The new error-minimization-based moving mesh method: theoretical and numerical analysis

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Motivation

Objective is solution of hyperbolic conservation laws:

$$\mathbf{U}_t + \nabla \cdot \vec{F}(\mathbf{U}) = 0 \quad \text{in} \quad \Omega(t) \times [0, T].$$

- 1D Burgers's equation:

$$\mathbf{U} = u, \qquad F(\mathbf{U}) = \frac{u^2}{2}$$

- 1D gas dynamics:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \qquad F(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ uE + pu \end{pmatrix}$$

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Motivation

- Eulerian methods
  - mesh is fixed
- Lagrangian methods
  - mesh moves with the fluid velocity
- Arbitrary Lagrangian-Eulerian (ALE) methods
  - mesh moves with an arbitrary velocity

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Motivation: decoupled ALE methods

1. Lagrangian step
   - explicit or implicit time integration of hyperbolic equations
2. Mesh motion step
   - Error-Minimization-Based (EMB) method
3. Remapping step
   - conservative interpolation onto the modified mesh

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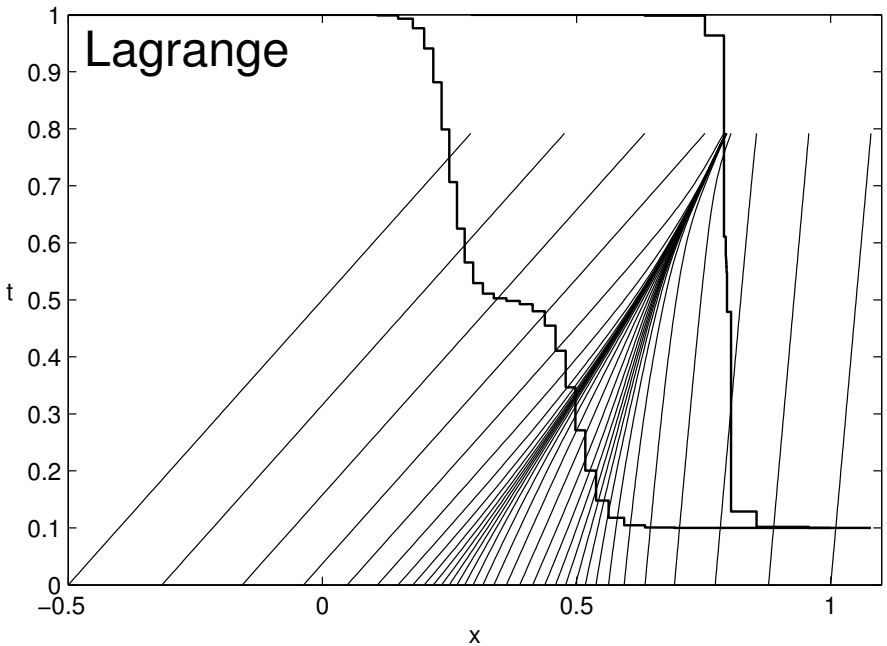
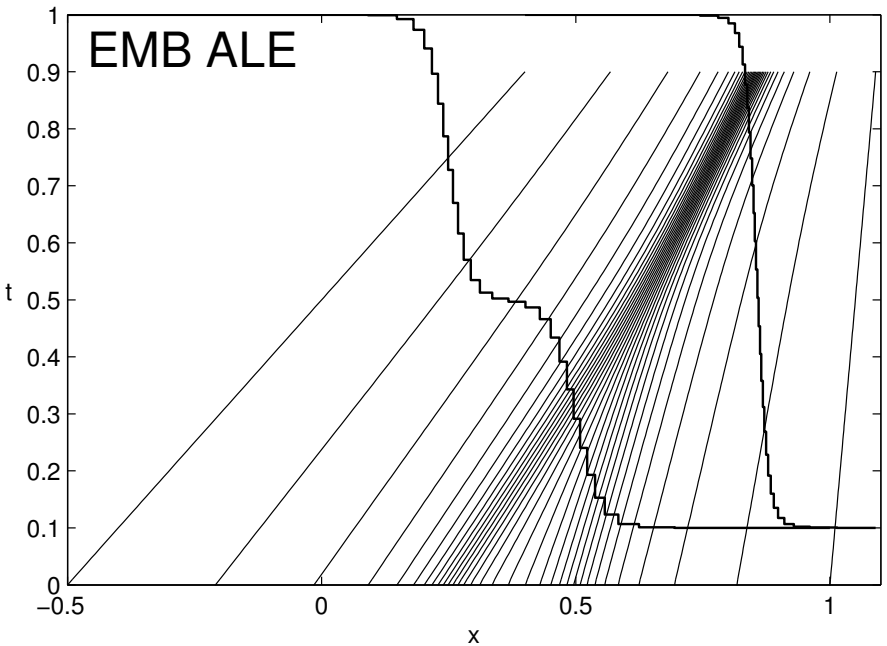
Motivation: goal-oriented ALE methods

- The *goal-oriented* mesh optimization saves the simulation



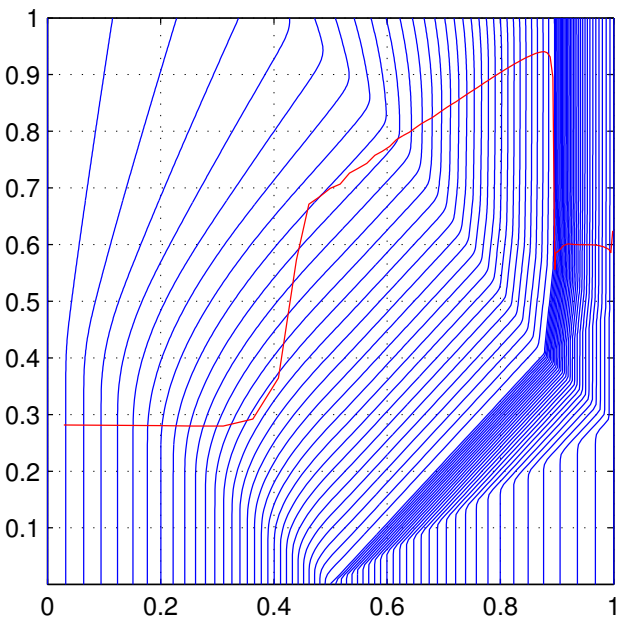
Lagrangian



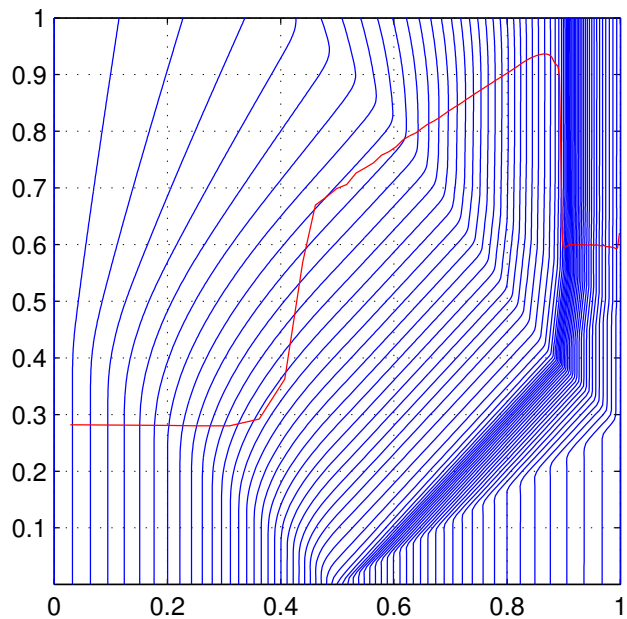
ALE (goal-oriented)

Motivation: goal-oriented ALE methods

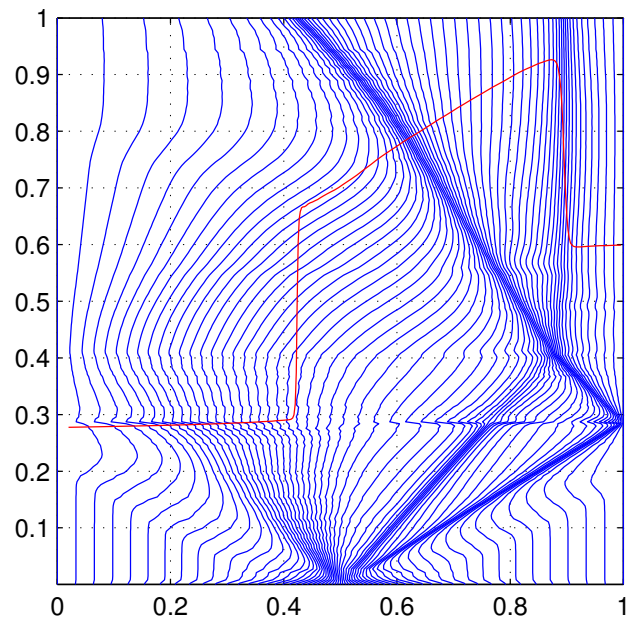
- The *geometric* mesh optimization may not improve solution accuracy
- The *goal-oriented* mesh optimization requires less time steps



Lagrangian
3800 time steps



ALE (geometric)
3417 time steps



ALE (goal-oriented)
2829 time steps

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Motivation: related methods

- decoupled ALE methods
  - variational methods (J.Brackbill, J.Saltzman, A.Winslow, etc.)
  - harmonic maps (B.Azarenok, S.Ivanenko, T.Tang, etc.)
  - monitor-based methods (T.Tang, W.Huang, R.Russell, etc.)
  - methods using physical analogies (spring systems, elastic media, etc)
- coupled ALE methods
  - moving finite elements (M.Baines, K.Miller, etc.)
  - moving finite differences (E.Dorfi, L.Drury)
  - moving mesh PDEs (W.Huang, R.Russell, etc.)
  - deformation methods (P.Bochev, B.Semper, G.Liao, etc.)
  - many others (J.Hyman, A.Harten, B.Perot, L.Petzold, etc.)

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The EMB method

Viscous Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \varepsilon \frac{\partial^2 u}{\partial x^2} \qquad \text{in} \quad (0,\ 1) \times [0, T].$$

- may develop shock-like solutions
- the solution is smooth
- easy to analyze and to present the big idea

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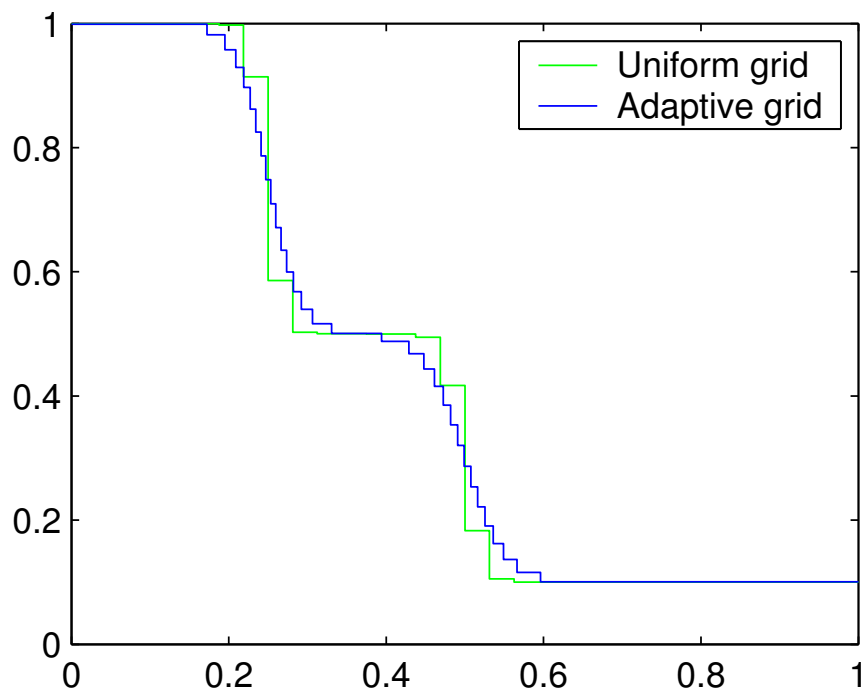
Viscous Burgers' equation

- Mesh: $x_0^n < \ldots < x_{M+1}^n$
- Unknowns: $u_{i+1/2}^n \approx \dfrac{1}{h_{i+1/2}^n} \displaystyle\int_{x_i^n}^{x_{i+1}^n} u(x, t^n)\,\mathrm{d}x$



