
- mimetic finite differences

$$\begin{aligned} \text{curl grad} &= 0 \\ \text{div curl} &= 0 \\ \text{grad} &= -\text{div}^* \end{aligned}$$

Objectives

- mimetic ideas in numerical analysis
  (e.g., variational methods)
- mimetic finite elements
  (e.g., Raviart-Thomas and Nedelec elements)
- mimetic finite differences

$$\mathrm{CURL}\ \mathrm{GRAD} = 0$$

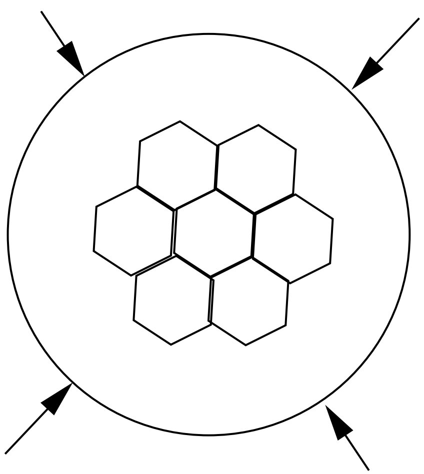
$$\mathrm{DIV}\ \mathrm{CURL} = 0$$

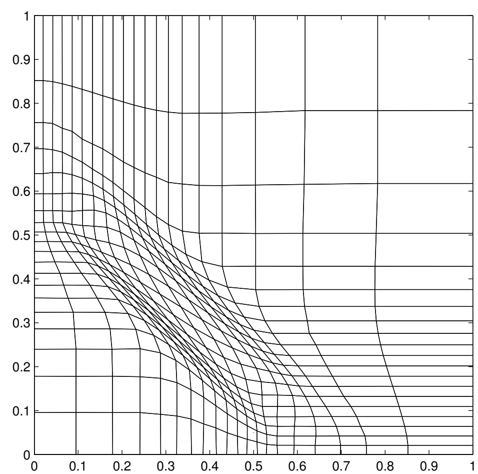
$$\mathrm{GRAD} = -\mathrm{DIV}^*$$

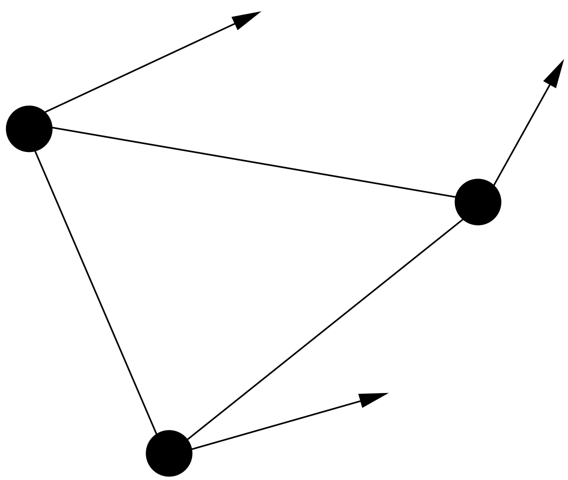
Objectives

Polygonal meshes are needed in ALE simulations:

- spherical contraction (polygon is more stable than triangle);
- front capturing (bad approximation near domain boundaries);
- deadlock in incompressible fluid simulations (triangle shape is too restrictive).



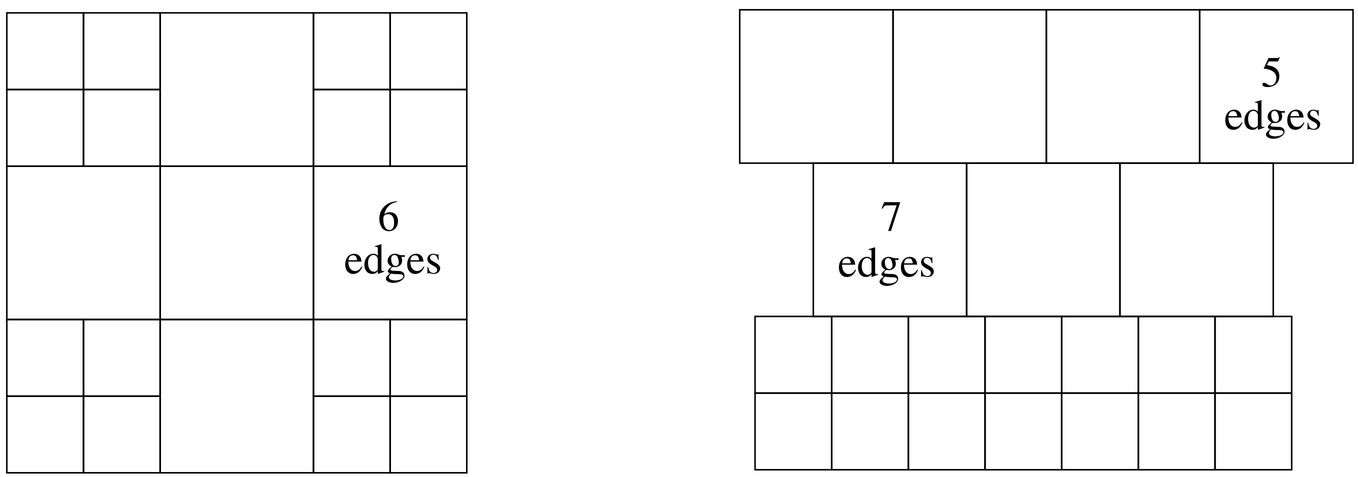




Objectives

Polygonal meshes are appeared in many applications:

- adaptive refinement with hanging nodes (degenerated polygons);
- non-matching grids (big aspect ratio of polygon edges).



Contents

- Support operator methodology
- Scalar product in space of fluxes
- Numerical experiments
- Conclusion

Support operators

$$\boxed{\int_\Omega \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x = -\int_\Omega p\, \mathrm{div}\, \boldsymbol{f}\, \mathrm{d}x \qquad \forall \boldsymbol{f} \in H_{div}(\Omega), \quad p \in H_0^1(\Omega)}$$

$$\downarrow$$

$$SO$$

$$\downarrow$$

$$\boxed{[\mathrm{GRAD}\, p^h,\, f^h]_X = -[p^h,\, \mathrm{DIV}\, f^h]_Q \qquad \forall f^h \in X_h, \quad p^h \in Q_h}$$

Support operators

$$\boxed{\int_\Omega \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x = -\int_\Omega p\, \mathrm{div}\, \boldsymbol{f}\, \mathrm{d}x \qquad \forall \boldsymbol{f} \in H_{div}(\Omega), \quad p \in H^1_0(\Omega)}$$

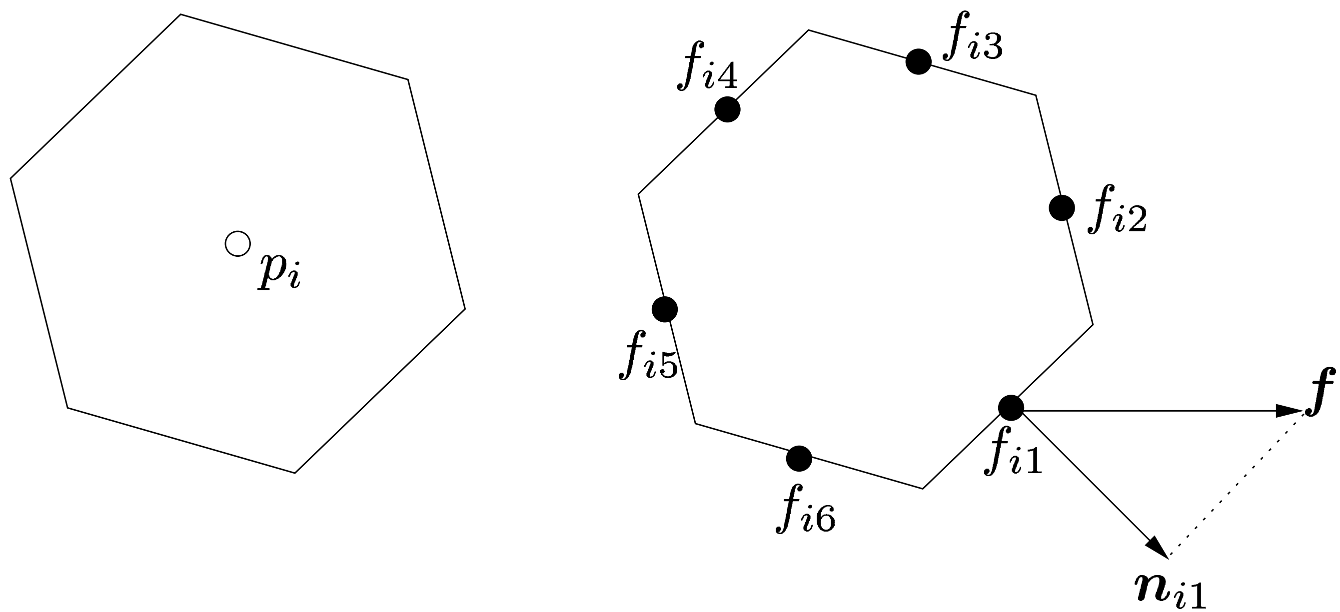
Support-operators (SO) methodology:

1. define degrees of freedom for p and $\boldsymbol{f}$;
2. discretize the divergence operator, $\mathrm{DIV} : X_h \to Q_h$;
3. equip discrete spaces with scalar products $[\cdot\,,\,\cdot]_Q$ and $[\cdot\,,\,\cdot]_X$;
4. derive the discrete gradient operator, $\mathrm{GRAD} : Q_h \to X_h$, from

$$\boxed{[\mathrm{GRAD}\, p^h,\ f^h]_X = -[p^h,\ \mathrm{DIV}\, f^h]_Q \qquad \forall f^h \in X_h, \quad p^h \in Q_h}$$

Support operators

Step 1. Define degrees of freedom for p and $\boldsymbol{f}$.



- p_i is defined at center of cell of e_i, $\quad p^h = (p_1, \ldots, p_N)$.
- $f_{i1}, \ldots, f_{i6}$ are defined at centers of cell edges, $\quad f^h = (f_1, \ldots, f_M)$.
  They approximate the normal components of $\boldsymbol{f}$, e.g.

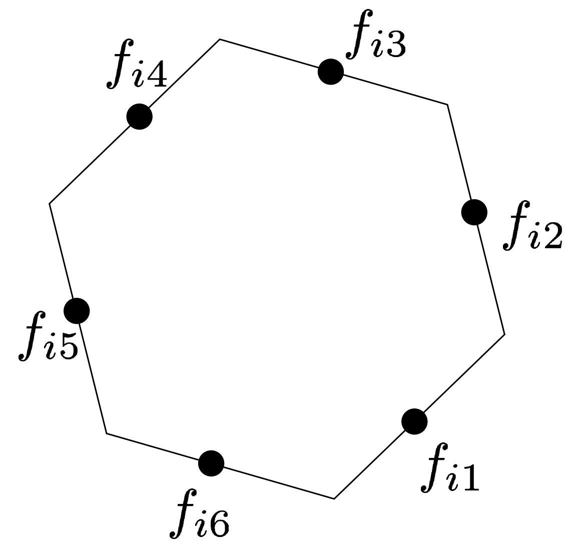
$$f_{i1} \approx \boldsymbol{f} \cdot \boldsymbol{n}_{i1}.$$

Support operators

Steps 2. Discretize the divergence operator.

Gauss' theorem:

$$\operatorname{div}\boldsymbol{f} = \lim_{|e|\to 0} \frac{1}{|e|} \oint_{\partial e} \boldsymbol{f}\cdot\boldsymbol{n}\, dl$$



Center-point quadrature gives

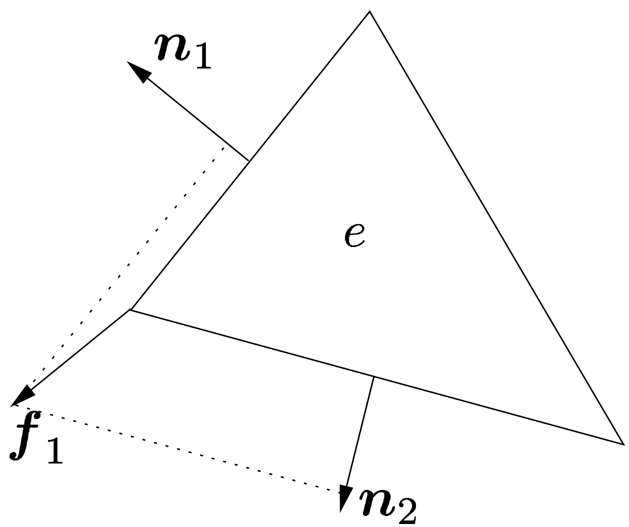
$$\left(\mathrm{DIV}\, f^h\right)_i = \frac{1}{|e_i|}\left(f_{i1}\,|\ell_1| + f_{i2}\,|\ell_2| + f_{i3}\,|\ell_3| + f_{i4}\,|\ell_4| + f_{i5}\,|\ell_5| + f_{i6}\,|\ell_6| \right)$$

Support operators

Step 3. Equip the discrete spaces with scalar products.