Viscous Burger's equation

Consider the simplest time integration scheme:

$$h_{i+1/2}^{n+1} u_{i+1/2}^{n+1} = h_{i+1/2}^{n} u_{i+1/2}^{n} + F_{i+1}^{n} - F_{i}^{n}$$

$$x_i^{n+1} = x_i^n + \Delta t^n u_i^n$$

where

$$F_i^n = \frac{1}{2}(u_i^n)^2 + \varepsilon \frac{u_{i+1/2}^n - u_{i-1/2}^n}{(h_{i+1/2}^n + h_{i-1/2}^n)/2}$$

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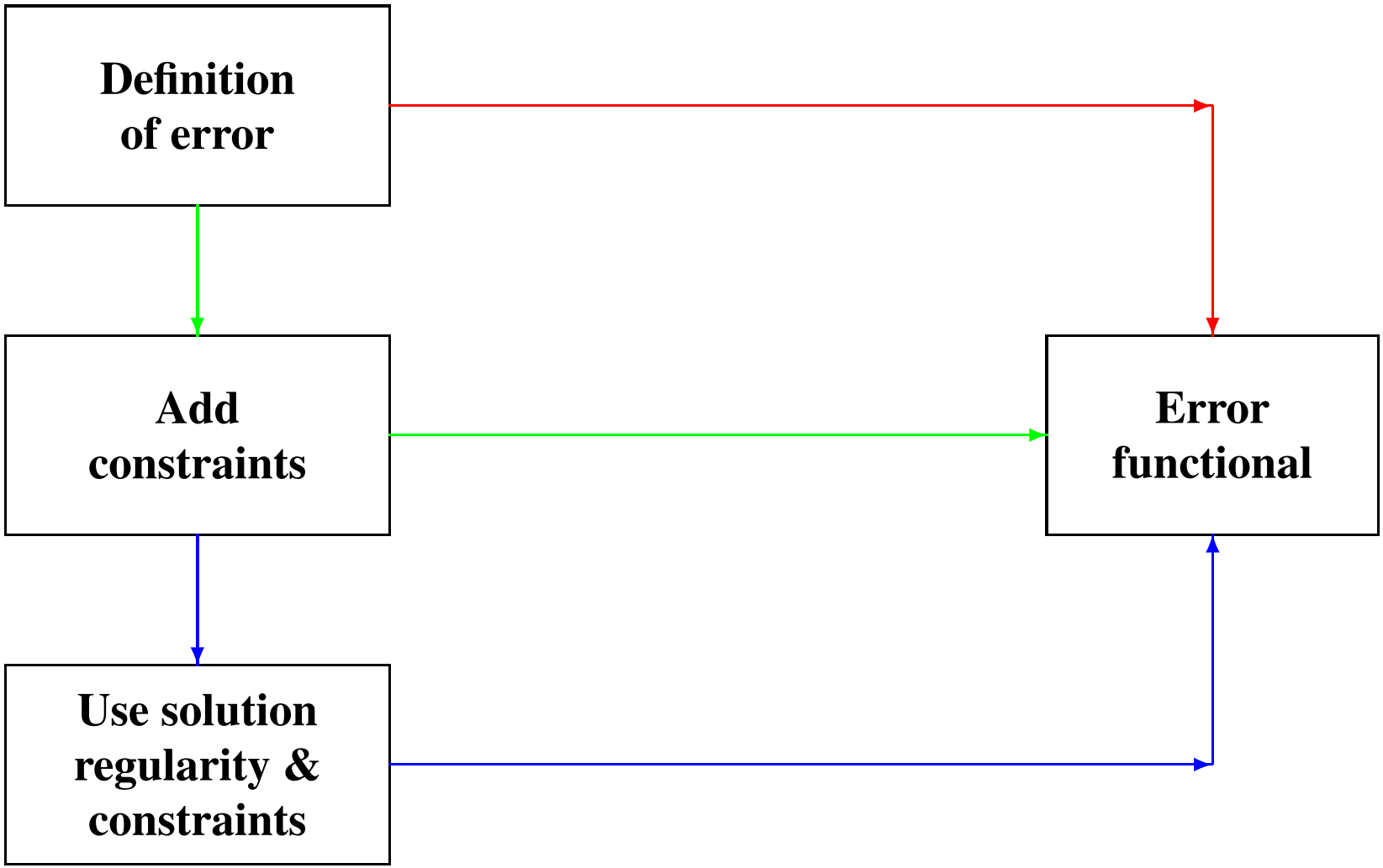
Design model for the EMB method



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Definition of error

The *ideal* error functional:

$$\Phi(\mathbf{x}^n) = \sum_{i=0}^{M} \int_{x_i^{n+1}}^{x_{i+1}^{n+1}} |u(x, t^{n+1}) - u_{i+1/2}^{n+1}|^2 \, \mathrm{d}x$$

where $\mathbf{x}^n = (x_1^n, \ldots, x_M^n)$.

- x_i^{n+1} and $u_{i+1/2}^{n+1}$ depends on $\mathbf{x}^n$.

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Definition of error

The *ideal* error functional:

$$\Phi(\mathbf{x}^n) = \sum_{i=0}^{M} \int_{x_i^{n+1}}^{x_{i+1}^{n+1}} |u(x, t^{n+1}) - u_{i+1/2}^{n+1}|^2 \, \mathrm{d}x$$

- space discretization error
- time integration error
- remapping error

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Definition of error

The *ideal* error functional:

$$\Phi(\mathbf{x}^n) = \sum_{i=0}^{M} \int_{x_i^{n+1}}^{x_{i+1}^{n+1}} |u(x, t^{n+1}) - u_{i+1/2}^{n+1}|^2 \, \mathrm{d}x$$

- iterative process
- *a posteriori* error estimates
- simplified for smooth solution

Look-ahead strategy

Add constraints

Minimize

$$\Phi(\mathbf{x}^n) = \sum_{i=0}^{M} \int_{x_i^{n+1}}^{x_{i+1}^{n+1}} |u(x, t^{n+1}) - u_{i+1/2}^{n+1}|^2 \, \mathrm{d}x$$

over a class of smooth meshes.

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Add constraints

1. mesh is smooth:

$$h^n_{i+1/2} \sim h, \qquad |h^n_{i+1/2} - h^n_{i-1/2}| \sim h^2$$

2. remapping is 2nd order accurate

3. solution is exact at time $t = t^n$:

$$u^n_{i+1/2} = \frac{1}{h^n_{i+1/2}} \int_{x^n_i}^{x^n_{i+1}} u(x, t^n)\, \mathrm{d}x$$

Use solution regularity & constraints

Main result:

$$\Phi(\mathbf{x}^n) = \Phi_0(\mathbf{x}^n) + O((h + \Delta t^n)^3)$$

where

$$\Phi_0(\mathbf{x}^n) = \sum_{i=0}^{M} \int_{x_i^n}^{x_{i+1}^n} |u(x, t^n) - u_{i+1/2}^n|^2 \, \mathrm{d}x.$$

- The problem of the *best piecewise constant fit* at time t^n (M. Baines).
- This justifies data lagging in mesh equation.

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Use solution regularity & constraints

$$\Phi_0(\mathbf{x}^n) = \Phi_1(\mathbf{x}^n) + O(h^3)$$

where

$$\Phi_1(\mathbf{x}^n) = \frac{1}{12}\sum_{i=0}^{M}\left[\frac{\delta u^n}{\delta x}\right]^2_{i+1/2}\left(h^n_{i+1/2}\right)^3$$

and

$$\left[\frac{\delta u^n}{\delta x}\right]_{i+1/2} = \frac{\partial u}{\partial x}(x^n_{i+1/2}) + O(h).$$

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Use solution regularity & constraints

Minimize

$$\Phi_1(\mathbf{x}^n) = \frac{1}{12} \sum_{i=0}^{M} \left[\frac{\delta u^n}{\delta x} \right]^2_{i+1/2} \left(h^n_{i+1/2} \right)^3$$

over a class of smooth meshes.

- the EMB method does not equidistribute the error

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Smooth meshes

Consider equation

$$(I - \alpha(\alpha+1)A)\begin{pmatrix} \tilde{h}_{1/2} \\ \vdots \\ \tilde{h}_{M+1/2} \end{pmatrix} = \begin{pmatrix} h_{1/2} \\ \vdots \\ h_{M+1/2} \end{pmatrix}$$

where

$$(A\tilde{\mathbf{h}})_{i+1/2} = \tilde{h}_{i-1/2} - 2\tilde{h}_{i+1/2} + \tilde{h}_{i+3/2}.$$

Then,

$$\frac{\alpha}{\alpha+1} \leq \frac{\tilde{h}_{i-1/2}}{\tilde{h}_{i+1/2}} \leq \frac{\alpha+1}{\alpha}.$$

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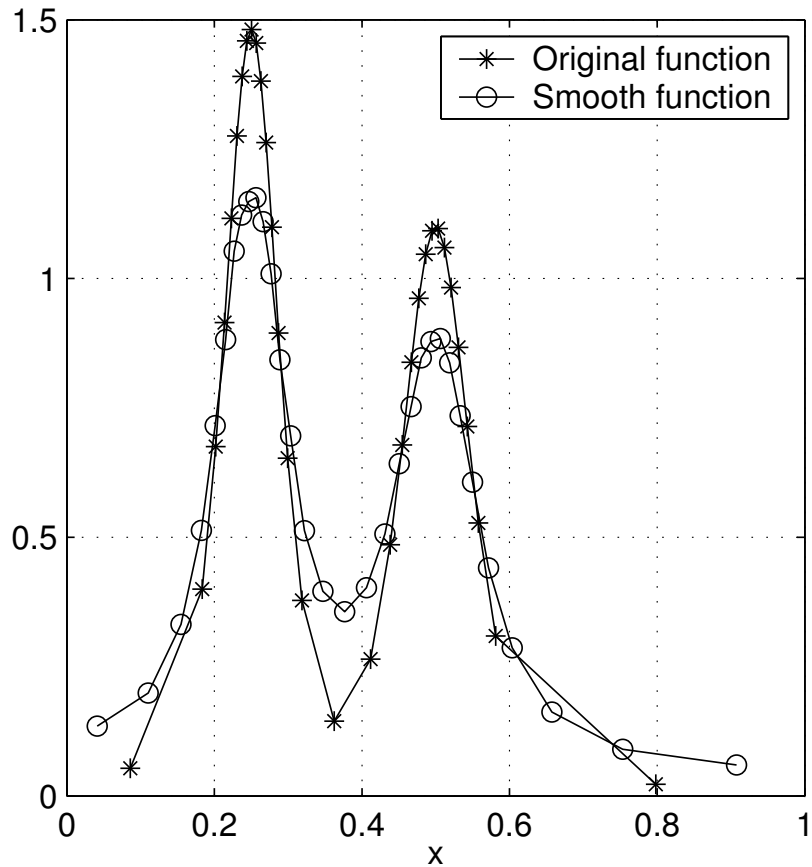
Smooth meshes

- smoothing preserves main features of **h**



- smoothing is performed in the logical space

Two algorithm

Algorithm 1

- find the minimizer of functional $\Phi_1(\mathbf{x}^n)$
- smooth the mesh

Algorithm 2

- modify the error functional
- prove that its minimizer is a smooth mesh

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Smoothed error functional

Minimize

$$\Phi_2(\mathbf{x}^n) = \frac{1}{12}\sum_{i=0}^{M}\left[\mathcal{S}^{-1}\left(\frac{\delta \mathbf{u}^n}{\delta x}\right)\right]_{i+1/2}^2 \left(h_{i+1/2}^n\right)^3$$

where $\mathcal{S} = I - \alpha(\alpha+1)A$.

Smoothed error functional

Minimize

$$\Phi_2(\mathbf{x}^n) = \frac{1}{12}\sum_{i=0}^{M}\left[\mathcal{S}^{-1}\left(\frac{\delta \mathbf{u}^n}{\delta x}\right)\right]_{i+1/2}^2 \left(h_{i+1/2}^n\right)^3$$

This solves the problem of minimizing

$$\Phi(\mathbf{x}^n) = \sum_{i=0}^{M}\int_{x_i^{n+1}}^{x_{i+1}^{n+1}} |u(x,t^{n+1}) - u_{i+1/2}^{n+1}|^2 \,\mathrm{d}x$$

over a class of smooth meshes.

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Smoothed error functional

Minimize

$$\Phi_2(\mathbf{x}^n) = \frac{1}{12}\sum_{i=0}^{M}\left[\mathcal{S}^{-1}\left(\frac{\delta \mathbf{u}^n}{\delta x}\right)\right]_{i+1/2}^2 \left(h_{i+1/2}^n\right)^3$$

- the EMB method equidistributes the smoothed error

$$\left[\mathcal{S}^{-1}(\cdot)\right]_{i+1/2}^{2/3} h_{i+1/2} = constant$$

- it produces a smooth mesh in regions where the error is 0.

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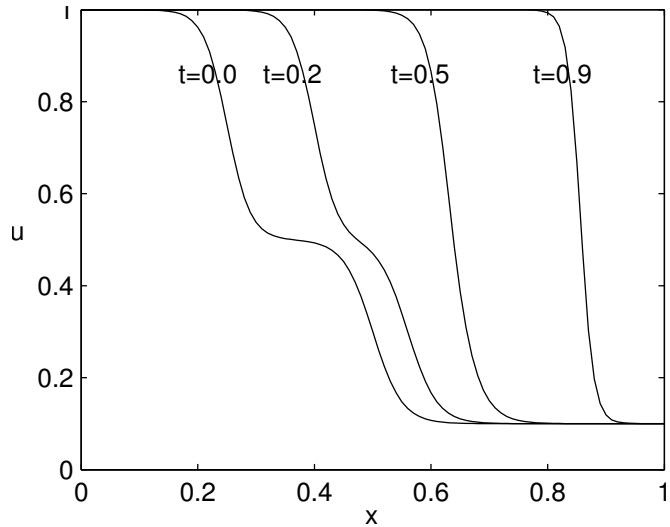
Numerical experiments

Viscous Burgers' equation



M	uniform mesh	optimal mesh	smooth mesh ($\alpha = 1$)
16	2.99e-2	1.01e-2	1.75e-2
32	1.59e-2	4.99e-3	6.28e-3
64	7.99e-3	2.48e-3	2.70e-3
128	4.00e-3	1.24e-3	1.28e-3

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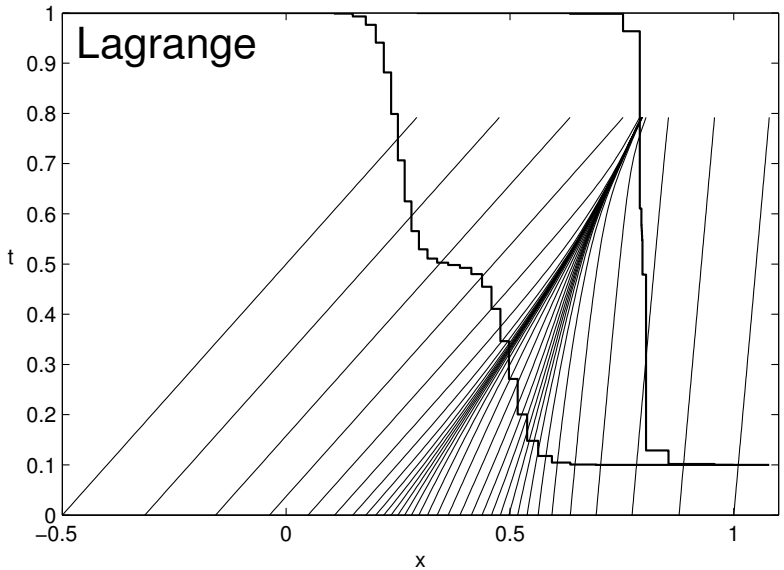
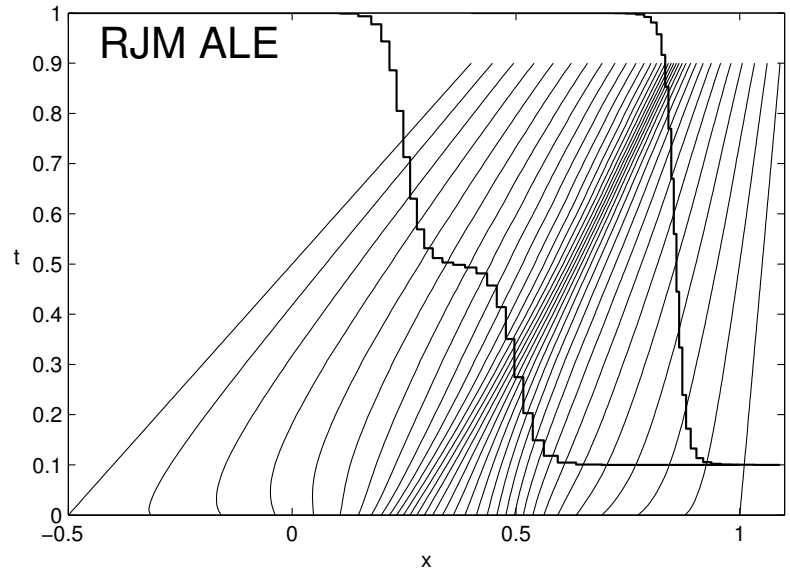
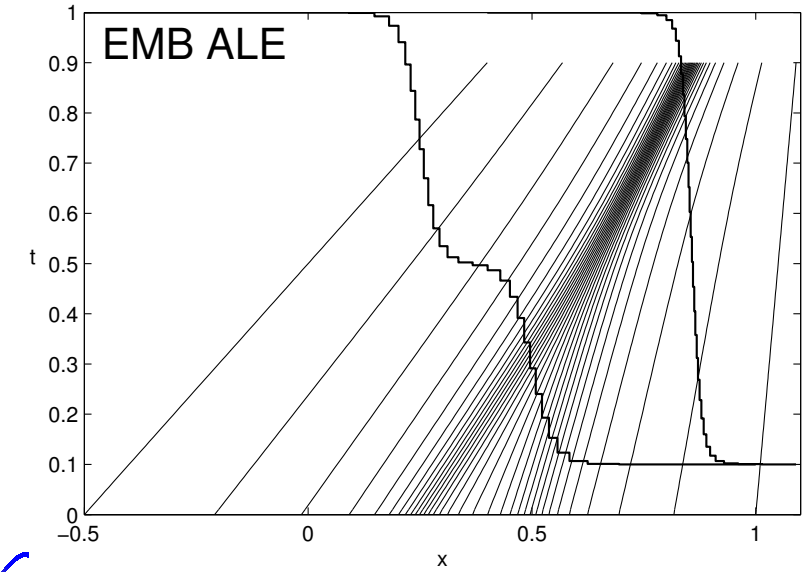
Viscous Burgers' equation

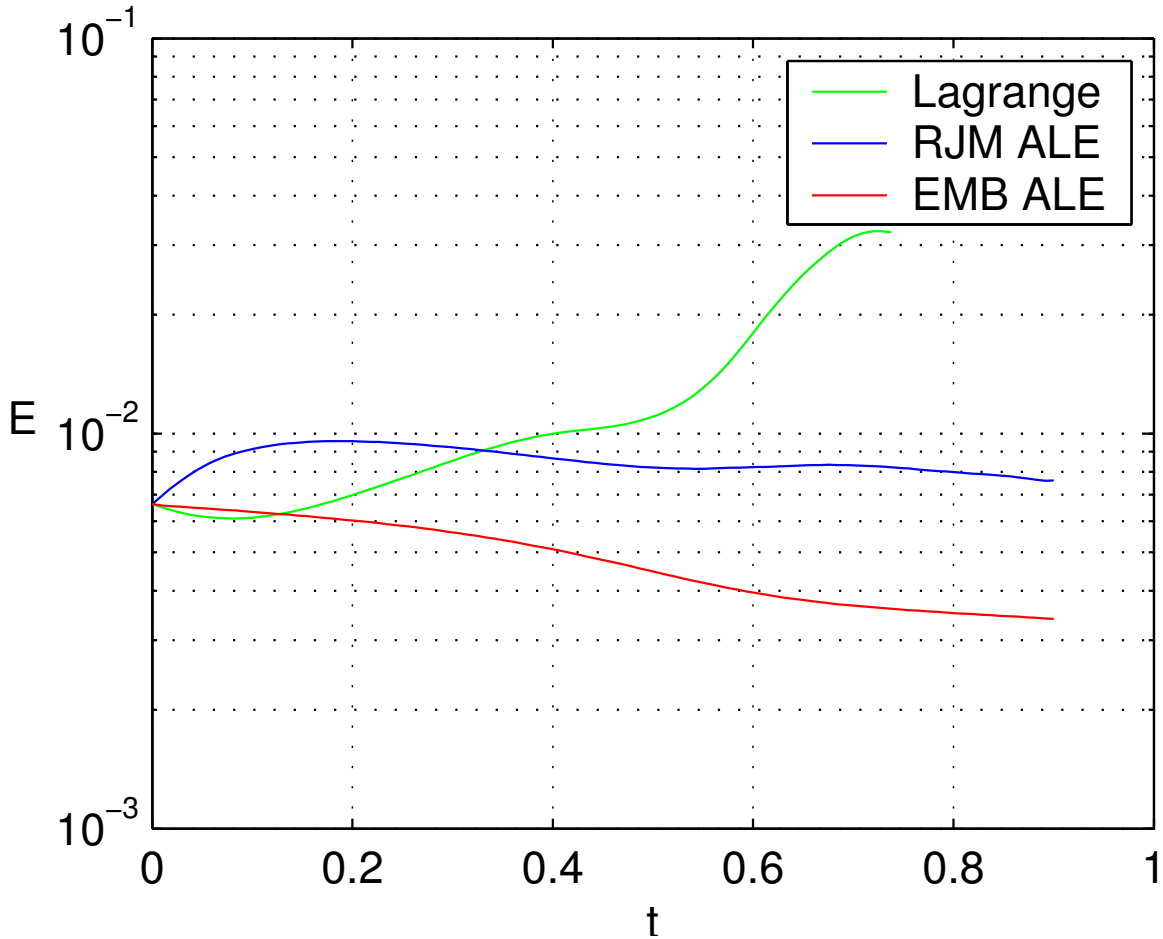






- Lagrangian method is not suitable for Burgers' equation

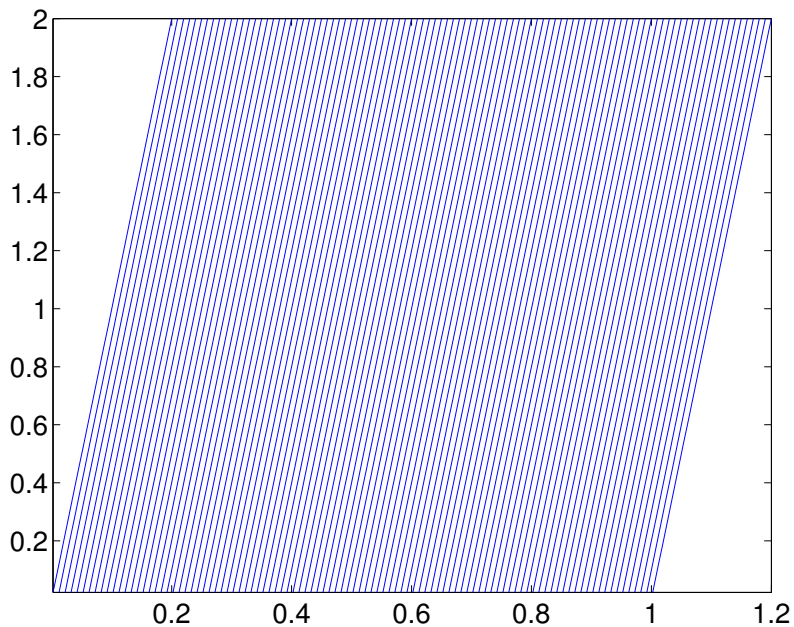
Viscous Burgers' equation



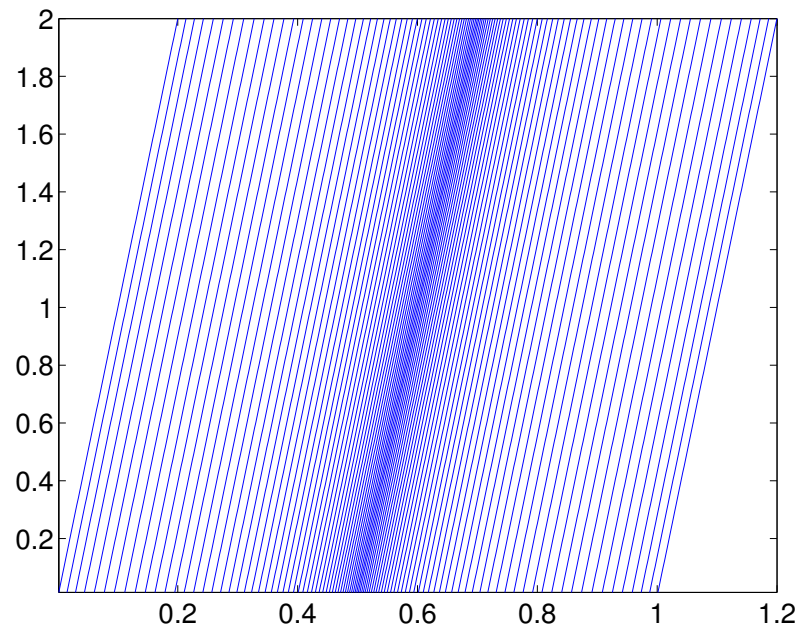
Error history for three simulations

Slowly moving contact

$$\begin{array}{l|l} \rho_L = 1.4 & \rho_R = 1.0 \\ p_L = 1 & p_R = 1 \\ u_L = 0.1 & u_R = 0.1 \end{array}$$