- $$[p^h,\, q^h]_Q = \sum_{cells} |e_i|\, p_i^h\, q_i^h \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}v.$$

- $$[\boldsymbol{f}^h,\, \boldsymbol{g}^h]_X = \sum_{cells} [\boldsymbol{f}^h,\, \boldsymbol{g}^h]_{X,e_i} \approx \sum_{cells} \int_{e_i} \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x.$$



The vectors can be recovered uniquely at 3 vertices of triangle e. Then

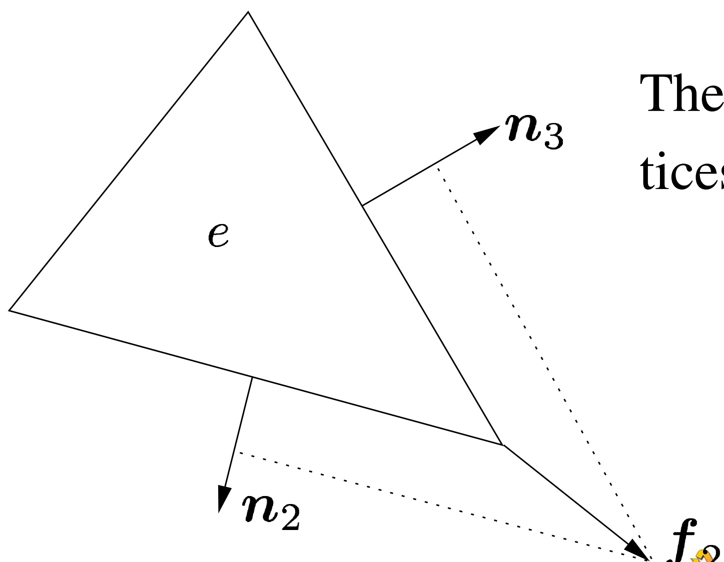
$$[\boldsymbol{f}^h,\, \boldsymbol{g}^h]_{X,e} = \frac{|e|}{3} \sum_{j=1}^{3} \boldsymbol{f}_j \cdot \boldsymbol{g}_j$$

Support operators

Step 3. Equip the discrete spaces with scalar products.

- $$[p^h,\, q^h]_Q = \sum_{cells} |e_i|\, p_i^h\, q_i^h \approx \int_\Omega p(x)\, q(x)\, \mathrm{d}v.$$

- $$[f^h,\, g^h]_X = \sum_{cells} [f^h,\, g^h]_{X,e_i} \approx \sum_{cells} \int_{e_i} \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x.$$



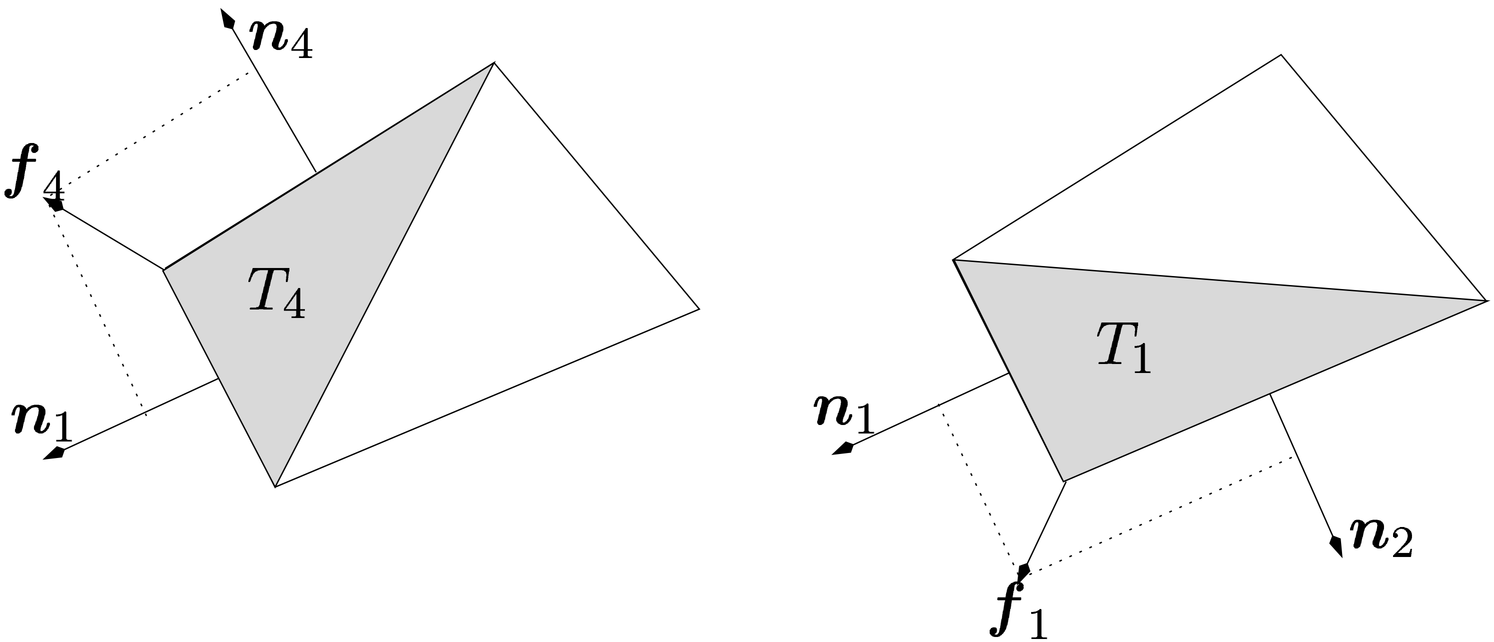
The vectors can be recovered uniquely at 3 vertices of triangle e. Then

$$[f^h,\, g^h]_{X,e} = \frac{|e|}{3} \sum_{j=1}^{3} \boldsymbol{f}_j \cdot \boldsymbol{g}_j$$

Support operators

Step 3 (cont.) Equip the discrete spaces with scalar products.

The vectors can be recovered uniquely at 4 vertices of non-degenerated quadrilateral e.

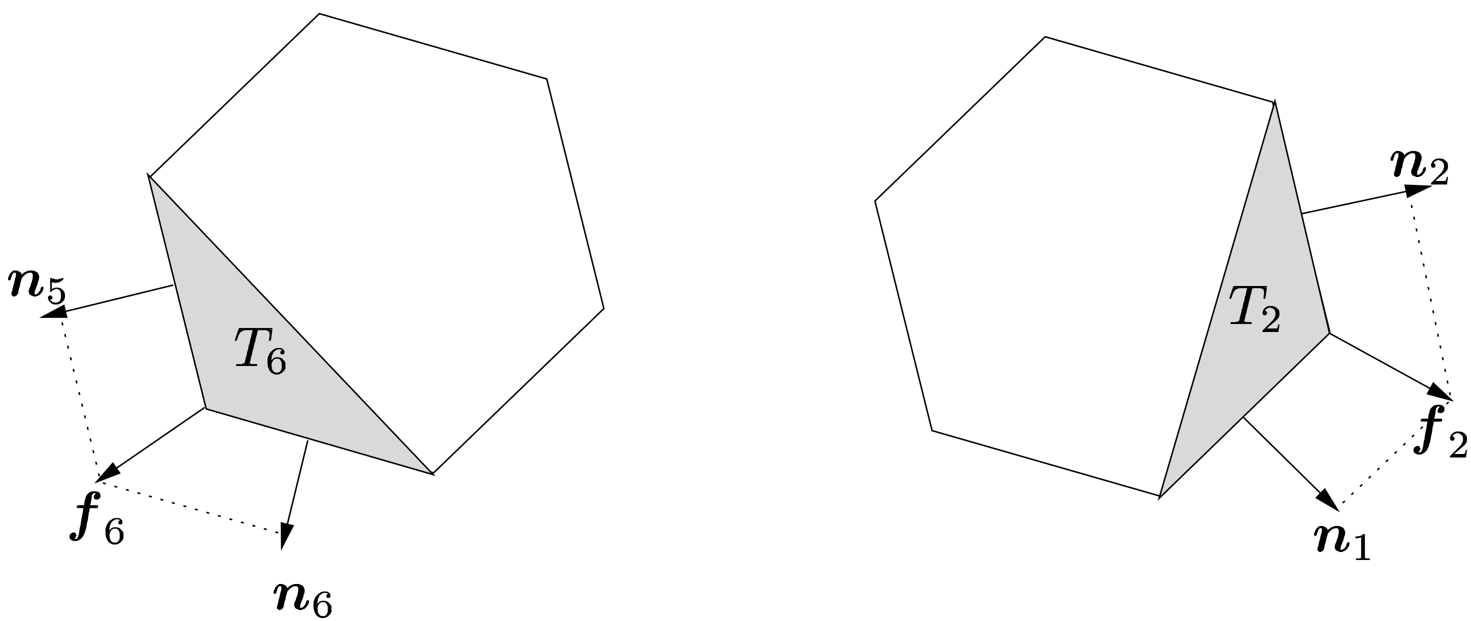


$$[f^h,\, g^h]_{X,e} = \frac{1}{2}\sum_{j=1}^{4} |T_j|\boldsymbol{f}_j \cdot \boldsymbol{g}_j$$

Support operators

Step 3 (cont.) Equip the discrete spaces with scalar products.

The vectors can be recovered uniquely at vertices of non-degenerated polygon e.



$$[f^h,\, g^h]_{X,e} = \frac{|e|}{\sum_j |T_j|} \sum_{j=1}^{6} |T_j| \boldsymbol{f}_j \cdot \boldsymbol{g}_j$$

Support operators

Steps 4. Derive the discrete gradient operator.

The derived operator GRAD is implicitly given by

$$[\mathrm{GRAD}\, p^h,\ f^h]_X = -[p^h,\ \mathrm{DIV}\ f^h]_Q \qquad \forall f^h \in X_h, \quad p^h \in Q_h$$

Support operators

$$\int_\Omega \operatorname{grad} p\, \boldsymbol{f}\, \mathrm{d}x = -\int_\Omega p \operatorname{div} \boldsymbol{f}\, \mathrm{d}x \qquad \forall \boldsymbol{f} \in H_{div}(\Omega), \quad p \in H_0^1(\Omega)$$

Support-operators (SO) methodology:

1. define degrees of freedom for p and $\boldsymbol{f}$;
2. discretize the divergence operator, DIV $: X_h \to Q_h$;
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$$[\mathrm{GRAD}\, p^h,\ f^h]_X = -[p^h,\ \mathrm{DIV}\, f^h]_Q \qquad \forall f^h \in X_h, \quad p^h \in Q_h$$

Mimetic discretizations

The stationary diffusion problem

$$\begin{aligned} -\mathrm{div}\,(\boldsymbol{K}\nabla p) &= Q \quad \text{in} \quad \Omega, \\ p &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

is rewritten as the 1st order system:

$$\boldsymbol{f} = -\boldsymbol{K}\nabla p, \qquad \mathrm{div}\,\boldsymbol{f} = Q,$$

and is discretized as follows:

$$f^h = -\mathrm{GRAD}\, p^h, \qquad \mathrm{DIV}\, f^h = Q^h.$$

Properties of GRAD operator

By the definition

$$[\text{GRAD}\, p^h,\, f^h]_X = -[p^h,\, \text{DIV}\; f^h]_Q.$$

Let $< \cdot\,,\, \cdot >$ be the usual vector dot product. Then

$$[p^h,\, q^h]_Q =< \mathcal{D}p^h,\, q^h >, \qquad [f^h,\, g^h]_X =< \mathcal{M} f^h,\, g^h > .$$

We get

$$\begin{aligned}[\text{GRAD}\, p^h,\, f^h]_X &= < \mathcal{M}\, \text{GRAD}\, p^h,\, f^h > \\ &= -[p^h,\, \text{DIV}\, f^h]_Q = - < \text{DIV}^t\, \mathcal{D}\, p^h,\, f^h > .\end{aligned}$$

Therefore,

$$\text{GRAD} = -\mathcal{M}^{-1}\, \text{DIV}^t\, \mathcal{D}.$$

is the non-local operator.

Local support operators

$$\boxed{\int_{e_i} \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x + \int_{e_i} p\, \mathrm{div}\, \boldsymbol{f}\, \mathrm{d}x = \int_{\partial e_i} p\, \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}s}$$

$$SO$$

$$\boxed{[\mathrm{GRAD}\, p^h,\, f^h]_{X,e_i} + [p^h,\, \mathrm{DIV}\, f^h]_{Q,e_i} =< p^h,\, f^h >_{W,e_i}.}$$

Local support operators

$$\int_{e_i} \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x + \int_{e_i} p\, \mathrm{div}\, \boldsymbol{f}\, \mathrm{d}x = \int_{\partial e_i} p\, \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}s$$

Support-operators (SO) methodology:

1. define degrees of freedom for p and $\boldsymbol{f}$;
2. discretize the divergence operator, DIV;
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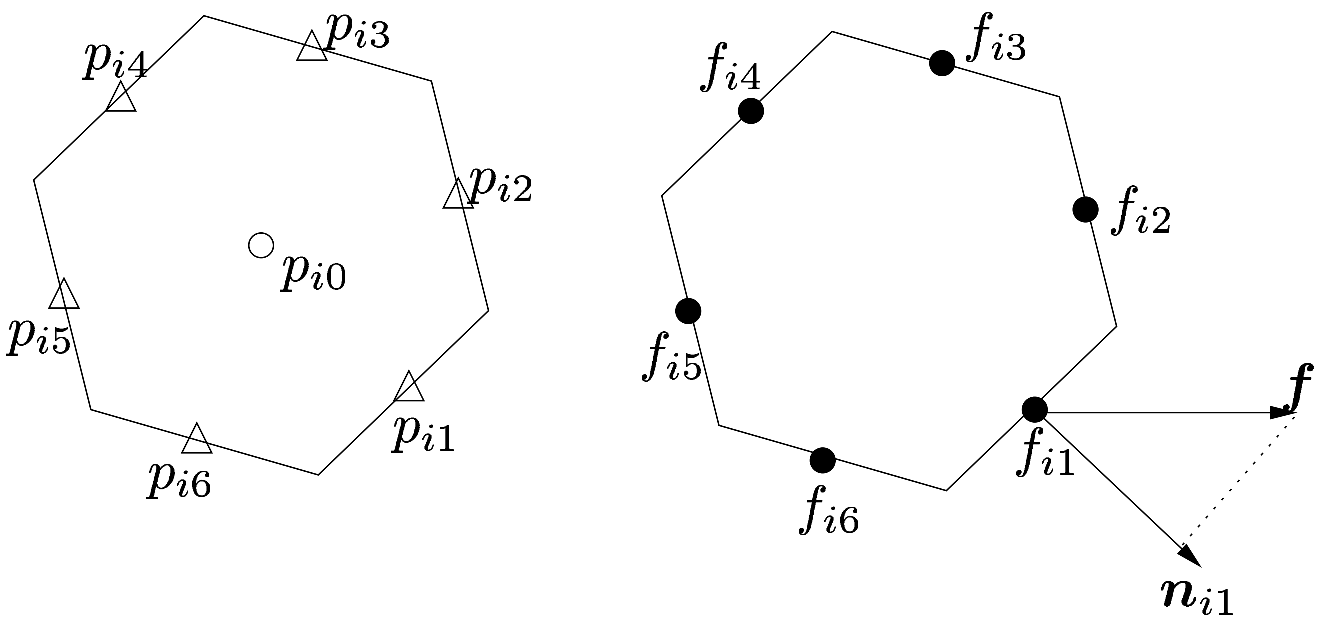
$$[\mathrm{GRAD}\, p^h,\, f^h]_{X,e_i} + [p^h,\, \mathrm{DIV}\, f^h]_{Q,e_i} =< p^h,\, f^h >_{W,e_i}.$$

Local support operators

Step 1. Define degrees of freedom for p and $\boldsymbol{f}$.