Lagrangian simulation

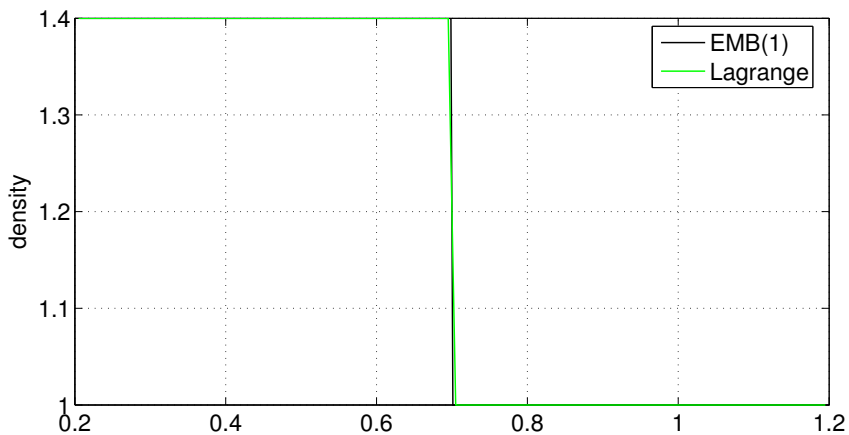


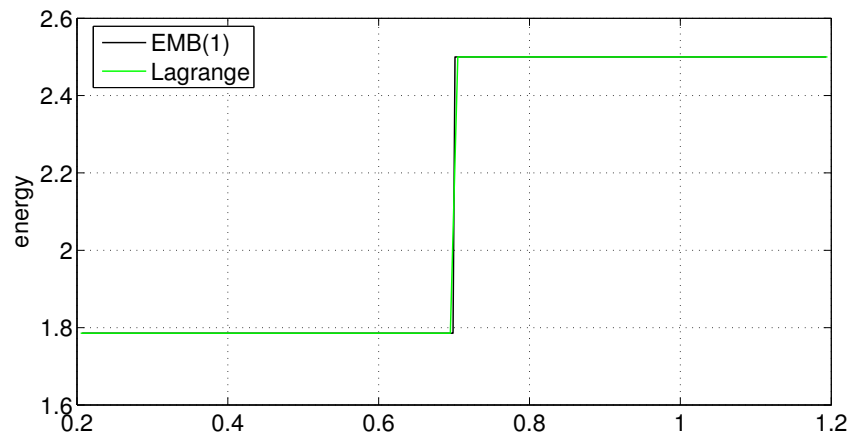
ALE simulation

- without space smoothing, all points will gather around the contact

Slowly moving contact: solution

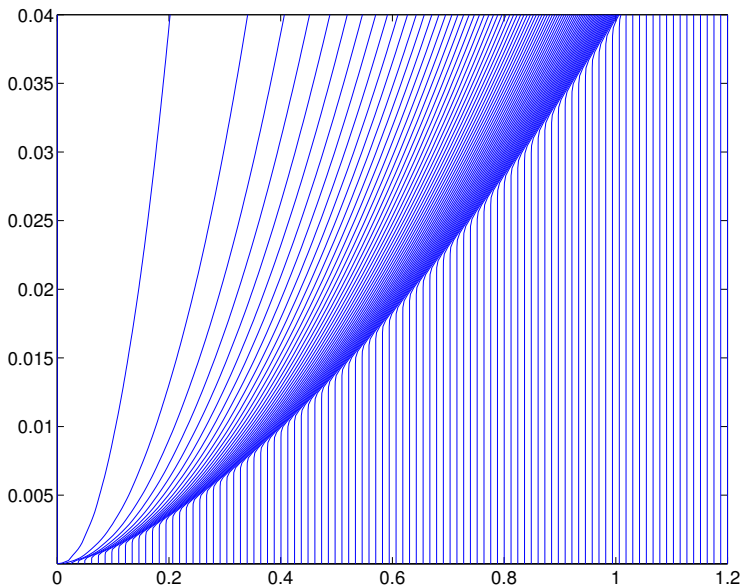
- profiles of density and internal energy
- solutions on the adaptive mesh are sharper



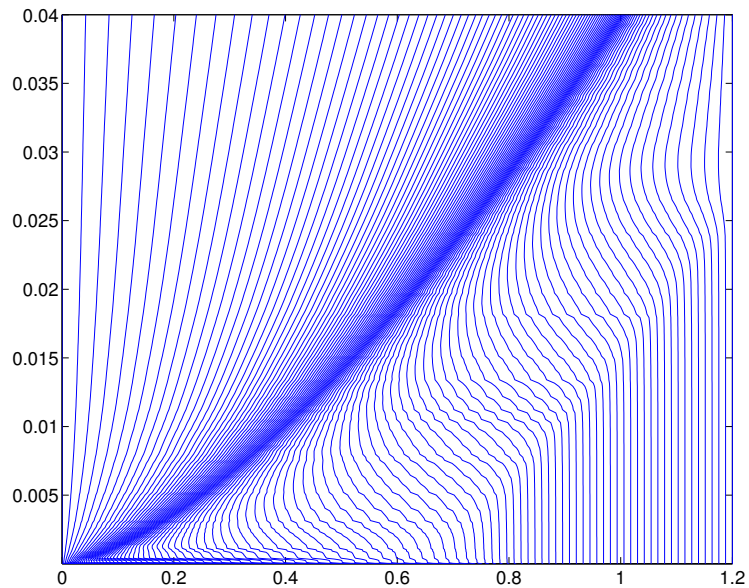


Sedov's problem

$$\begin{array}{l|l} \rho_L = 1 & \rho_R = 1 \\ e_L = 340000 & p_R = 0.00001 \\ u_L = 0 & u_R = 0 \end{array}$$



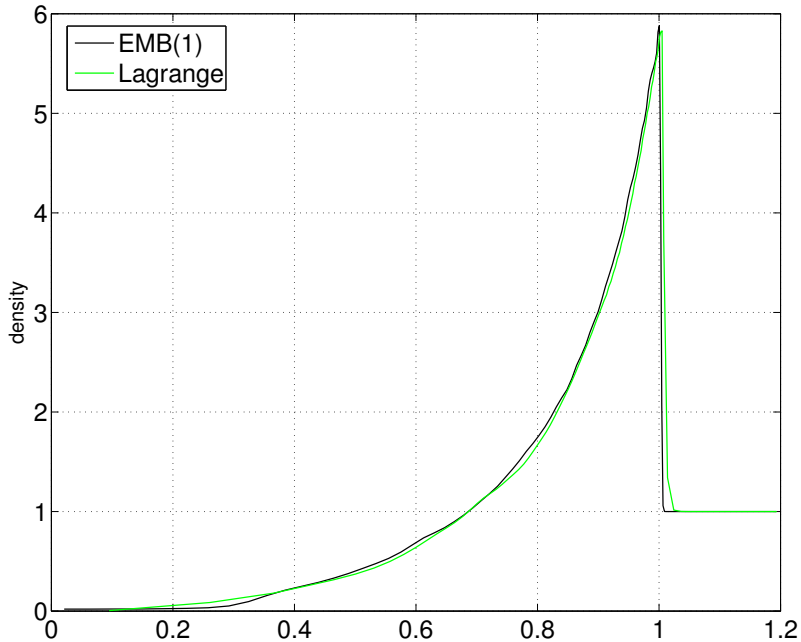
Lagrangian simulation
3999 time steps

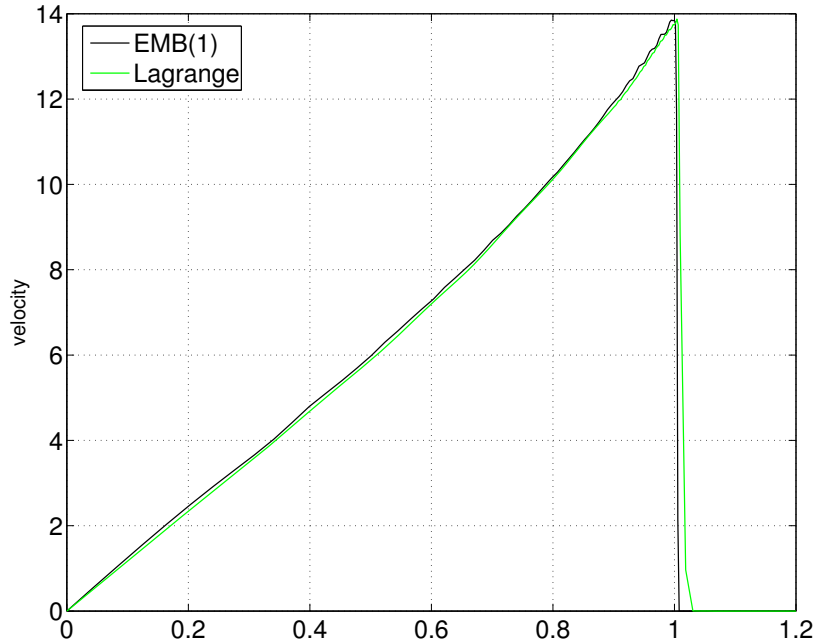


ALE simulation
3784 time steps

Sedov's problem: solution

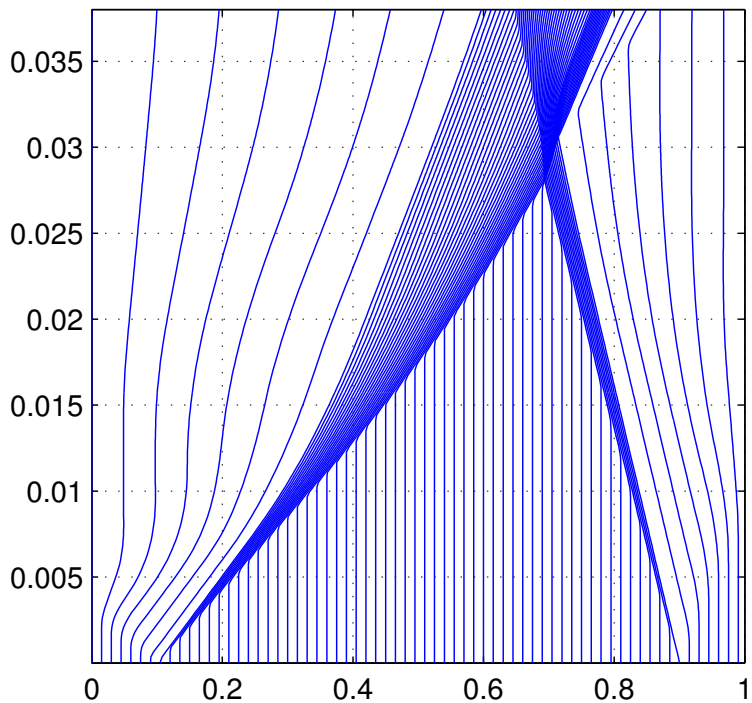
- profiles of density and velocity
- density is 2.2 times more accurate (velocity - 1.6 times)





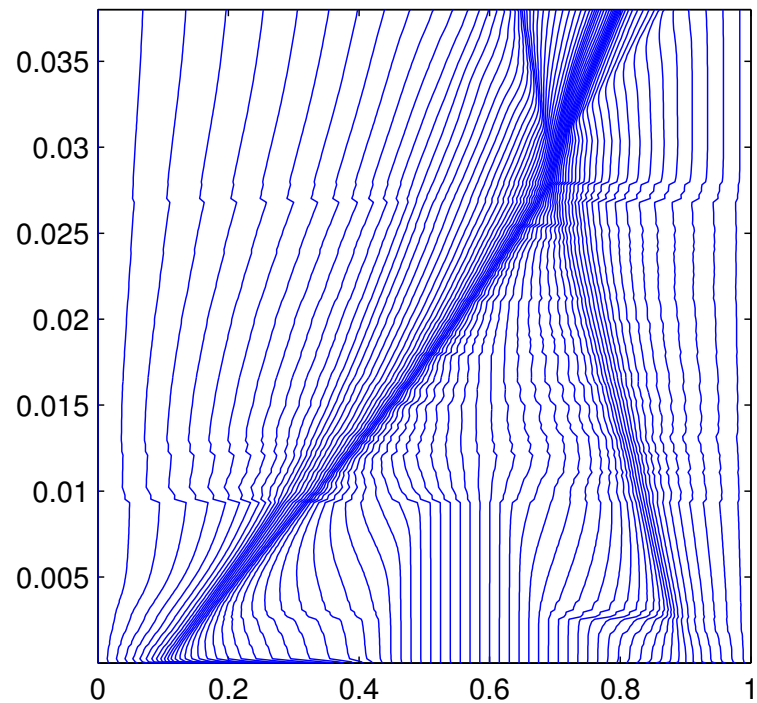
Woodward-Collela's problem

$\rho_L = 1$	$\rho_C = 1$	$\rho_R = 1$
$e_L = 2500$	$e_C = 0.025$	$e_R = 250$
$u_L = 0$	$u_C = 0$	$u_R = 0$



Lagrangian simulation

1716 time steps

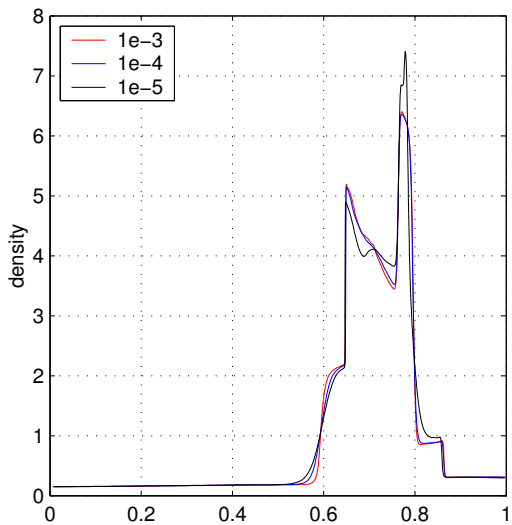


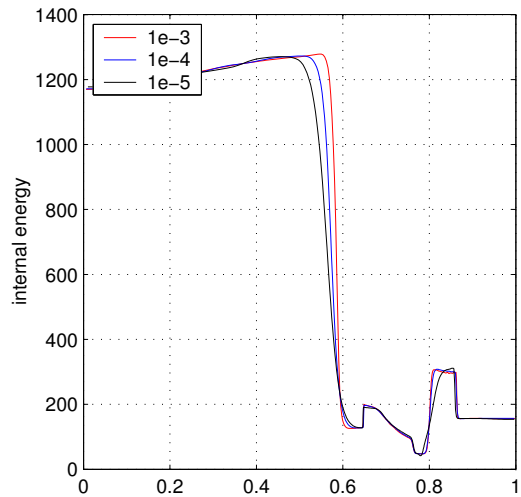
ALE simulation

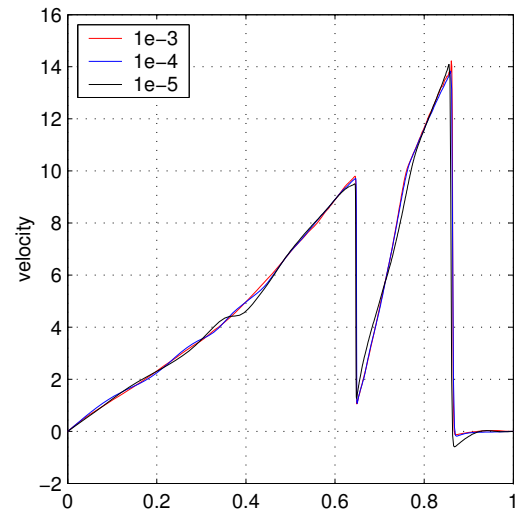
2145 time steps

Woodward-Collela: non-linear CG

	Gauss-Seidel			CG		
1e-3	27	9	43.6%	21	9	43.6%
1e-4	269	13	43.8%	109	13	44.5%
1e-5	979	16	47.2%	412	16	46.1%







- the mesh equation can be solved approximately

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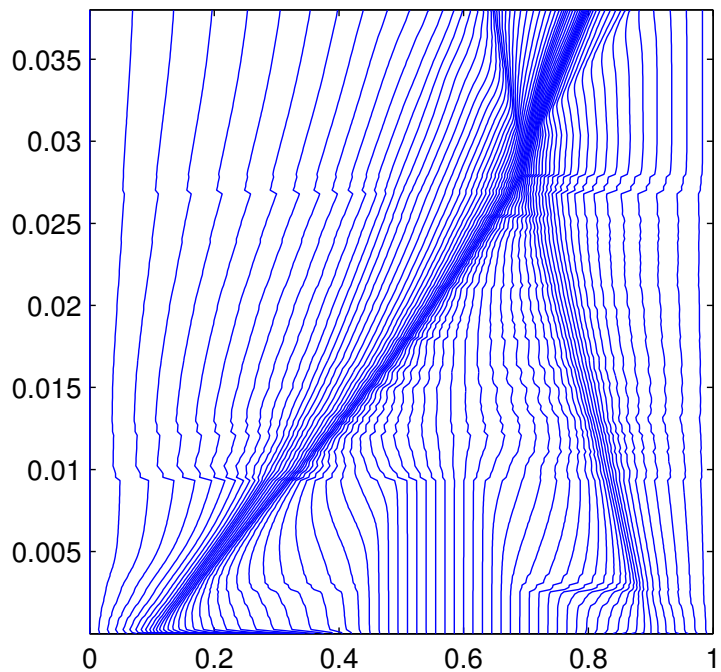
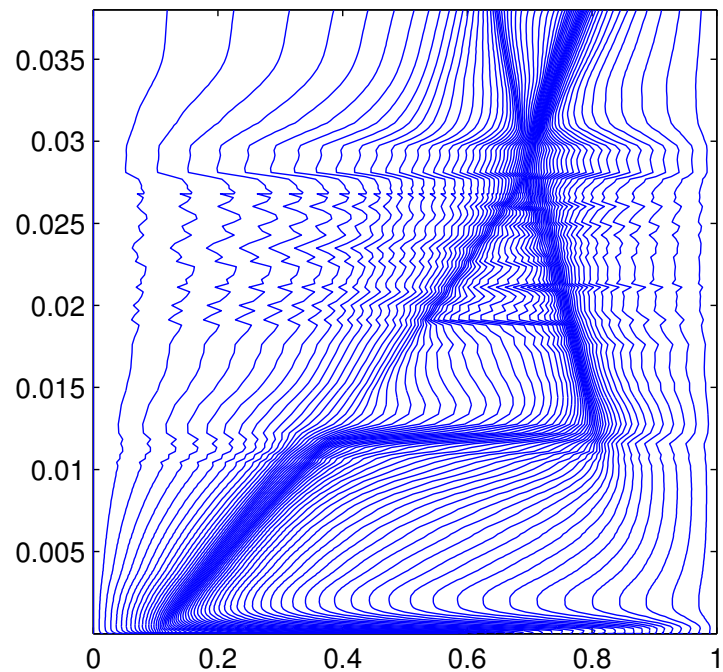
Woodward-Collela: non-linear CG



$TOL = 10^{-3}$



$TOL = 10^{-5}$

- under-resolved weak shocks
- simple time smoothing, $\mathbf{x}^n := \gamma \mathbf{x}^n + (1-\gamma)\mathbf{x}^{n-1}$, may not be sufficient

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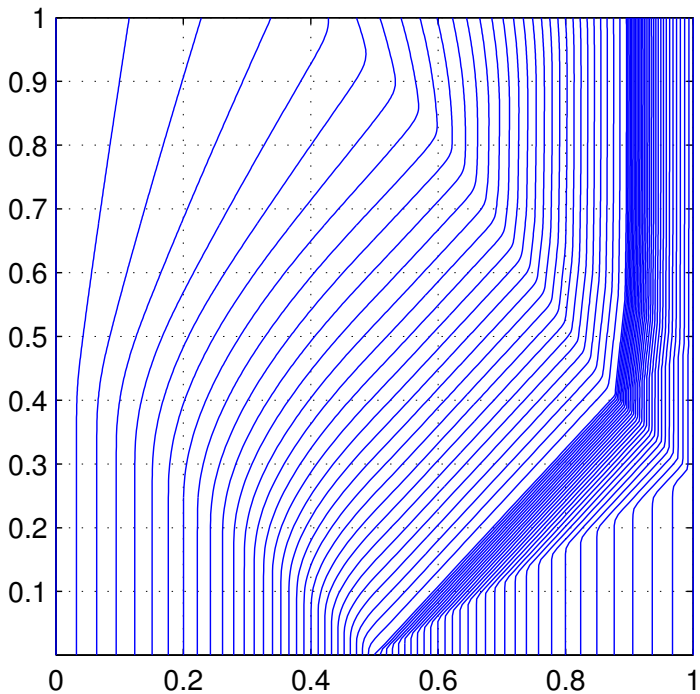
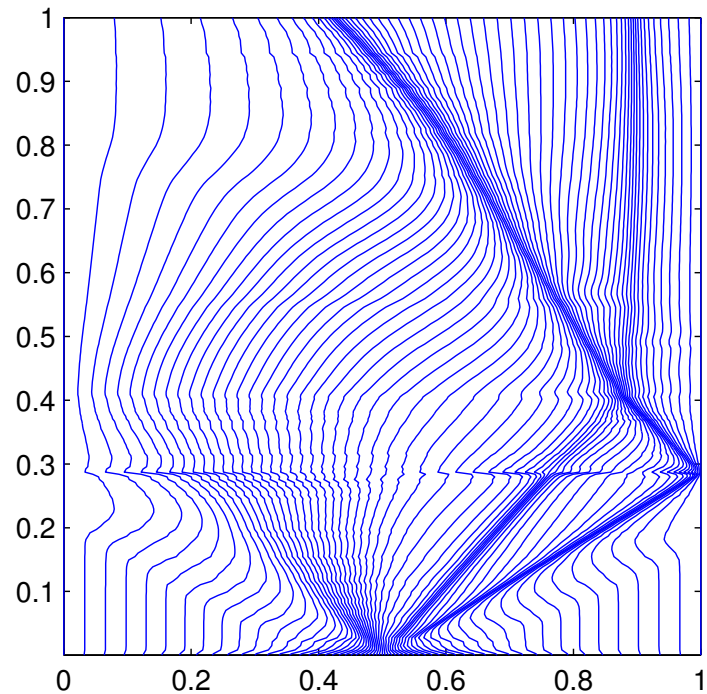
Sod's shock tube

$$\begin{array}{l|l} \rho_L = 1 & \rho_R = 0.125 \\ p_L = 1 & p_R = 0.1 \\ u_L = 0 & u_R = 0 \end{array}$$