- p_{i0} is defined at center of cell e_i.
  $p_{i1}, \ldots, p_{i6}$ are defined at centers of cell edges, $\quad p^h = (p_0, \ldots, p_6)$.
- $f_{i1}, \ldots, f_{i6}$ are defined at centers of cell edges, $\quad f^h = (f_1, \ldots, f_6)$.
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Local support operators

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Gauss' theorem:

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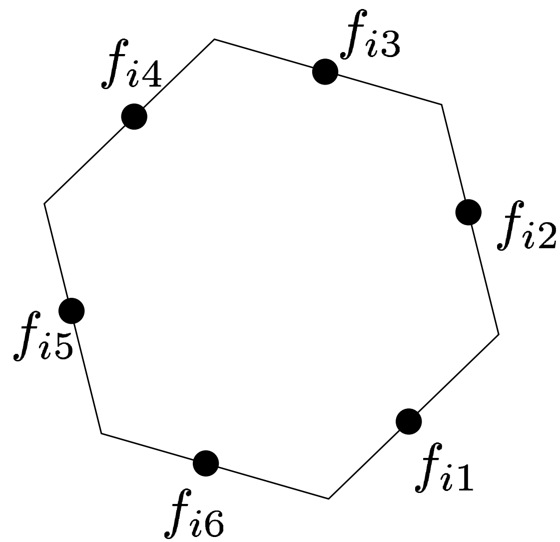


Center-point quadrature gives

$$\left(\mathrm{DIV}\, f^h\right)_i = \frac{1}{|e_i|} \left(f_{i1}\, |\ell_1| + f_{i2}\, |\ell_2| + f_{i3}\, |\ell_3| + f_{i4}\, |\ell_4| + f_{i5}\, |\ell_5| + f_{i6}\, |\ell_6| \right)$$

Local support operators

Step 3. **Equip the discrete spaces with scalar products.**

- $$[p^h,\, q^h]_{Q,e_i} = |e_i|\, p_{i0}\, q_{i0} \approx \int_{e_i} p(x)\, q(x)\, \mathrm{d}v.$$

- $$[f^h,\, g^h]_{X,e_i} = \frac{|e_i|}{\sum_j |T_{ij}|} \sum_{j=1}^{s} |T_{ij}|\, \boldsymbol{f}_{ij} \cdot \boldsymbol{g}_{ij} \approx \int_{e_i} \boldsymbol{f}(x) \cdot \boldsymbol{g}(x)\, \mathrm{d}x.$$

- $$< p^h,\, q^h >_{W,e_i} = \sum_{j=1}^{s} |\ell_{ij}|\, p_{ij}\, q_{ij} \approx \int_{\partial e_i} p(x)\, q(x)\, \mathrm{d}x.$$

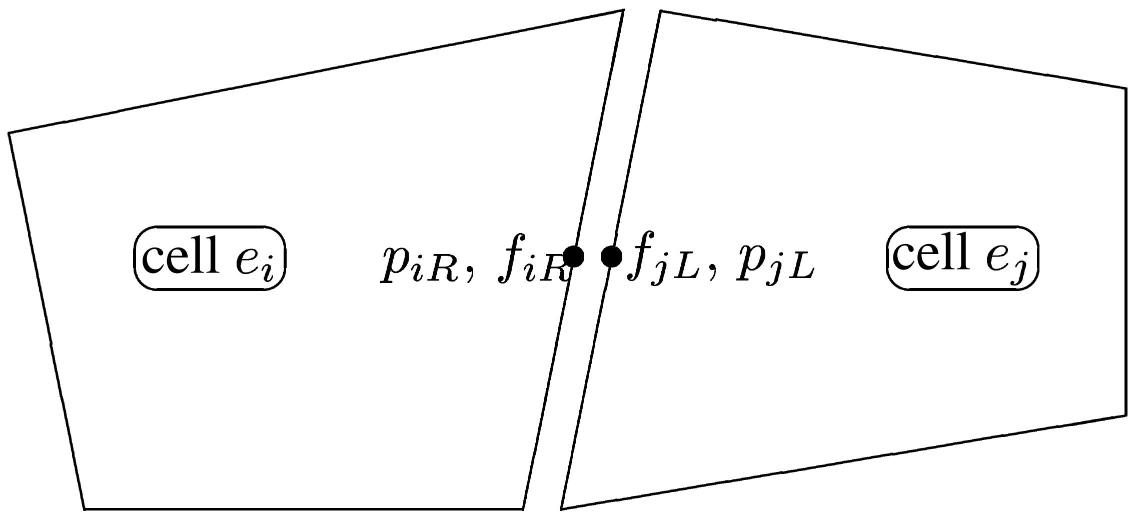
Local support operators

Steps 4. Derive the discrete gradient operator.

The derived local operator GRAD is implicitly given by

$$[\mathrm{GRAD}\, p^h,\, f^h]_{X,e_i} + [p^h,\, \mathrm{DIV}\, f^h]_{Q,e_i} = < p^h,\, f^h >_{W,e_i} \qquad \forall f^h,\, p^h.$$

Continuity conditions



The global discretization is achieved by imposing the continuity of fluxes

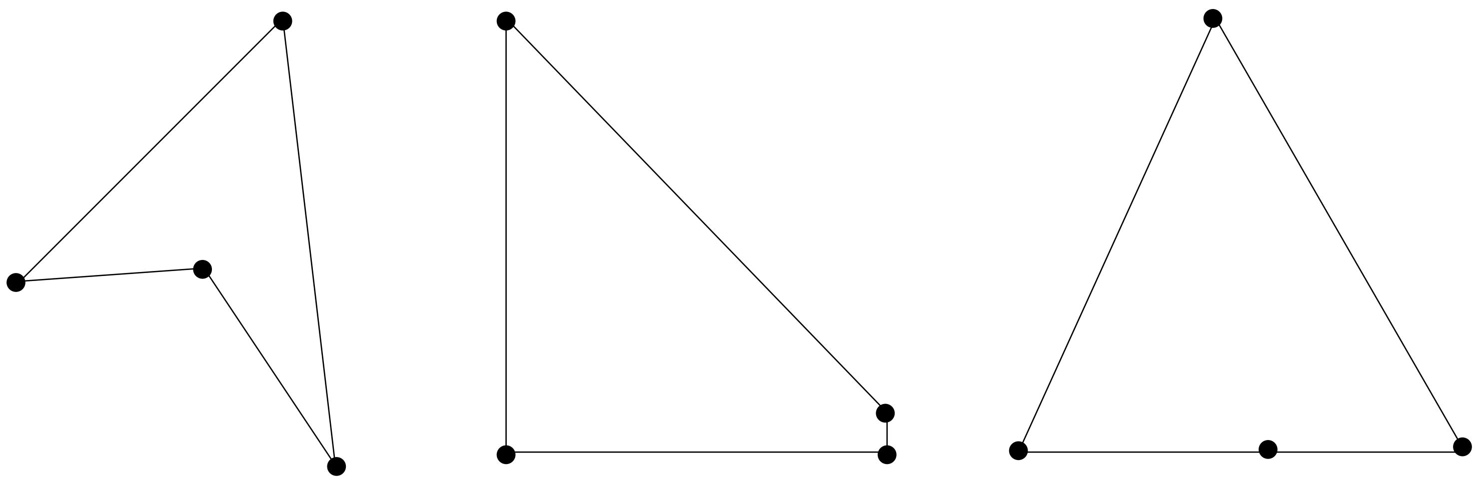
$$f_{iR} = -f_{jL}$$

and interface intensities

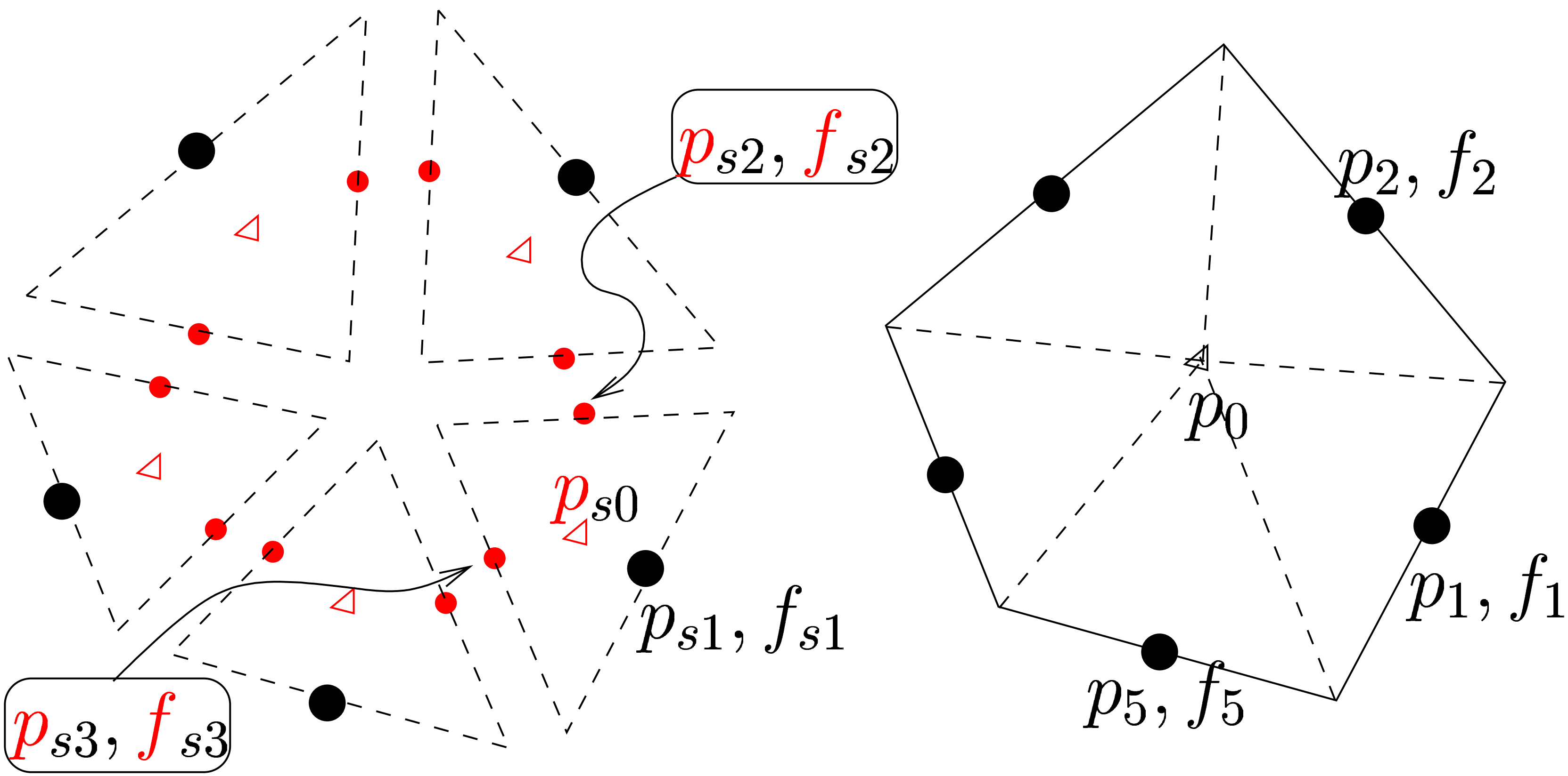
$$p_{iR} = p_{jL}.$$

Scalar product of fluxes

- triangle ✔✔✔
- quadrilateral ✔✔ (strict convexity is required, linear solution is preserved)
- polygon ✔ (strict convexity is required, linear solution is NOT preserved)



Scalar product of fluxes



- d.o.f. for polygonal scheme differ from d.o.f. for triangular scheme;
- the added d.o.f. can not be explicitly eliminated.