Lagrangian simulation

3800 time steps

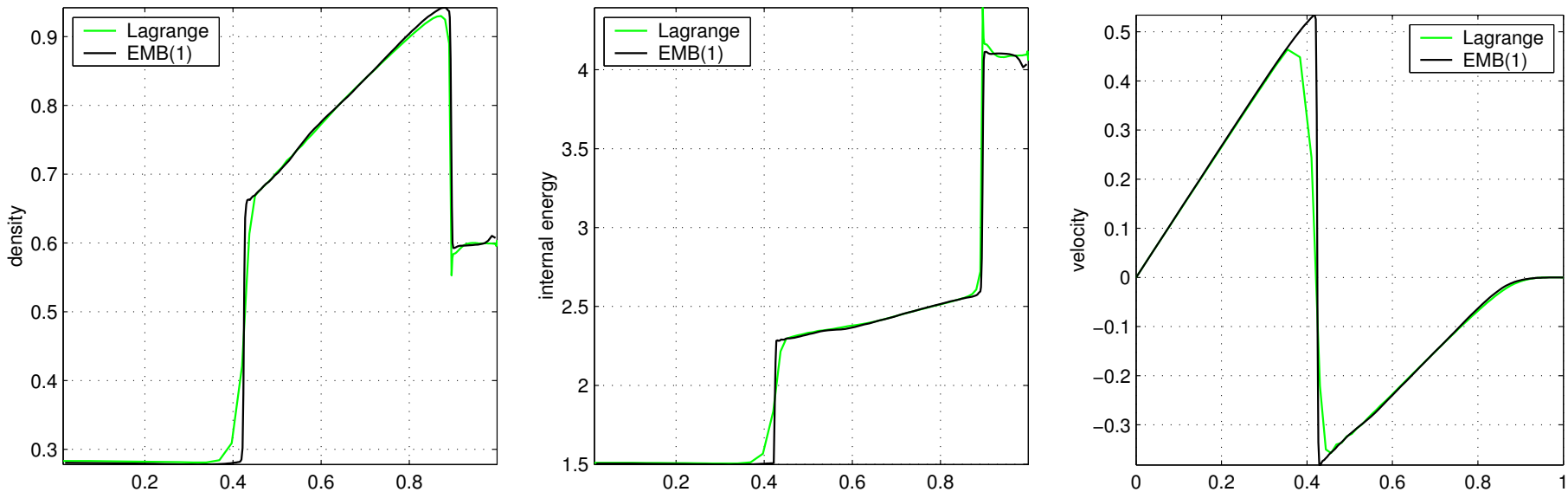


ALE simulation

2829 time steps

Sod's shock tube: solution

- profiles of density, internal energy and velocity



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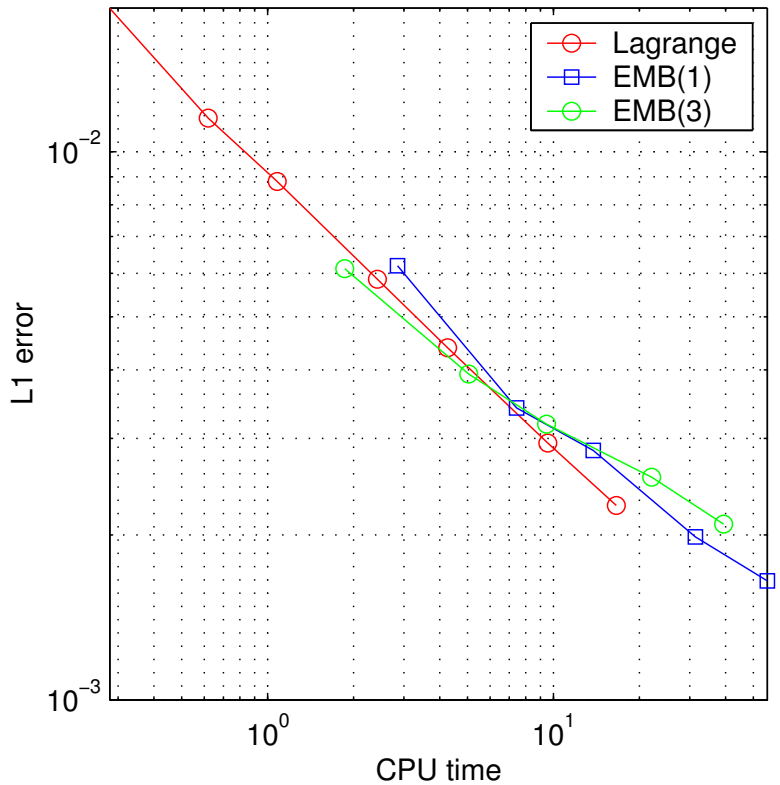
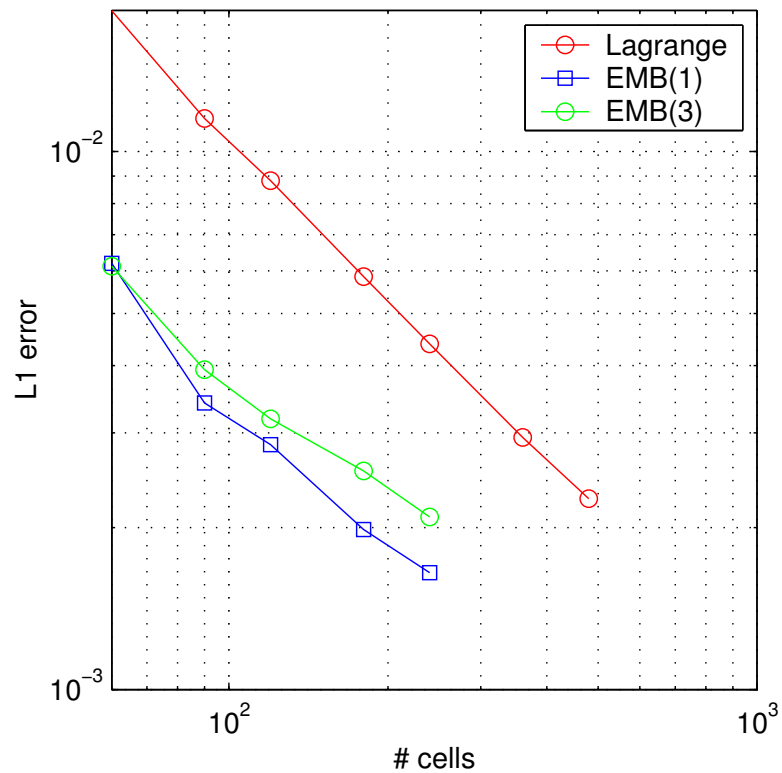
Sod's shock tube: efficiency

- initial mesh is uniform

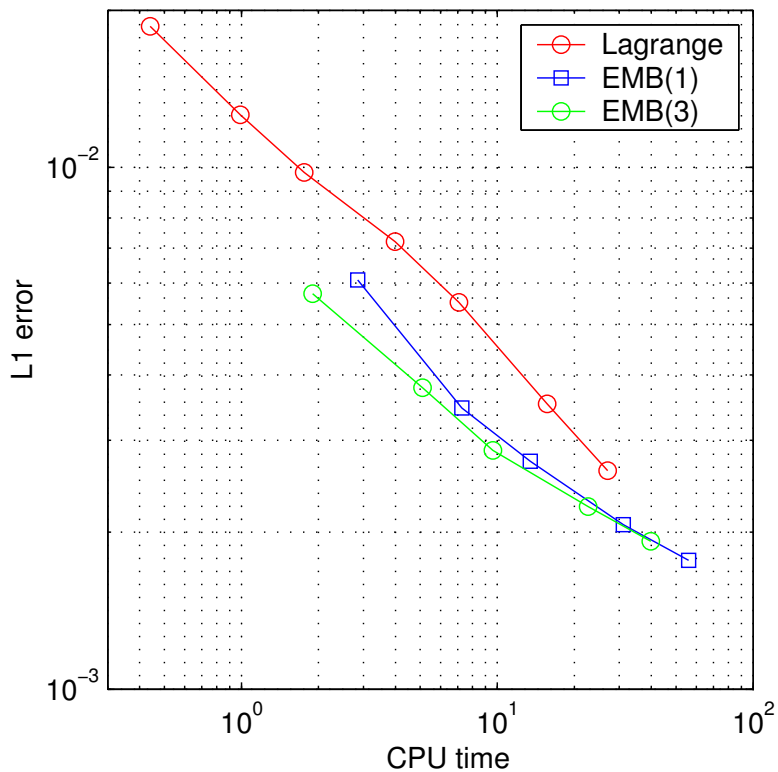


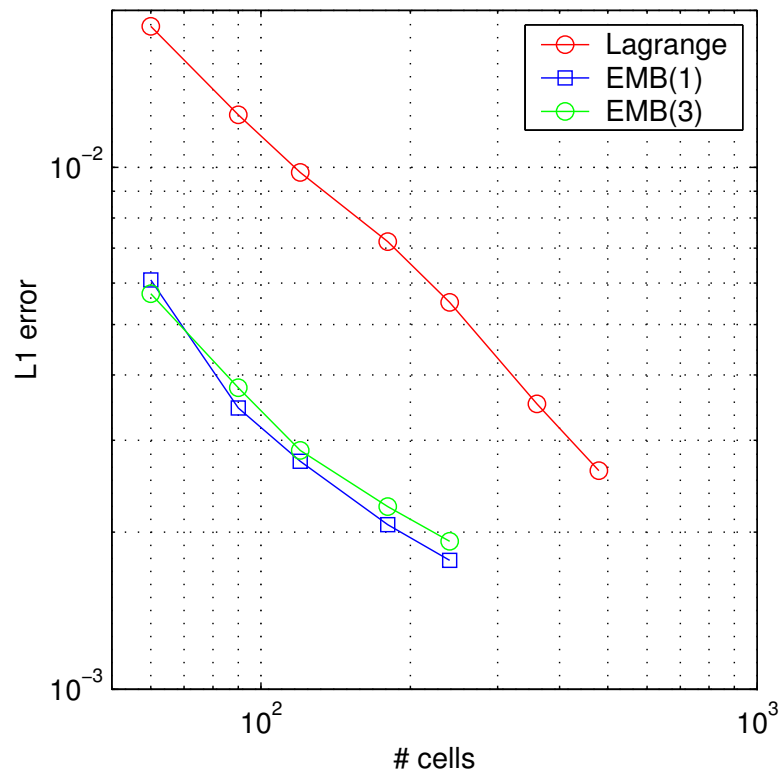


- the EMB method requires tuning for the computational efficiency

Sod's shock tube: efficiency

- initial mesh is adapted to the density profile

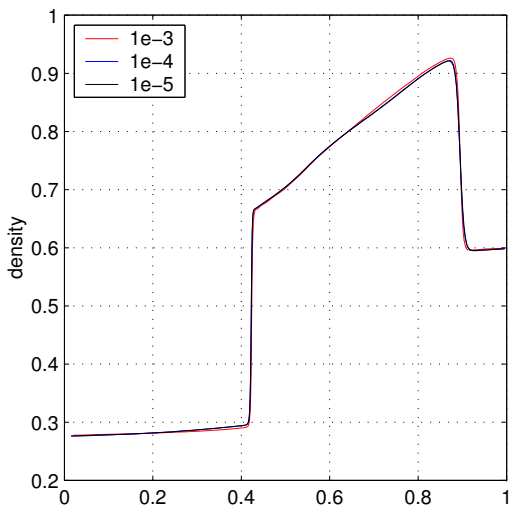


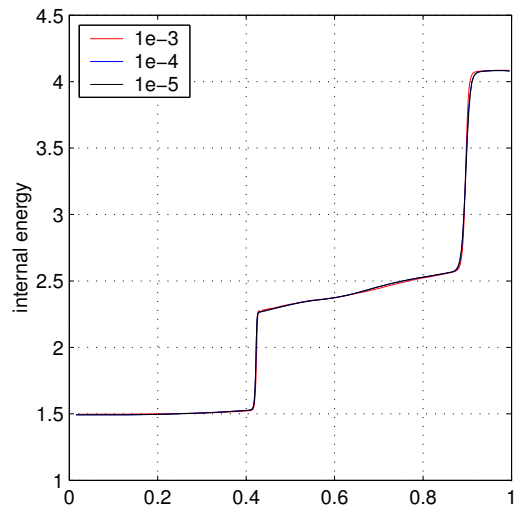


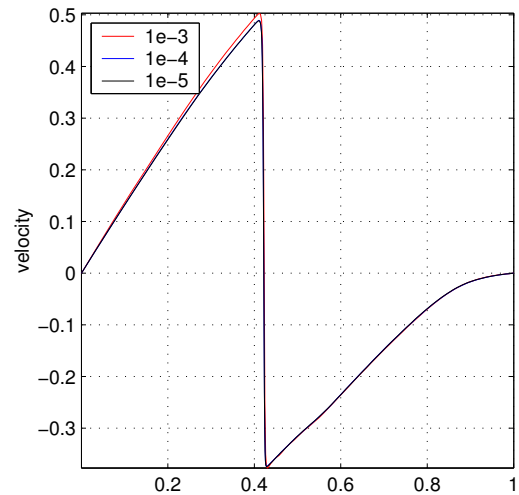
- for CPU time $t = 10s$, the Lagrangian and EMB errors are 0.0035 and 0.0021 on meshes with 300 and 110 cells, respectively.

Sod's shock tube: non-linear CG

	Gauss-Seidel			CG			Newton-CG		
1e-3	22	8	18.8%	18	8	18.7%	41	7	16.3%
1e-4	297	10	18.1%	93	12	17.7%	239	10	17.9%
1e-5	487	15	18.3%	224	15	17.5%	no data		







- the mesh equation can be solved approximately

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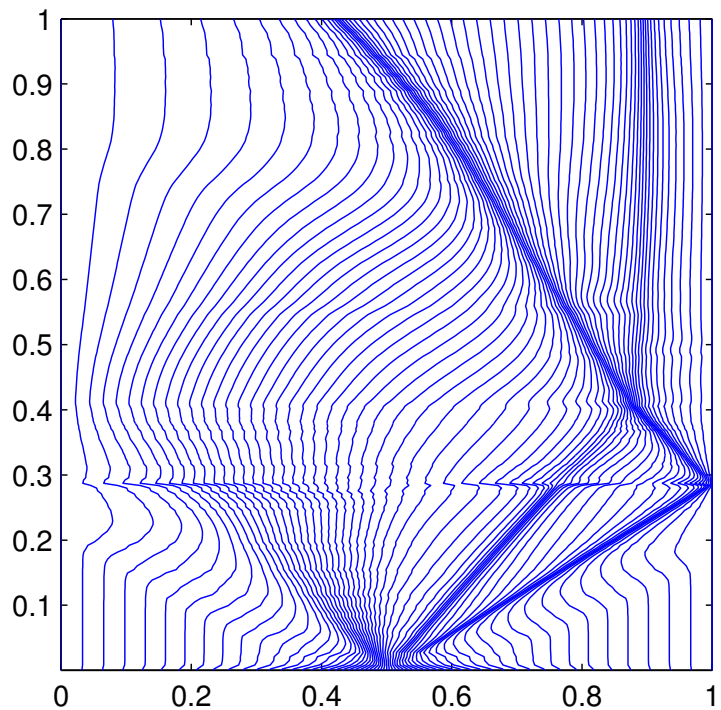
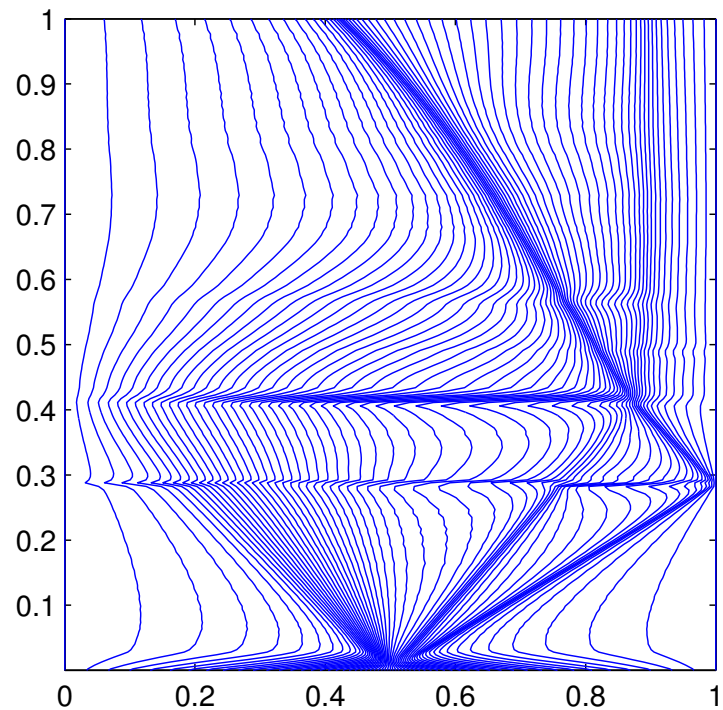
Sod's shock tube: non-linear CG



$TOL = 10^{-3}$

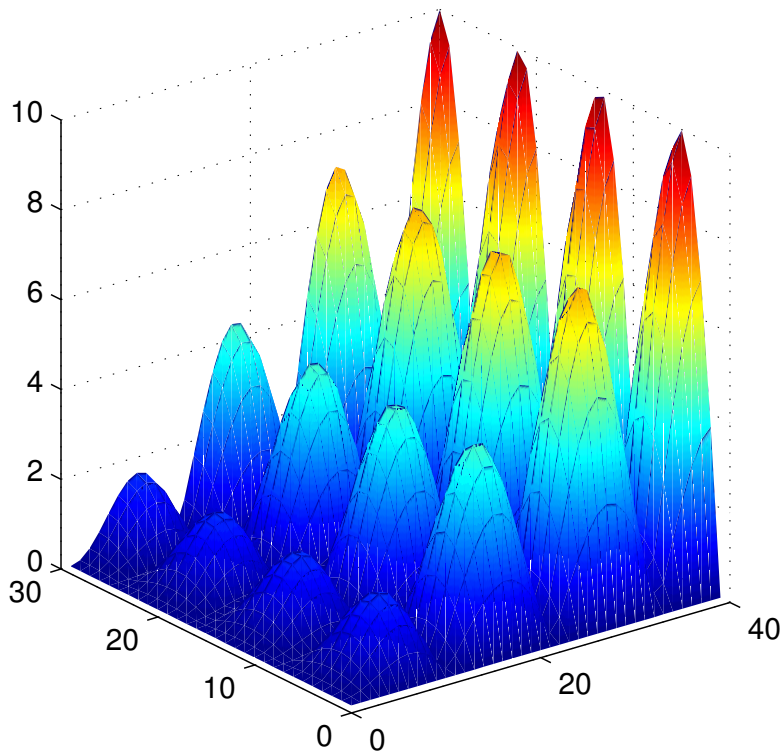


$TOL = 10^{-5}$

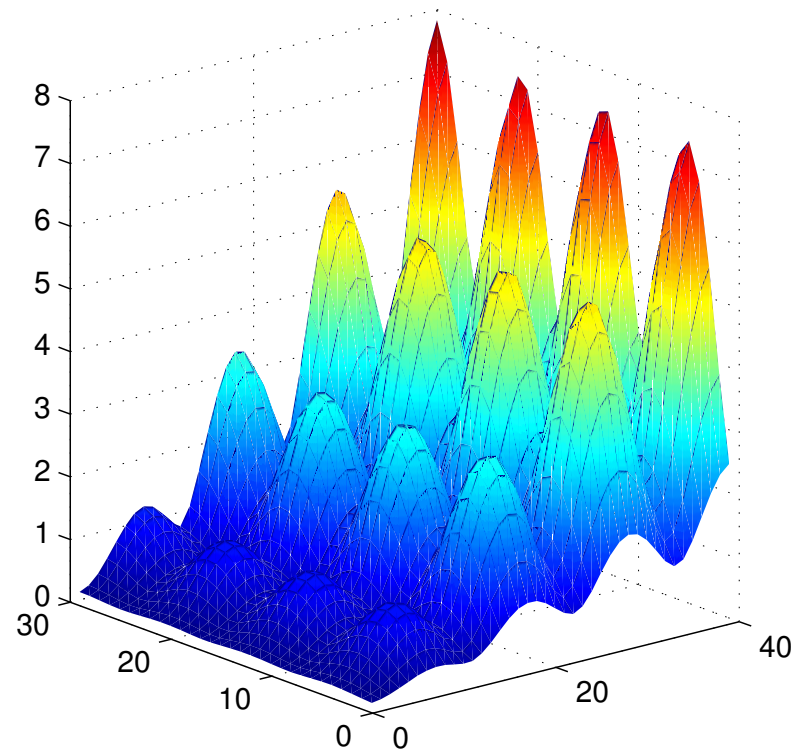
- error dynamics does not match solution dynamics

2D extension: error smoothing

- smoothing preserves main features of $\mathbf{h}$



$2.7 \cdot 10^{-6} \leq [\mathbf{h}] \leq 3.7 \cdot 10^{5}$



$0.57 \leq [\tilde{\mathbf{h}}] \leq 1.8$

- smoothing is performed in the logical space (M-matrix close to I)

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2D extension: constraint minimization

Minimize

$$\Phi_2(\mathbf{x}^n, \mathbf{y}^n) = \sum_{cells} \left[\mathcal{S}^{-1} \left(\frac{\delta \mathbf{u}^n}{\delta x}, \frac{\delta \mathbf{u}^n}{\delta y} \right) \right]_c^a V_c^b$$

over a class of meshes with convex cells.

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2D extension: constraint minimization

- The idea is to minimize

$$\Phi_3(\mathbf{x}^n, \mathbf{y}^n) = \sum_{cells} \left[\mathcal{S}^{-1} \left(\frac{\delta \mathbf{u}^n}{\delta x}, \frac{\delta \mathbf{u}^n}{\delta y} \right) \right]_c^a V_c^b \mathcal{C}_c$$

where $\mathcal{C}_c \geq 1$ measures the cell convexity.

- If the equidistribution principle holds, we get

$$V_c^b \mathcal{C}_c \sim V_n^b \mathcal{C}_n.$$

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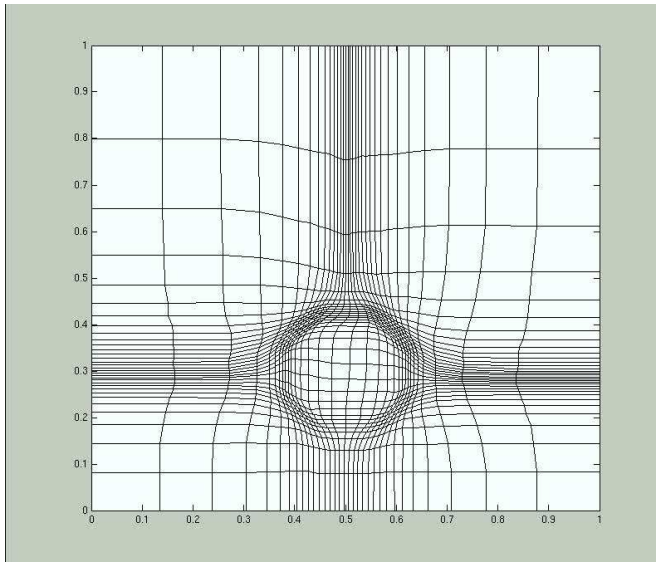
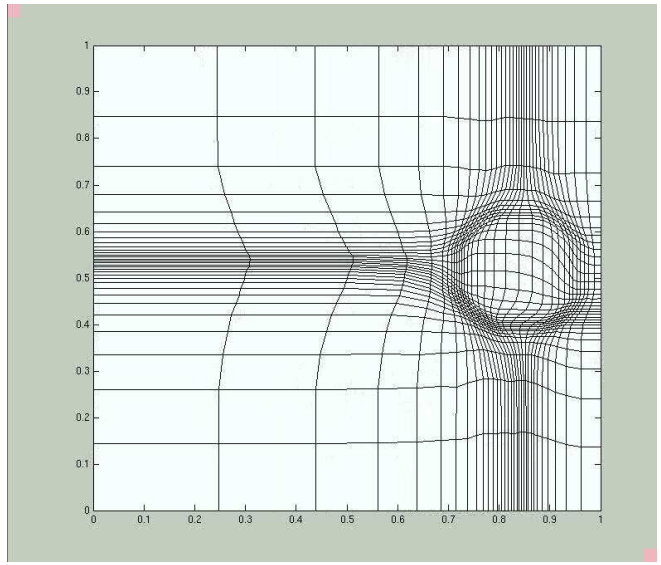
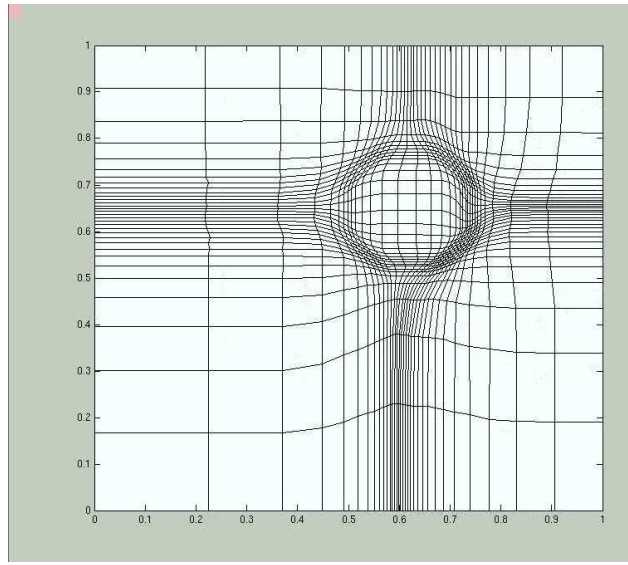
2D extension: examples