Scalar product of fluxes

Let
$$f^h = (f^h,\ f^h_{int}).$$
Then

$$[\mathrm{GRAD}p^h,\ f^h]_{X,\Delta_s} + [p^h,\ \mathrm{DIV}\ f^h]_{Q,\Delta_s} =< p^h,\ f^h >_{W,\Delta_s}\ .$$

- the formulas are exact for linear solutions;
- our goal is to find $[\cdot,\ \cdot]_{X,e}$ such that

$$[\mathrm{GRAD}p^h,\ f^h]_{X,e} + [p^h,\ \mathrm{DIV}\ f^h]_{Q,e} =< p^h,\ f^h >_{W,e}$$

  is exact for linear solution;
- it can be done if

$$(\mathrm{DIV}\,f^h)_{\Delta_s} = (\mathrm{DIV}\,f^h)_e \qquad \forall \Delta_s \subset e.$$

Scalar product of fluxes

$$\boxed{(\mathrm{DIV}\, f^h)_{\Delta_s} = (\mathrm{DIV}\, f^h)_e \qquad \forall \Delta_s \subset e}$$

We rewrite:

$$[\mathrm{GRAD} p^h,\ f^h]_{X,\Delta_s} + [p^h,\ \mathrm{DIV}\ f^h]_{Q,\Delta_s} = < p^h,\ f^h >_{W,\Delta_s} \ .$$

as

$$[g^h,\ f^h]_{X,\Delta_s} + (\mathrm{DIV}\, f^h)_{\Delta_s}\, p_{s0}\, |\Delta_s| = \sum_{k=1}^{3} f_{sk}\, p_{sk}\, |\ell_{sk}|.$$

Assembling the Green formulas, we get:

$$\sum_{\Delta_s \subset e} [g^h,\ f^h]_{X,\Delta_s} + (\mathrm{DIV}\, f^h)_e \sum_{\Delta_s \subset e} p_{s0} |\Delta_s| = \sum_{\ell_k \subset \partial e} f_k\, p_k\, \ell_k .$$

Scalar product of fluxes

$$\sum_{\Delta_s \subset e} [g^h,\, f^h]_{X,\Delta_s} + (\mathrm{DIV}\, f^h)_e \sum_{\Delta_s \subset e} p_{s0} |\Delta_s| = \sum_{\ell_k \subset \partial e} f_k\, p_k\, \ell_k.$$

For linear solutions we have

$$p_0 |e| = \sum_{\Delta_s \subset e} p_{s0} |\Delta_s|.$$

Thus, formula

$$\sum_{\Delta_s \subset e} [g^h,\, f^h]_{X,\Delta_s} + [\mathrm{DIV}\, f^h,\, p^h]_{Q,e} =< p^h,\, f^h >_{W,e}$$

is exact for linear solutions.

Scalar product of fluxes

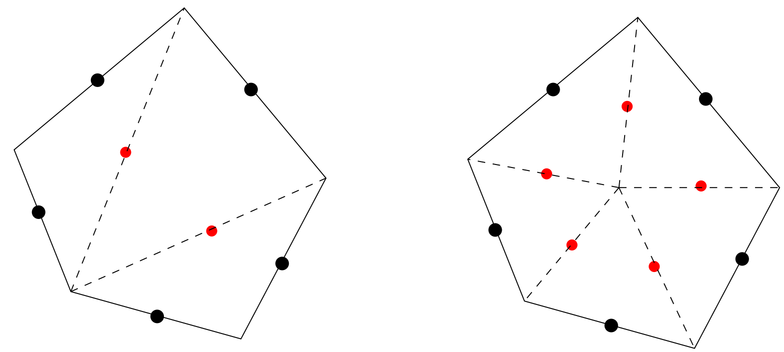
Lemma. *The system*

$$(\mathrm{DIV}\, f^h)_{\Delta s} = (\mathrm{DIV}\, f^h)_e, \qquad s = 1, \ldots, S,$$

has at least one solution.

Example.

- T - number of triangles ($T = 3$ or 5);
- B - number of boundary edges ($B = 5$);
- I - number of interior edges ($I = 2$ or 5).



$$C_2\, f^h_{int} = C_1\, f^h, \qquad C_2 \in \Re^{T\times I}, \qquad C_1 \in \Re^{T\times B}$$

In both cases:

$$f^h_{int} = C\, f^h.$$

Scalar product of fluxes

Let

$$\sum_{\Delta_s \subset e} [g^h, f^h]_{X,\Delta_s} = [g^h]^T \, M \, f^h.$$

where

$$f^h = \begin{pmatrix} f^h \\ f^h_{int} \end{pmatrix}.$$

Substituting

$$f^h_{int} = C \, f^h \qquad \text{and} \qquad g^h_{int} = C \, g^h,$$

we get

$$[g^h]^T \, M \, f^h = [g^h]^T \, \tilde{M} \, f^h \equiv [g^h, \, f^h]_{X,e}.$$

Local support operators

$$\boxed{\int_{e_i} \mathrm{grad}\, p\, \boldsymbol{f}\, \mathrm{d}x + \int_{e_i} p\, \mathrm{div}\, \boldsymbol{f}\, \mathrm{d}x = \int_{\partial e_i} p\, \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}s}$$

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1. define degrees of freedom for p and $\boldsymbol{f}$;
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Numerical experiments

I Cartesian geometry

- New scalar product vs the old one
- Polygonal meshes vs locally refined meshes
- Non-matching meshes
- Rayleigh-Taylor instability mesh

II Cylindrical geometry

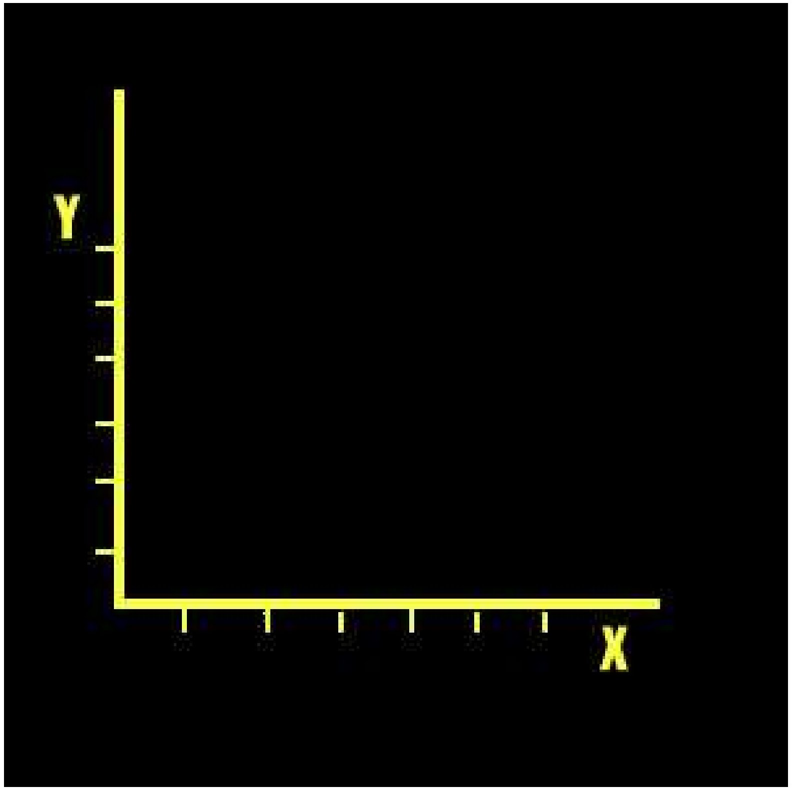
- Polygonal meshes vs locally refined meshes
- Symmetry breaking
- Non-matching meshes
- Rayleigh-Taylor instability mesh

III Mimetic and some other discretizations

Numerical experiments

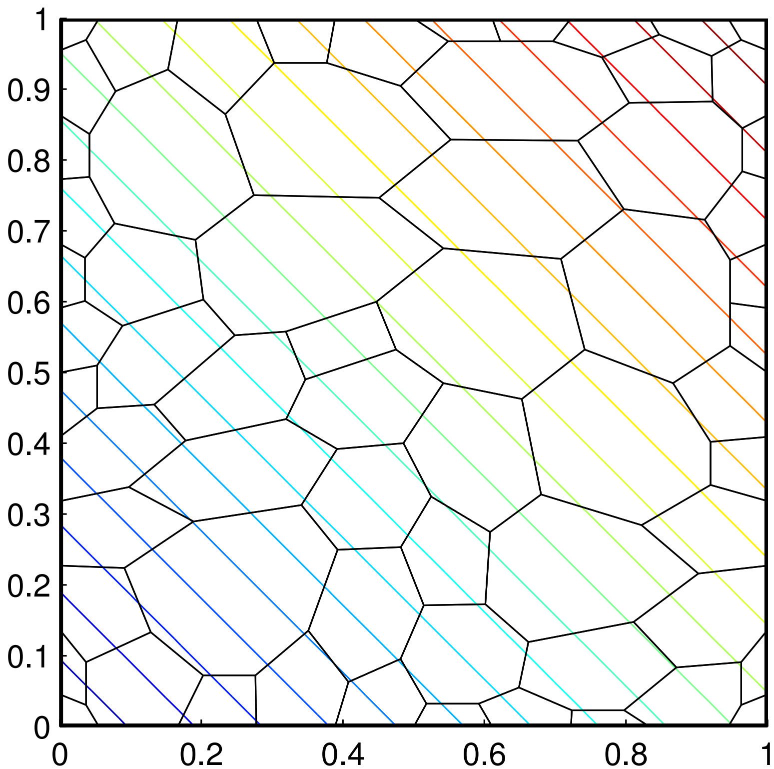
Cartesian

Geometry

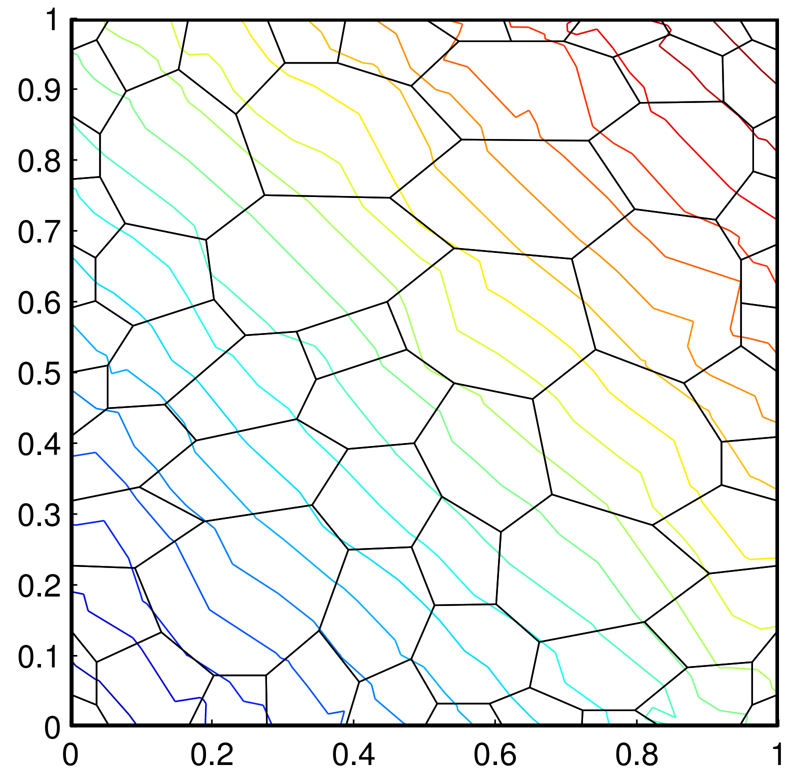


New scalar product vs old one

Cartesian geometry: exact solution is $p(x,\, y) = x + y$.



NEW



OLD

Summary of properties

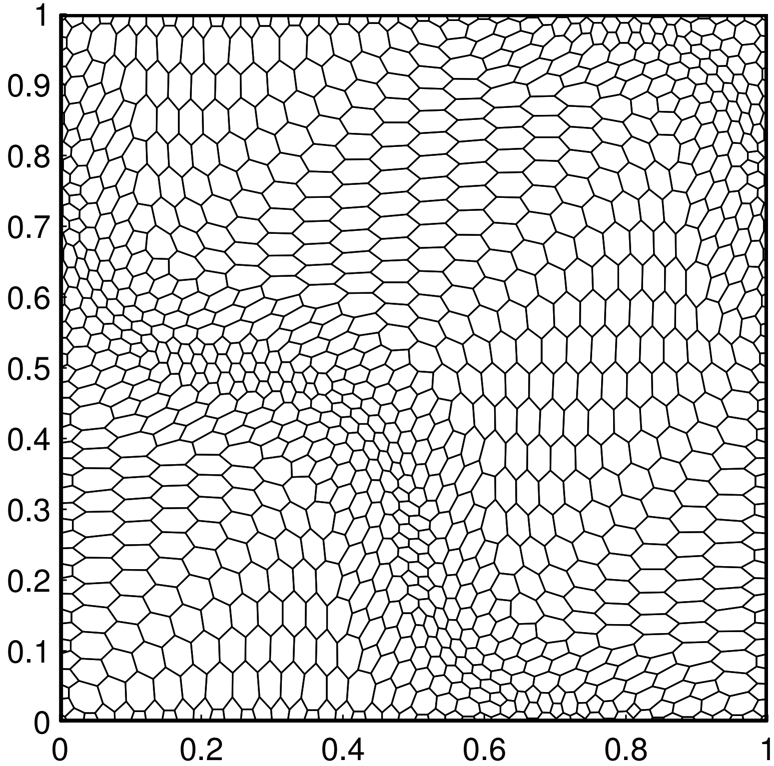
The new mimetic scheme:

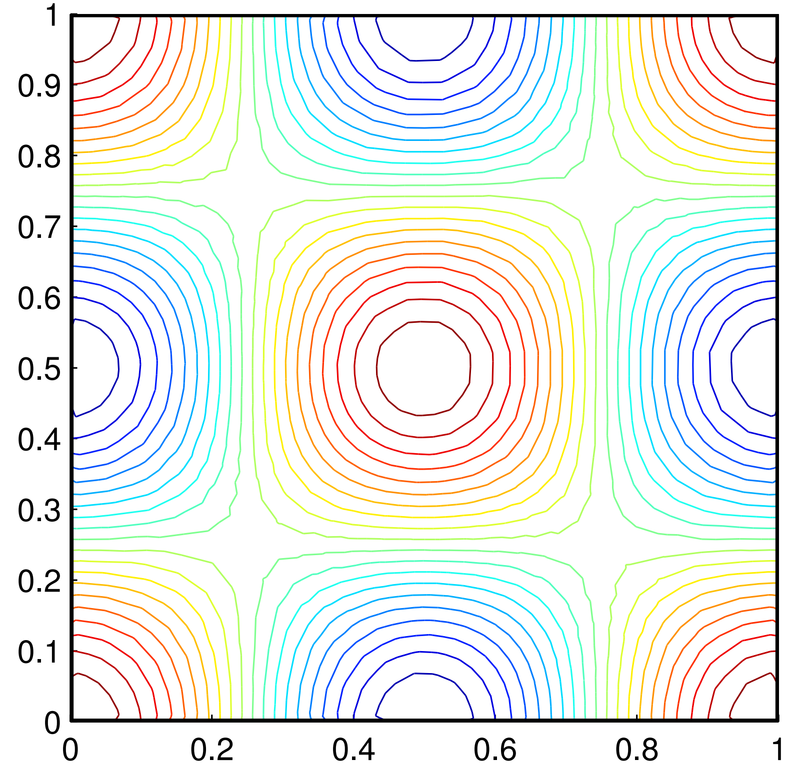
- exact for linear solutions.