- The analytical solution is given
  - example I: bouncing ball



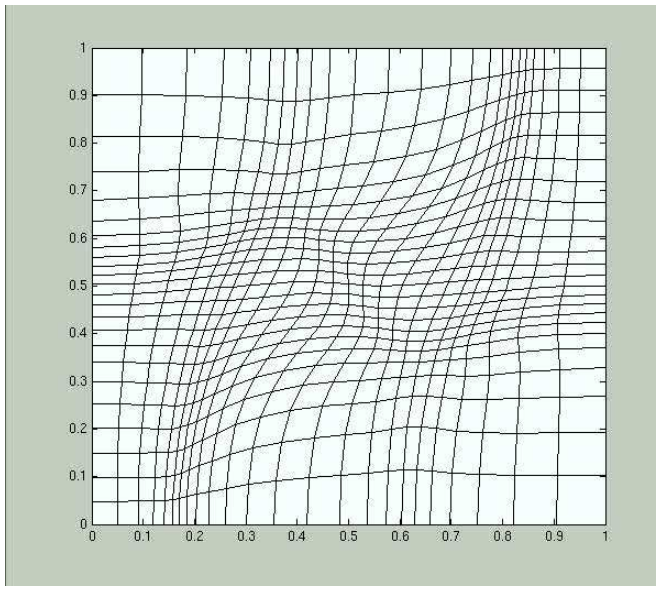


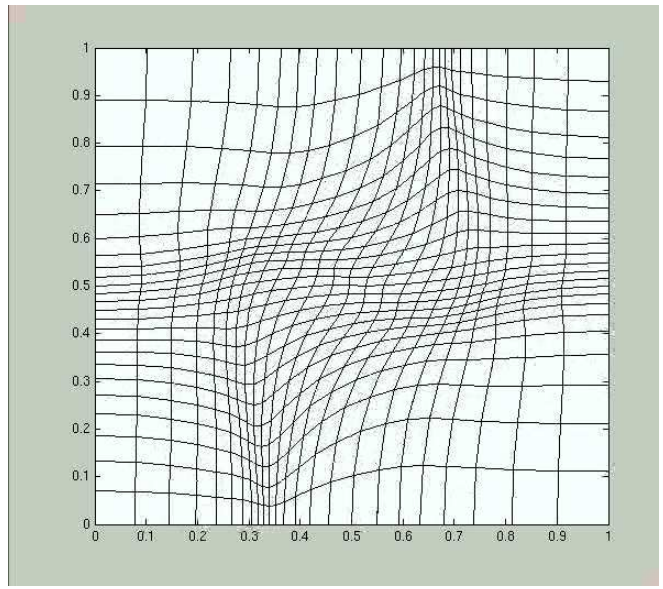


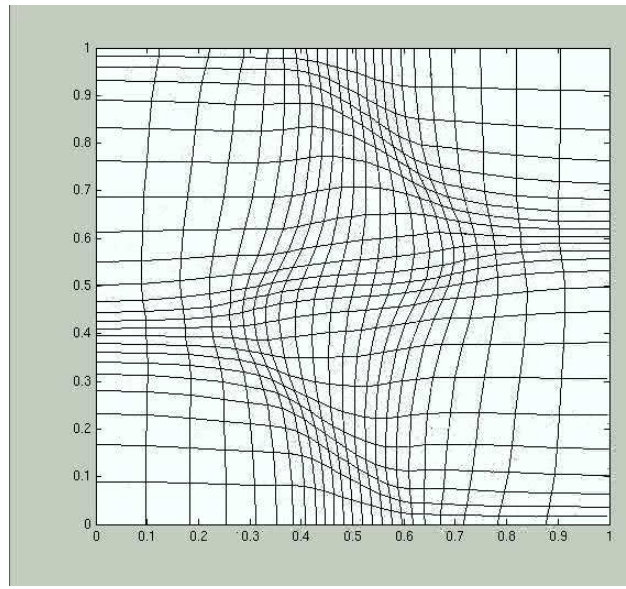
  - solution is constant everywhere expect a small region around the circle.

2D extension: examples

- The analytical solution is given
  - example II: rotating saddle







  - velocity field makes the problem difficult for Lagrangian simulations.
  - solution is constant everywhere except a small region around the saddle.

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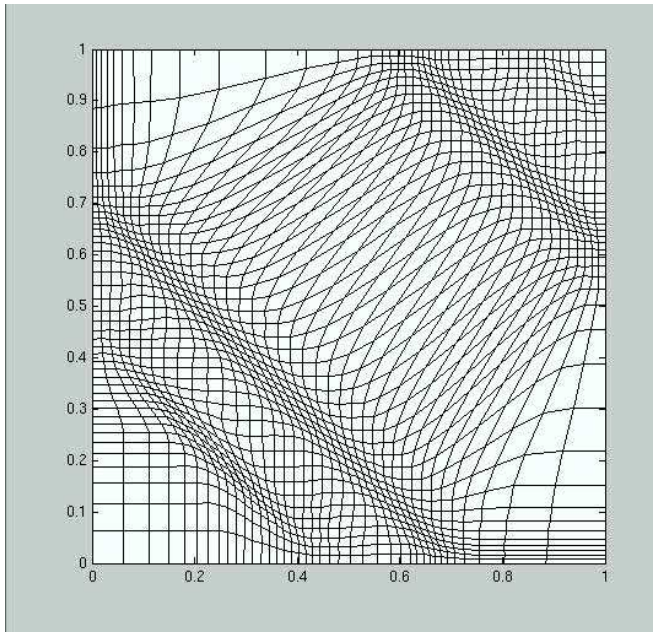
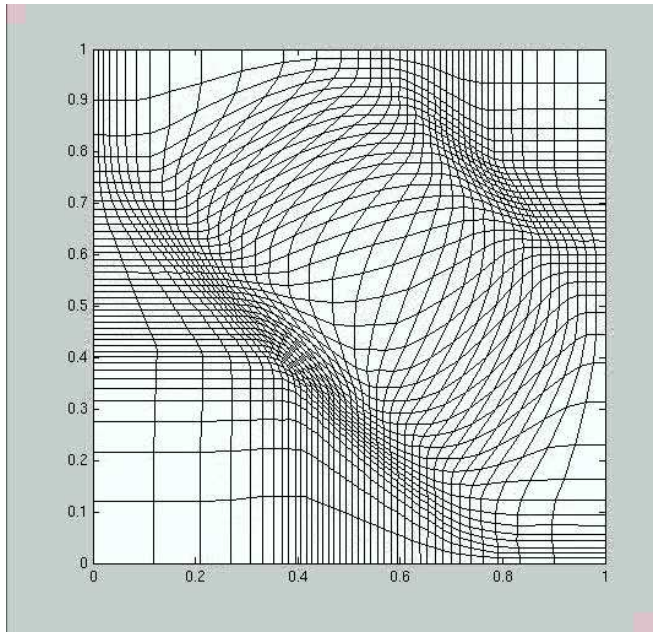
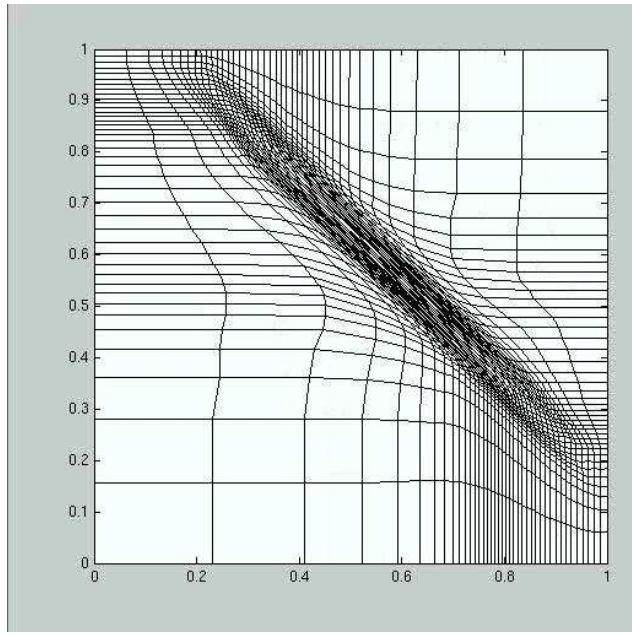
2D extension: examples

- The analytical solution is given
  - example III: two parallel waves







  - mesh cells become stretched but still convex

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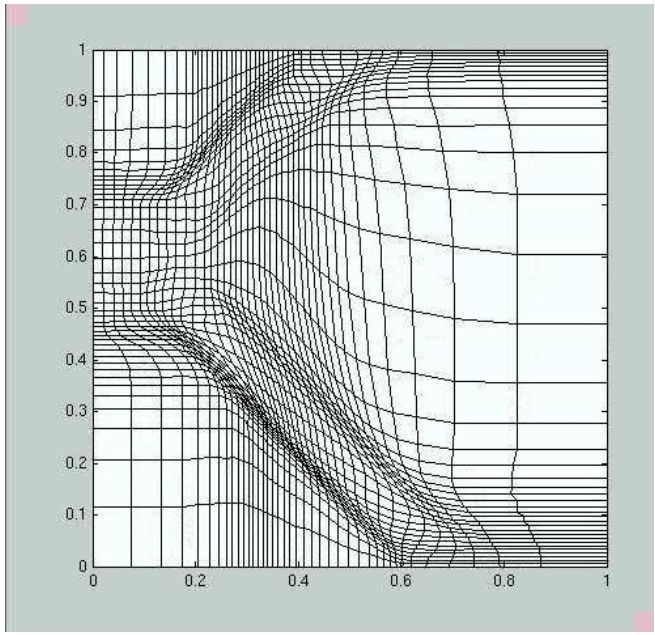
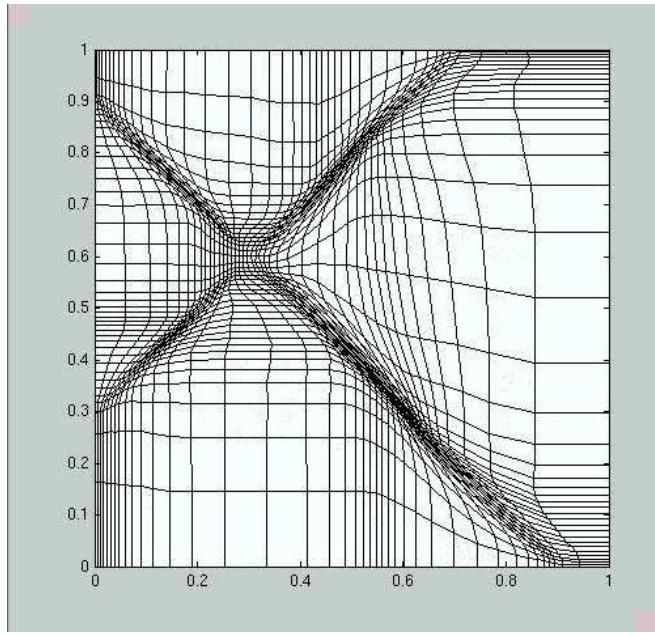
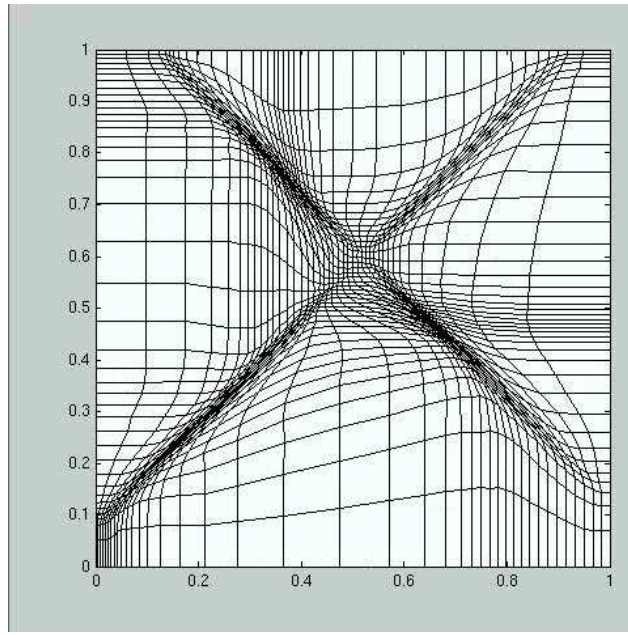
2D extension: examples

- The analytical solution is given
  - example IV: two perpendicular waves







  - the boundary cells are controlled by the smoothing parameter α

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Conclusion

- The analysis explains why the mesh equation can be solved very approximately and input data can be lagged.
- Space smoothing is necessary for problems with smooth solutions.
- Time smoothing is crucial for problems with shocks.
- In Lagrangian gasdynamics, there exists a class of problems where the moving mesh methods save simulation or improve accuracy of solutions.

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