New scalar product vs old one

Cartesian geometry: exact solution is

$$p(x,\, y) = \cos(2\pi x)\cos(2\pi y).$$





(The grids were generated by R. Loubere, T-7.)

New scalar product vs old one (cont.)

Cartesian geometry: exact solution is

$$p(x,\, y) = \cos(2\pi x)\cos(2\pi y).$$

m	New scalar product		Old scalar product	
	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$
64	1.07e-1	3.68e-1	1.81e-1	4.57e-1
256	2.60e-2	1.64e-1	3.39e-2	2.52e-1
1024	5.11e-3	8.28e-2	6.64e-3	1.72e-1
4096	1.05e-3	4.29e-2	1.51e-3	1.20e-1
rate	2.23	1.03	2.31	0.63

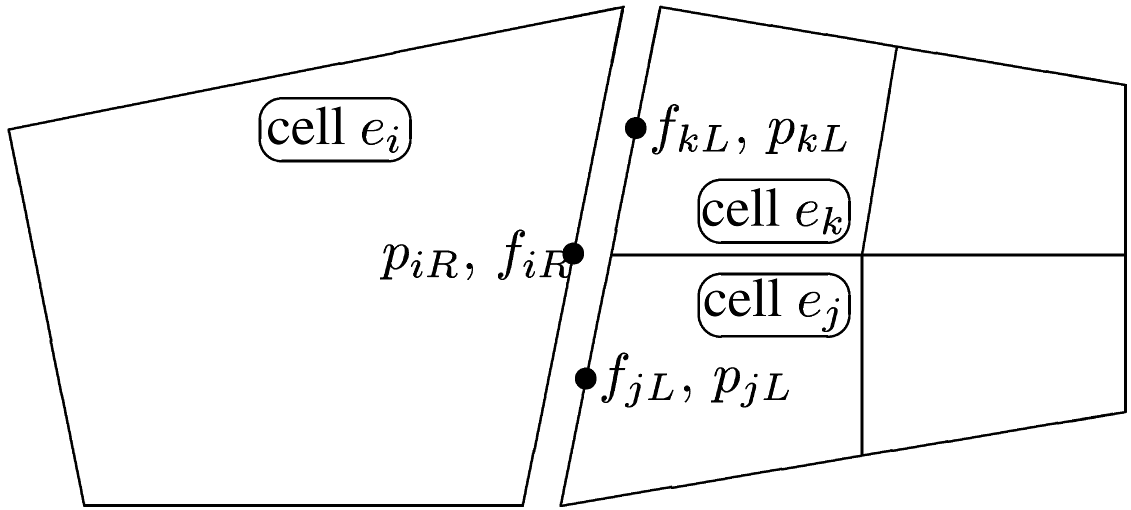
Summary of properties

The new mimetic scheme:

- exact for linear solutions;
- provides more accurate pressures and fluxes than the old scheme.

Polygonal meshes vs AMR meshes

Continuity conditions for AMR meshes: joint work with Jim Morel, CCS-4.



The global discretization is achieved by imposing the continuity of fluxes:

$$f_{iR} = -f_{jL} = -f_{kL}$$

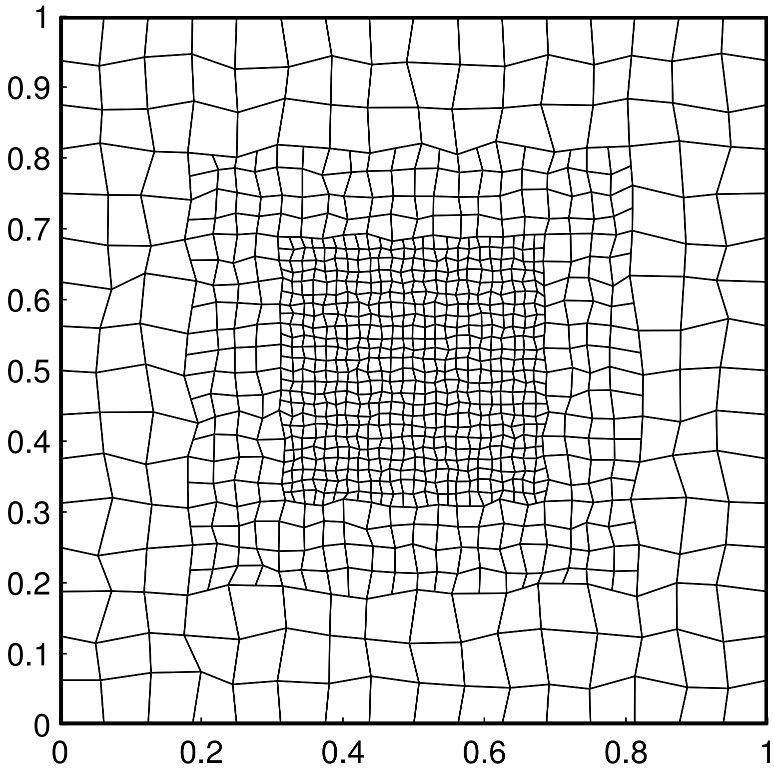
and interface pressures:

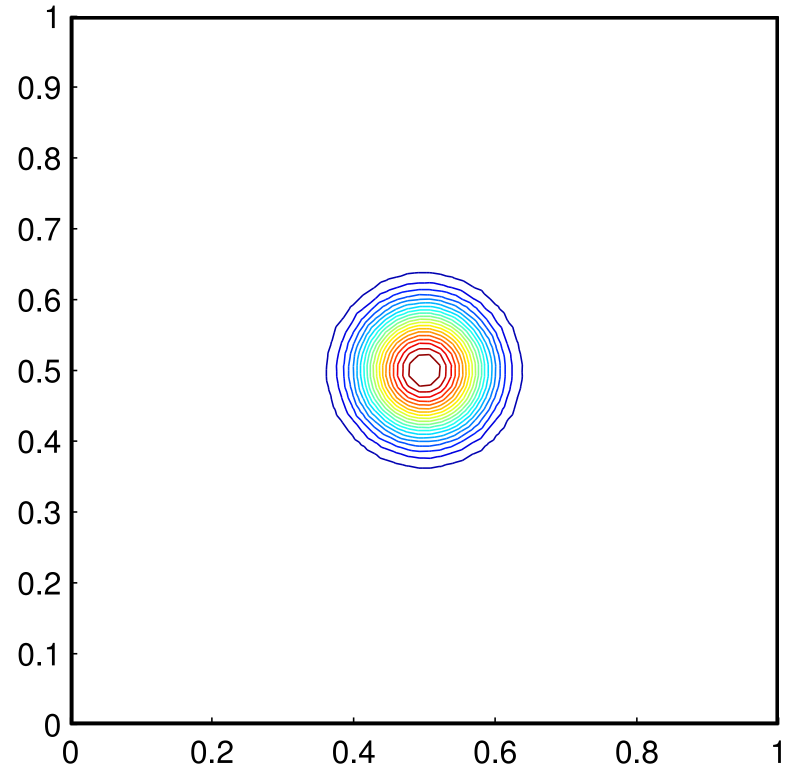
$$|\ell_{iR}|\, p_{iR} = |\ell_{jL}|\, p_{jL} + |\ell_{kL}|\, p_{kL}.$$

Polygonal meshes vs AMR meshes

Cartesian geometry: exact solution is

$$p(x, y) = 1 - \tanh\left[\frac{(x-0.5)^2 + (y-0.5)^2}{0.01}\right].$$





Polygonal meshes vs AMR meshes

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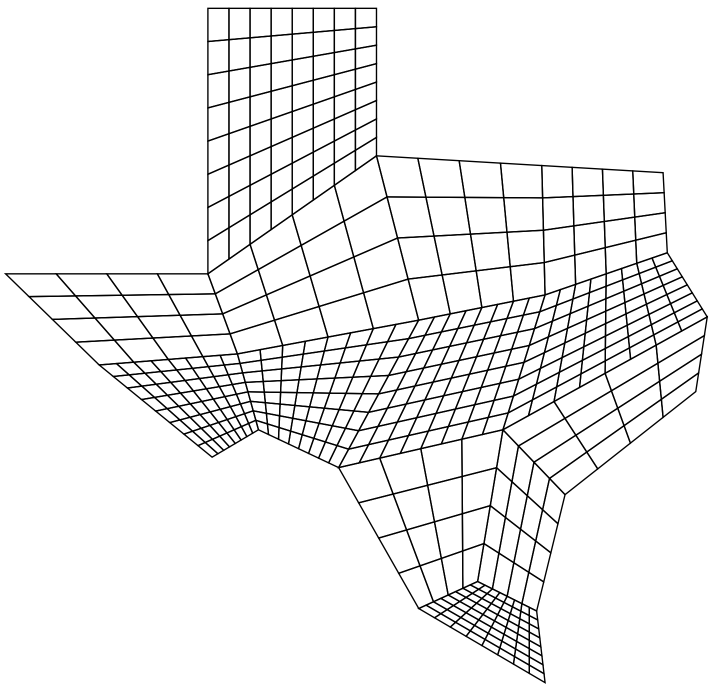
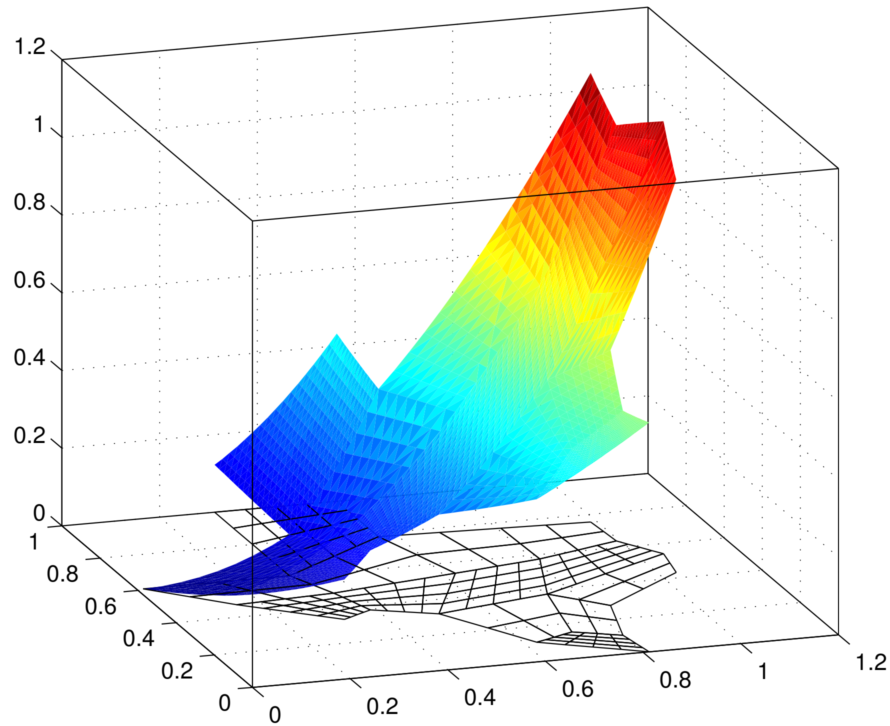
l	m	Polygonal grids		Hanging nodes	
		$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$
0	256	5.34e-2	8.26e-2	7.11e-2	9.25e-2
1	556	1.01e-2	3.28e-2	1.47e-2	3.25e-2
2	988	2.71e-3	1.16e-2	3.96e-3	1.55e-2
3	3952	6.92e-4	5.56e-3	9.60e-4	7.89e-3
4	15808	1.73e-4	2.84e-3	2.45e-4	3.69e-3
	rate	2.67	1.58	2.66	1.47

Polygonal meshes vs AMR meshes

Cartesian geometry: diffusion tensor and exact solution are

$$\boldsymbol{K}(x,\,y) = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}, \qquad p(x,\,y) = x^3\,y^2 + x\,\cos(xy)\,\sin(x).$$





Polygonal meshes vs AMR meshes

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$1/h$	Polygonal grids		Hanging nodes	
	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$
16	2.97e-4	1.37e-2	4.37e-4	9.15e-3
32	7.37e-5	4.64e-3	1.11e-4	2.96e-3
64	1.87e-5	1.68e-3	2.77e-5	9.88e-4
128	4.72e-6	7.23e-4	6.90e-6	3.38e-4
256	1.19e-6	2.42e-4	1.72e-6	1.17e-4
rate	1.99	1.43	2.00	1.57

Summary of properties

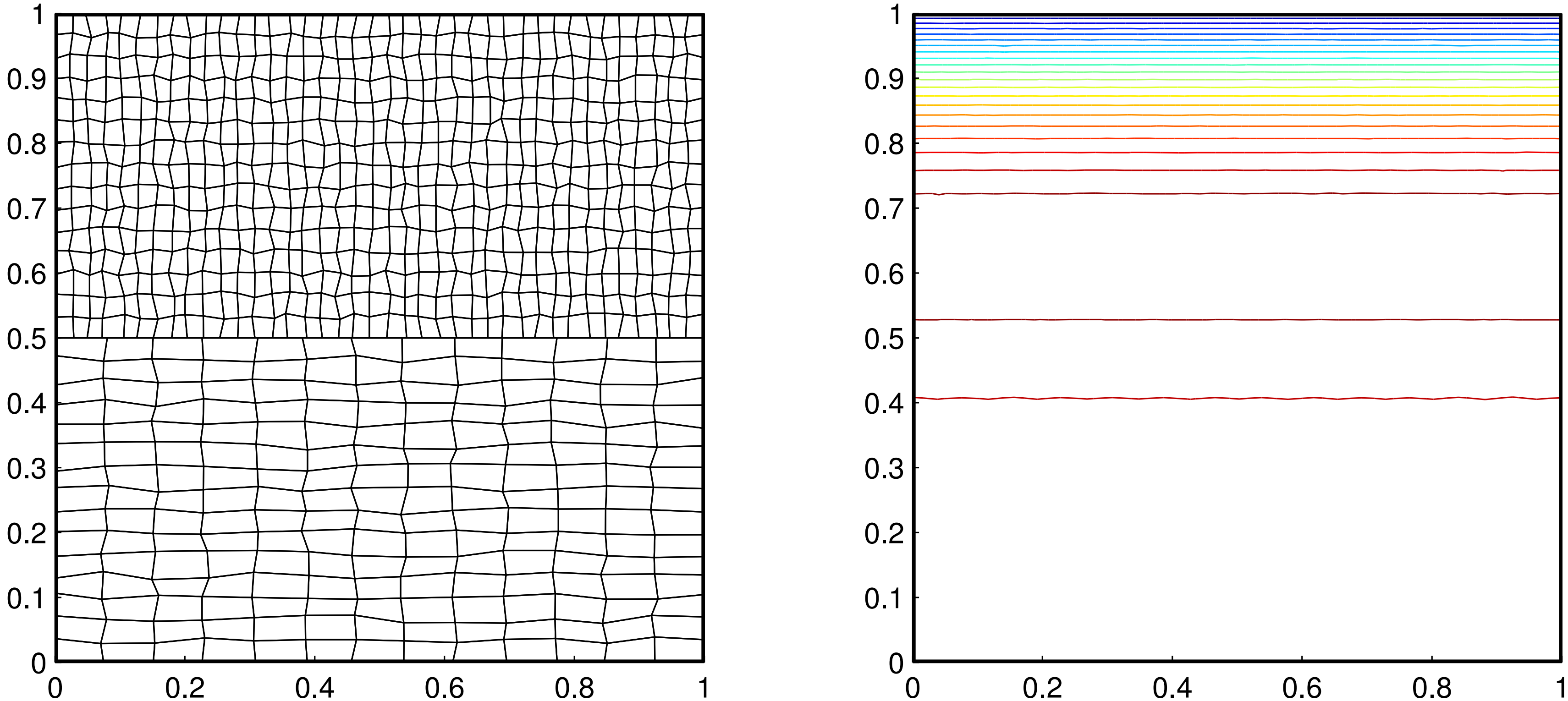
The new mimetic scheme:

- exact for linear solutions;
- provides more accurate pressures than old schemes;
- choice of a scalar product of fluxes is seems crucial for superconvergence results.

Non-matching meshes

Cartesian geometry: diffusion coefficient and exact solution are

$$\boldsymbol{K}(x,\,y) = \begin{cases} 4, \\ 1, \end{cases} \qquad p(x,\,y) = \begin{cases} 5/48 + 8y^2/3, & y < 0.5, \\ y - y^4, & y \geq 0.5. \end{cases}$$



Non-matching meshes

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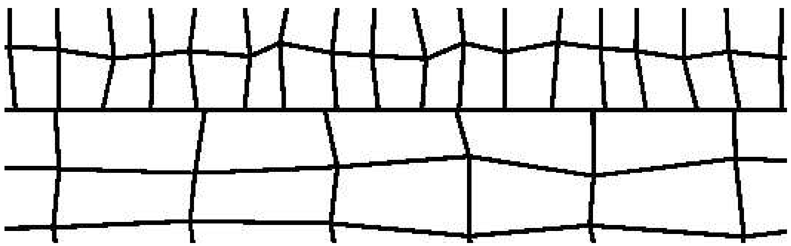
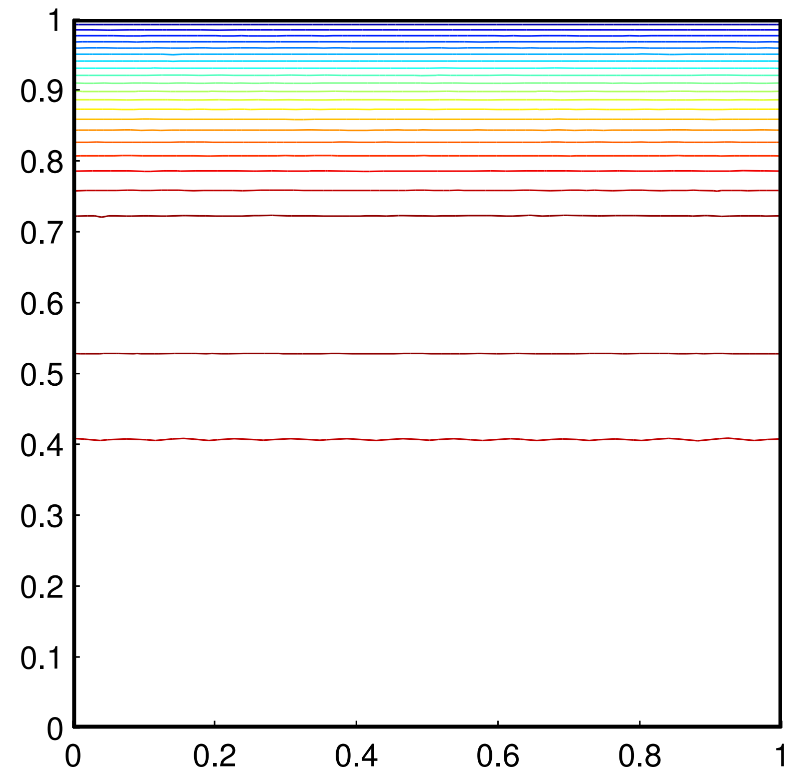


Zoom of the mesh



Non-matching meshes

Cartesian geometry: diffusion coefficient and exact solution are

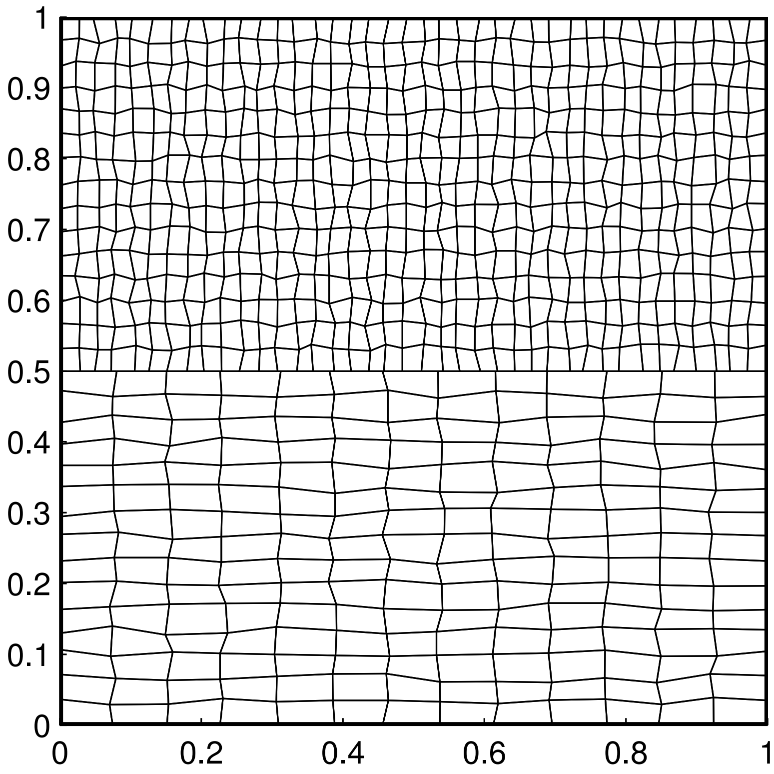
$$\boldsymbol{K}(x,\,y) = \begin{cases} 4, \\ 1, \end{cases} \qquad p(x,\,y) = \begin{cases} 5/48 + 8y^2/3, & y < 0.5, \\ y - y^4, & y \geq 0.5. \end{cases}$$

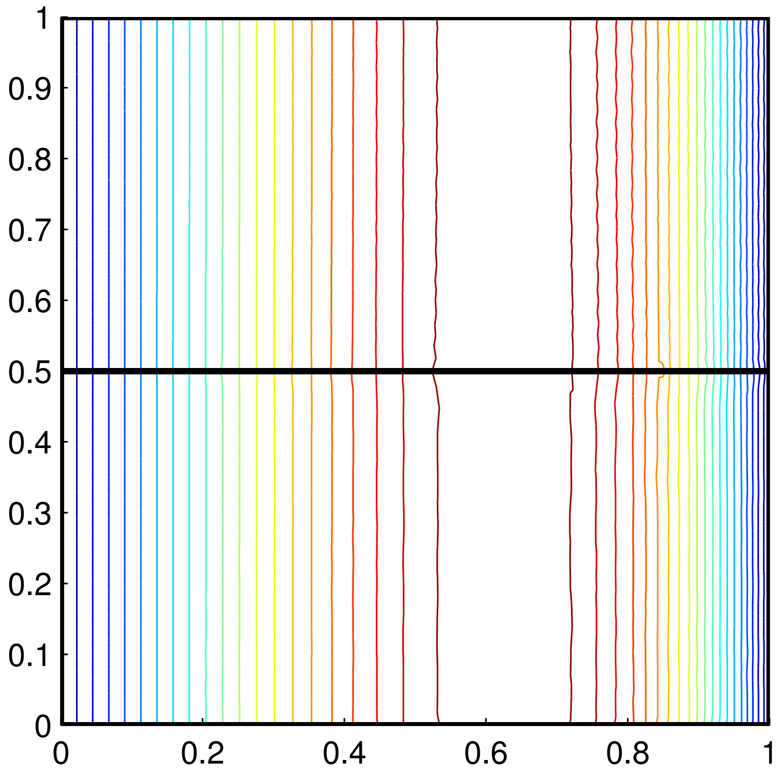
m	$\lVert\mathcal{P}_h p - p^h\rVert$	$\lVert\Pi_h \boldsymbol{f} - f^h\rVert$	$\max_i \rho_i$	$\#itr$	CPU, s
175	4.91e-3	1.39e-2	167.3	11	0.02
780	1.12e-3	6.35e-3	267.1	13	0.11
3286	2.70e-4	2.89e-3	159.5	12	0.61
13482	6.63e-5	1.45e-3	612.1	14	3.14
54610	1.64e-5	7.22e-4	2024.	14	13.2
rate	1.98	1.03			

Non-matching meshes

Cartesian geometry: exact solution is

$$p(x, y) = x - x^4.$$





Non-matching meshes

Cartesian geometry: exact solution is

$$p(x, y) = x - x^4.$$

m	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\max_i \rho_i$	$\#itr$	CPU, s
175	2.14e-2	4.16e-2	167.3	12	0.03
780	4.94e-3	1.30e-2	267.1	13	0.13
3286	1.19e-3	5.79e-3	159.5	15	0.71
13482	2.83e-4	2.15e-3	612.1	17	3.02
54610	7.04e-5	1.04e-3	2024.	17	14.8
rate	1.99	1.28			

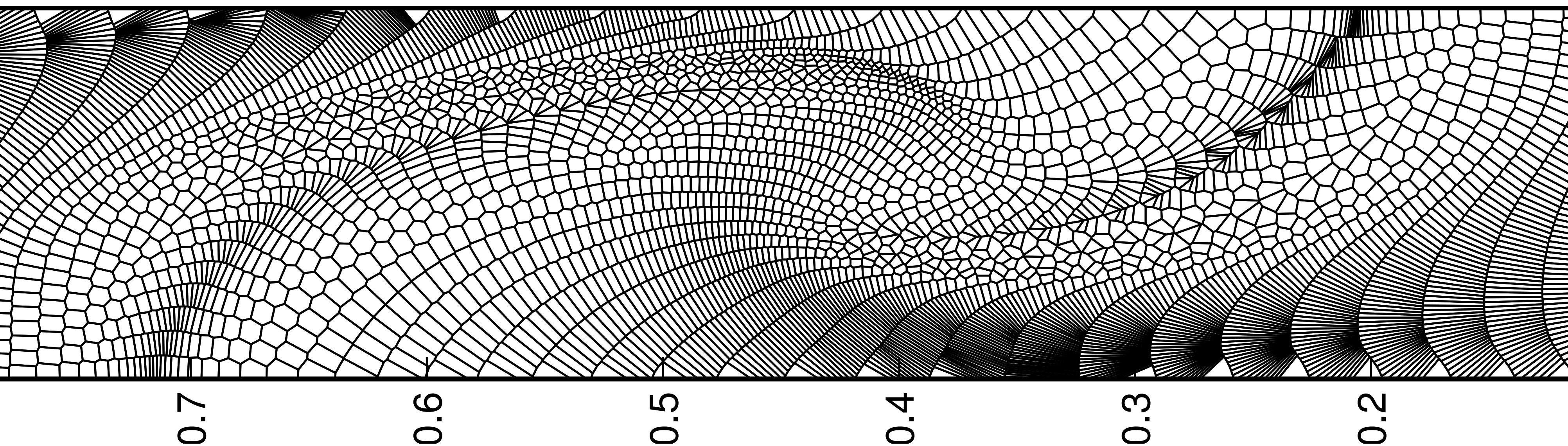
Summary of properties

The new mimetic scheme:

- exact for linear solutions;
- provides more accurate pressures than old schemes;
- choice of a scalar product of fluxes is seems crucial for superconvergence results;
- solver and discretization are stable for degenerated polygons.

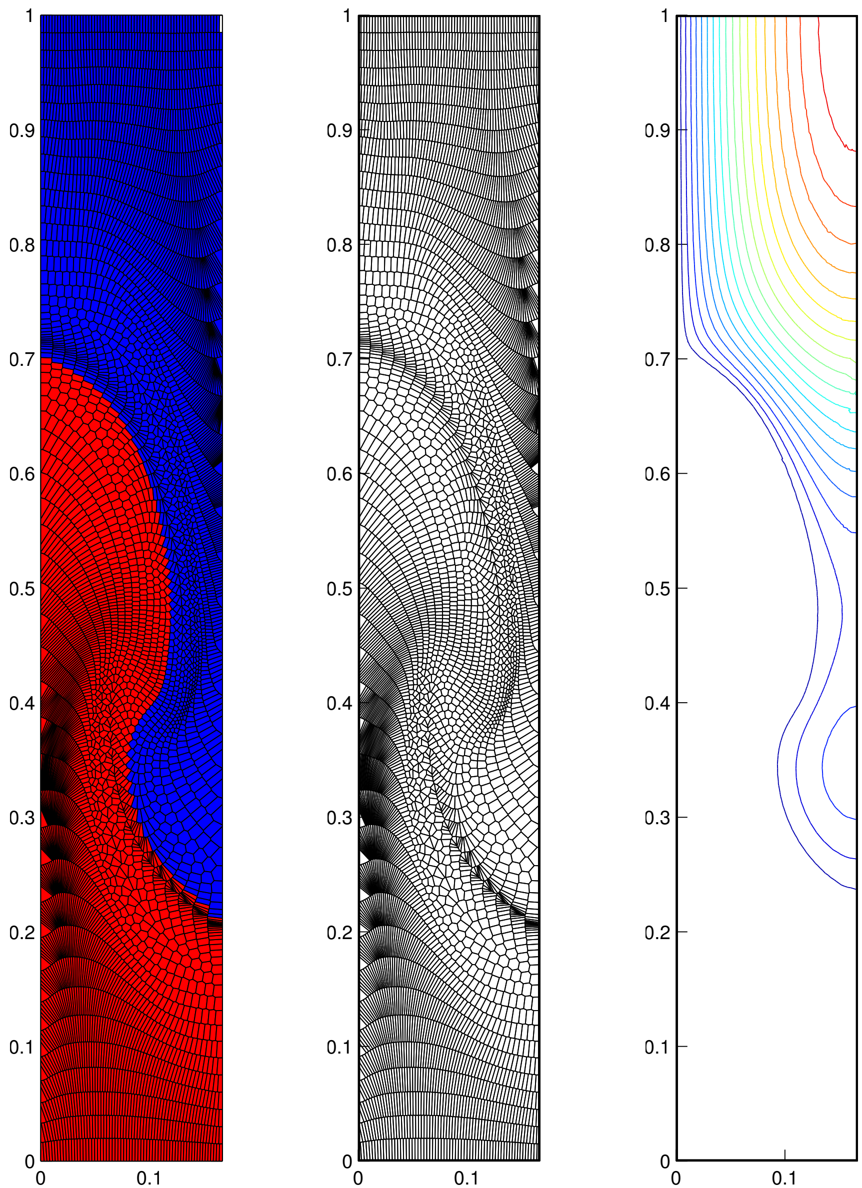
Rayleigh-Taylor instability mesh

Cartesian geometry



- zoom of $[0, 0.166667] \times [0, 1]$ computational domain rotated counter clockwise;
- 64×64 mesh cells.

Rayleigh-Taylor instability mesh



- **Cartesian geometry**;
- homogeneous Dirichlet b.c. on the left wall;
- homogeneous Neumann b.c. on the other walls;
- constant source term;
- diffusion coefficient is constant in the colored zones and jumps 100 times on the interface;
- logically "rectangular" mesh with 64×64 elements has aspect ratio ρ,

$$\rho = \max_{cells} \frac{\max_i |\ell_i|}{\min_i |\ell_i|} \approx 4.5 \cdot 10^4;$$

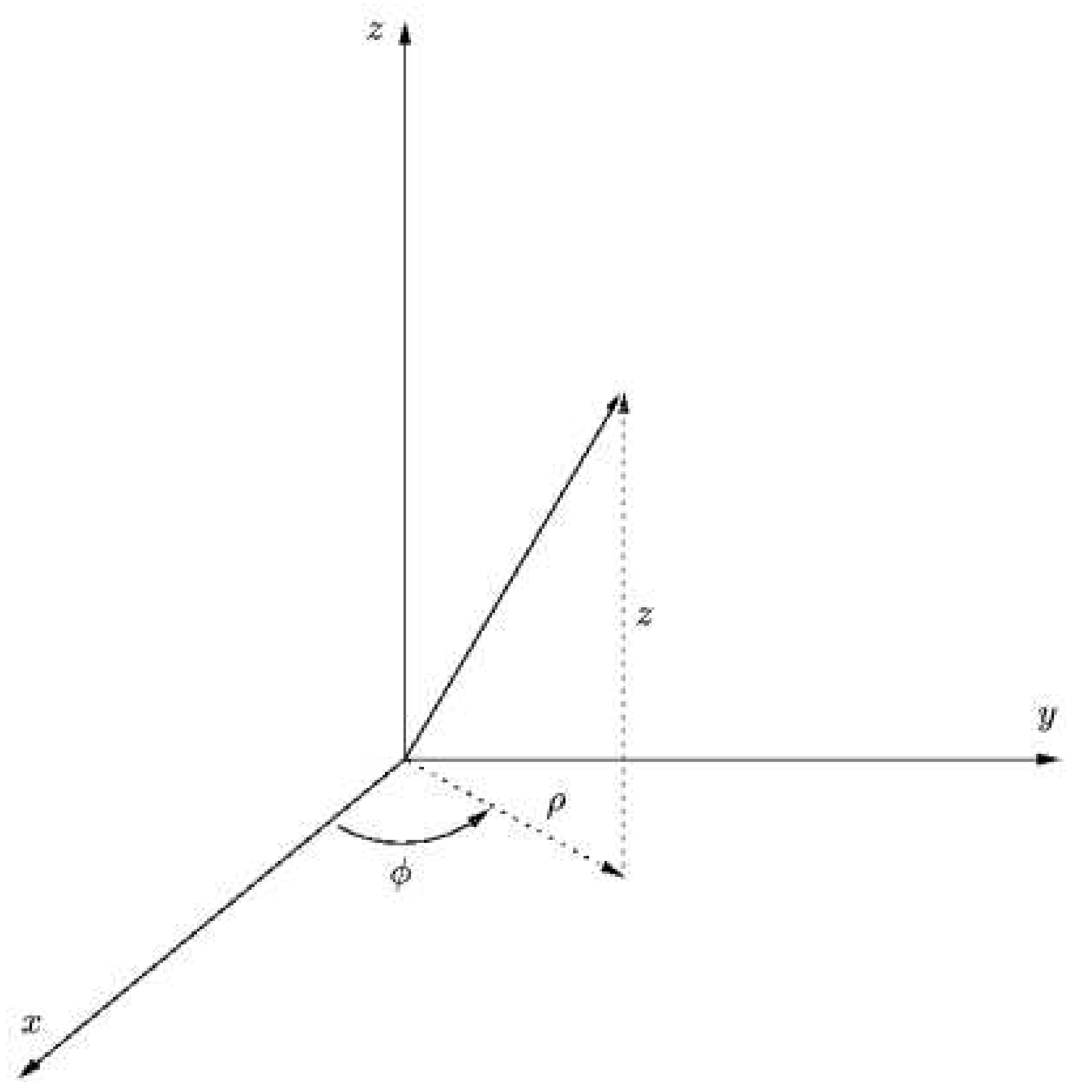
- 39 PCG iterations for the relative tolerance 10^{-12}.

Numerical experiments

Cylindrical

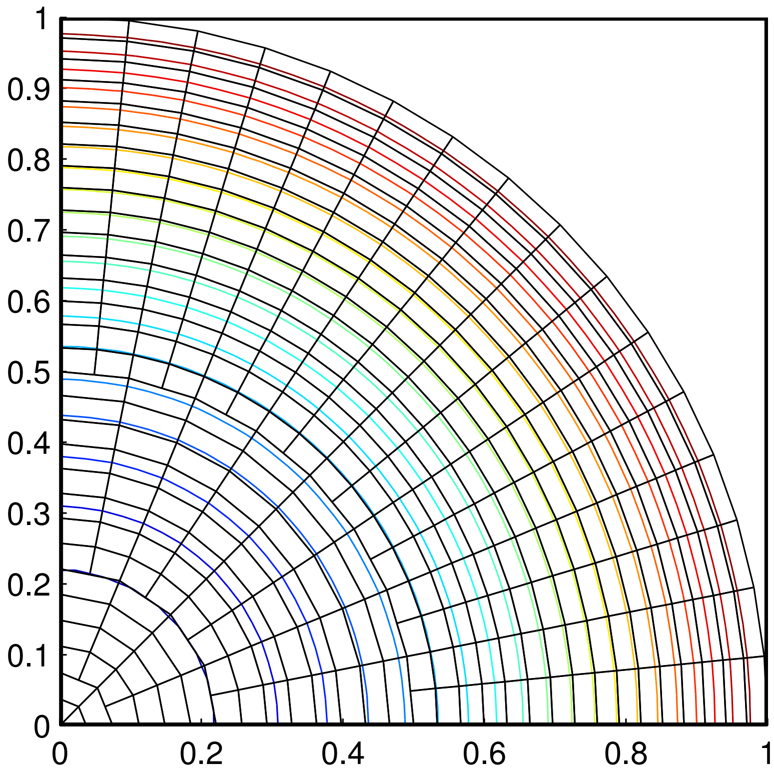
$(r - z)$

Geometry

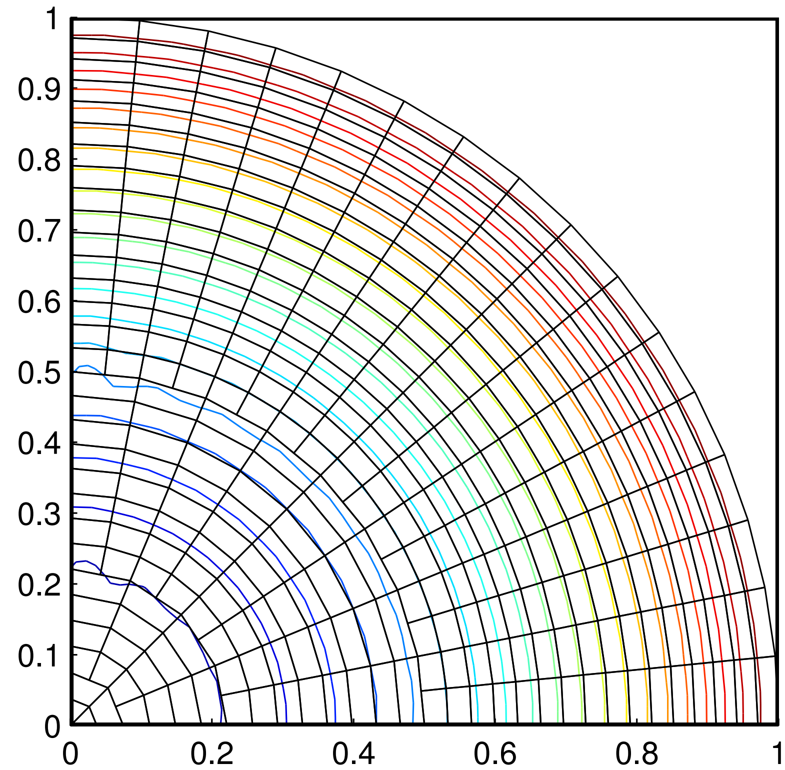


Polygonal meshes vs AMR meshes

Cylindrical geometry: exact solution is $p(r,\, z) = r^2 + z^2 = R^2$.



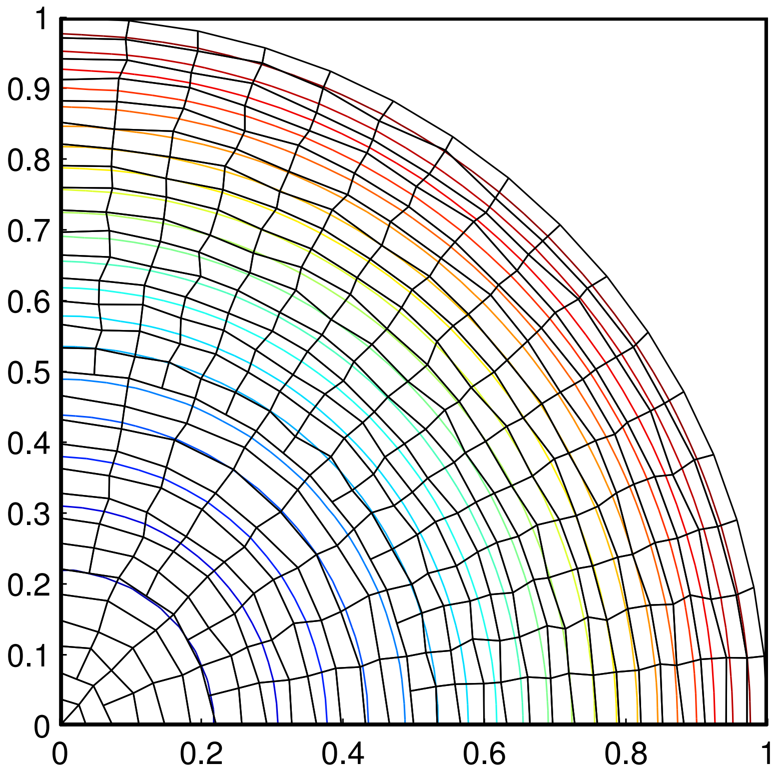
NEW



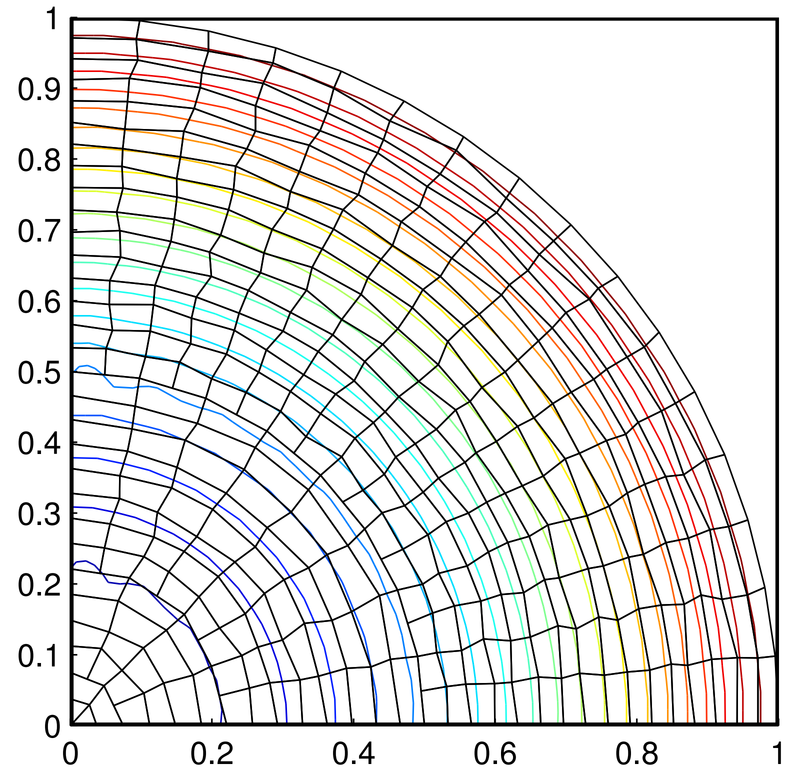
OLD/AMR

Polygonal meshes vs AMR meshes

Cylindrical geometry: exact solution and is $p(r,\, z) = r^2 + z^2 = R^2$.



NEW



OLD/AMR

Summary of properties

The new mimetic scheme:

- exact for linear solutions;
- provides more accurate pressures than old schemes;
- choice of a scalar product of fluxes is seems crucial for superconvergence results;
- solver and discretization are stable for degenerated polygons;

- results in a more symmetric solution.

Polygonal meshes vs AMR meshes

Cylindrical geometry: exact solution is

$$p(r,\, z) = r^2 + z^2 = R^2.$$

m	Polygonal grids		Hanging nodes	
	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$
340	1.03e-3	4.78e-4	2.16e-3	9.91e-3
1360	2.57e-4	1.24e-4	5.25e-4	3.95e-3
5440	6.64e-5	3.52e-5	1.34e-4	1.98e-3
rates	1.97	1.87	2.00	1.16

Polygonal meshes vs AMR meshes

Cylindrical geometry: diffusion coefficient and exact solution are

$$\boldsymbol{K}(r,z) = \begin{cases} 1, \\ 20, \end{cases} \qquad p(r,z) = \begin{cases} 1007/1280 - R^2/6 - R^4/20, & R < 0.5, \\ 893/1200 - R^2/120 - R^4/400, & R \geq 0.5. \end{cases}$$

m	Polygonal grids		Hanging nodes	
	$\lVert\mathcal{P}_h p - p^h\rVert$	$\lVert\Pi_h \boldsymbol{f} - \boldsymbol{f}^h\rVert$	$\lVert\mathcal{P}_h p - p^h\rVert$	$\lVert\Pi_h \boldsymbol{f} - \boldsymbol{f}^h\rVert$
340	1.46e-4	2.96e-3	1.05e-4	1.15e-2
1360	3.65e-5	8.94e-4	2.66e-5	4.42e-3
5440	9.13e-6	3.08e-4	6.68e-6	1.68e-3
rate	2.00	1.63	1.99	1.39

Summary of properties

The new mimetic scheme:

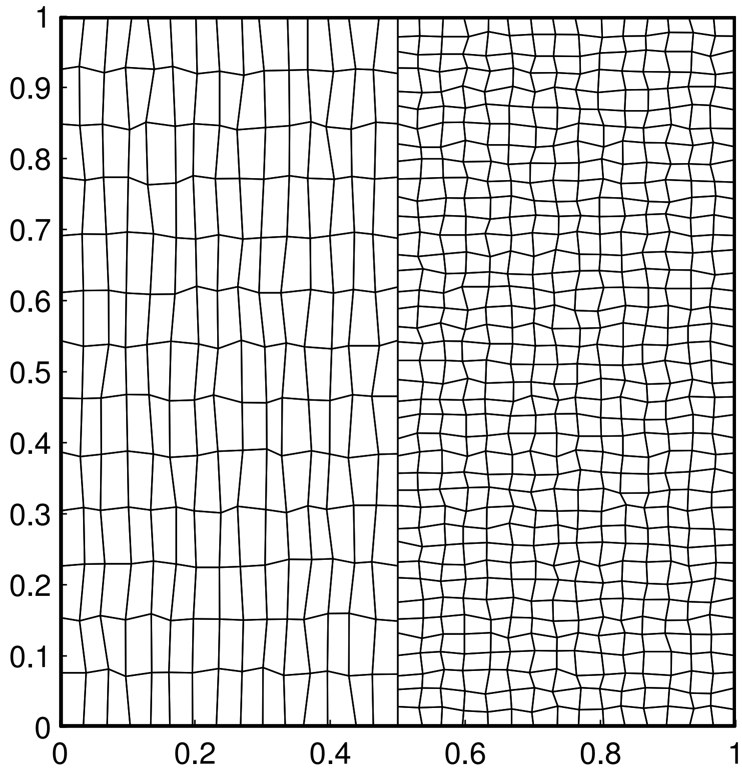
- exact for linear solutions;
- provides more accurate pressures than old schemes;
- choice of a scalar product of fluxes is seems crucial for superconvergence results;
- solver and discretization are stable for degenerated polygons;

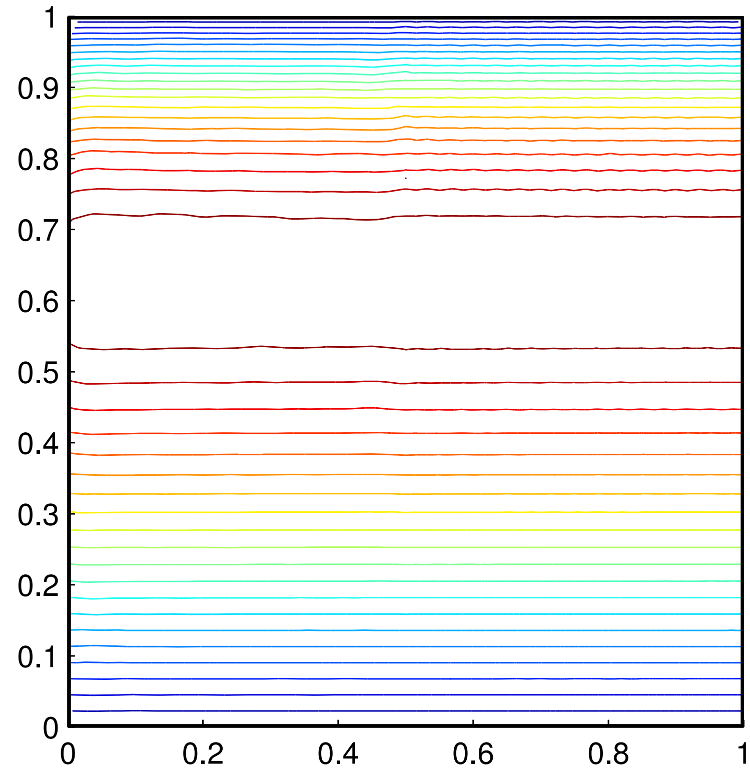
- results in a more symmetric solution;
- provides more accurate pressures and fluxes.

Non-matching meshes

Cylindrical geometry: exact solution is

$$p(r, z) = z - z^4.$$





Non-matching meshes

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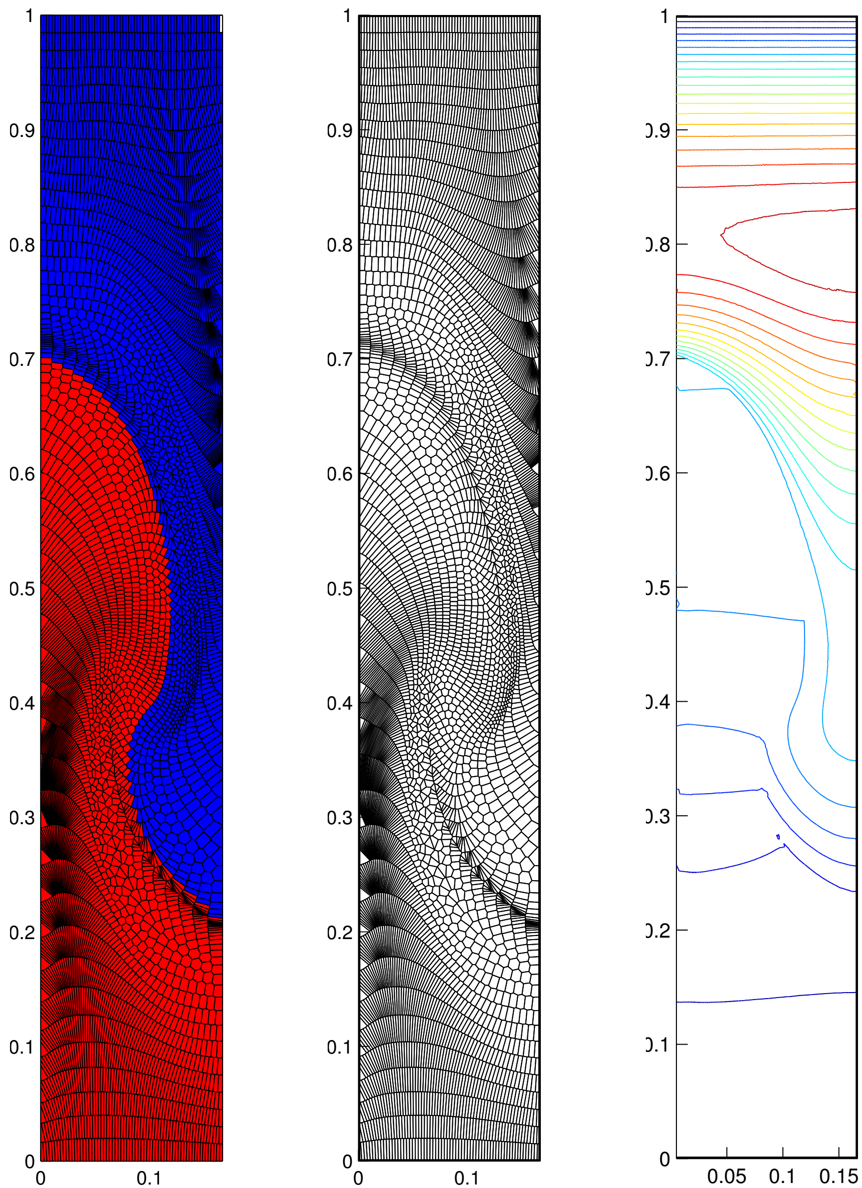
m	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$	$\#itr$	CPU, s
175	1.15e-2	3.59e-2	13	0.02
780	2.78e-3	1.21e-2	13	0.11
3286	6.56e-4	4.35e-3	15	0.68
13482	1.63e-4	1.97e-3	18	3.65
54610	4.01e-5	8.63e-4	18	15.3
rate	1.97	1.29		

Rayleigh-Taylor instability mesh



- **Cylindrical geometry**;
- homogeneous Dirichlet b.c. on the left and right walls;
- homogeneous Neumann b.c. on the other walls;
- constant source term;
- diffusion coefficient is constant in the colored zones and jumps 100 times on the interface;
- logically “rectangular” mesh with 64×64 elements;
- 44 PCG iterations for the relative tolerance 10^{-12}.

Two slides left

2 slides left

Mimetic & some other discretizations

Numerical schemes which are exact for linear solutions on polygonal meshes:

Scheme	"$p = x + y$"	Symmetric	$\|\mathcal{P}_h p - p^h\|$	$\|\Pi_h \boldsymbol{f} - f^h\|$
Mimetic FD	✔	✔	2	1-2?
Mixed FE (Y.Kuznetsov)	✔	✔	2	1-2?
CV (T.Palmer)	✔		2	—

Conclusion

The new mimetic scheme:

- exact for linear solutions;
- provides more accurate pressures than old schemes;
- choice of a scalar product of fluxes is seems crucial for superconvergence results;
- solver and discretization are stable for degenerated polygons;

- results in a more symmetric solution;
- provides more accurate pressures and fluxes;
- solver and discretization are stable for degenerated polygons;

- extension to 3D is possible;
- symmetry breaking & superconvergence of fluxes are challenging problems.