A new discretization method for solving pressure equation on arbitrary meshes.

Konstantin Lipnikov

Los Alamos National Laboratory
lipnikov@lanl.gov

Collaborators

- **Franco Brezzi**
  Istituto Universitario di Studi Superiori, Pavia, Italy
  brezzi@imati.cnr.it

- **Mikhail Shashkov**
  Los Alamos National Laboratory
  shashkov@lanl.gov

- **Valeria Simoncini**
  Dipartimento di Matematica, Università di Bologna, Bologna, Italy
  valeria@dm.unibo.it

Acknowledgment

- It would be impossible to test new discretization methods without **MSTK library** written by Rao Garmella, T-7.

Talk flow

1. Motivation
2. Mimetic philosophy
3. MFD method for pressure equation
4. Important details
5. High-order MFD methods
6. New MPFA-MFD methods
7. Concluding remarks

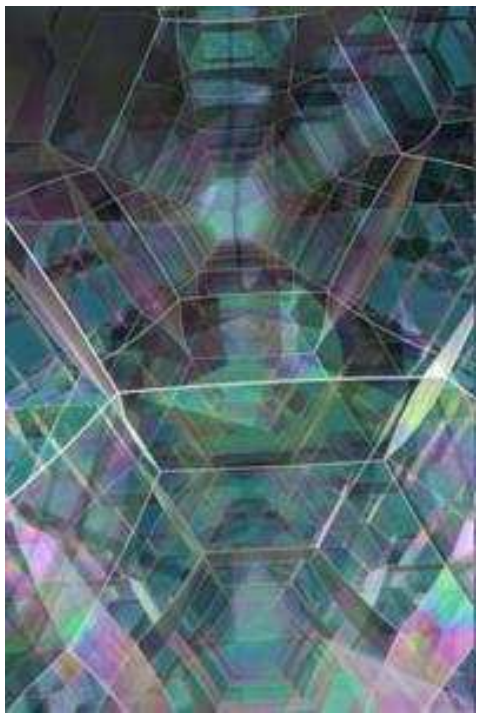
Increase Predictiveness of Simulations

by

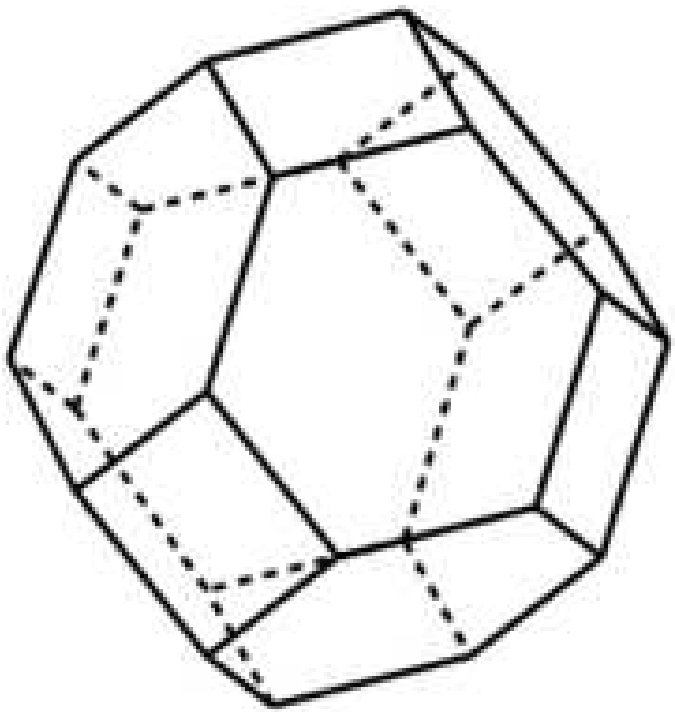
Incorporating Physical Principles into Algorithms

Polyhedra in nature

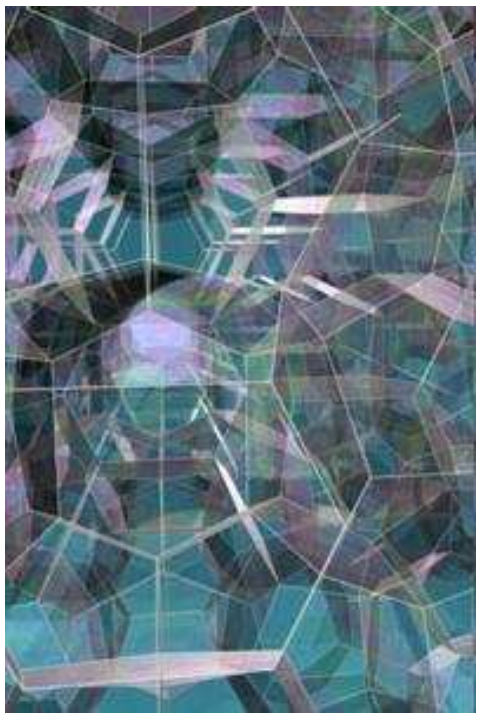
What partition of the 3D space into cells of equal volume minimizes the surface area of cell faces?



Kelvin (1887)

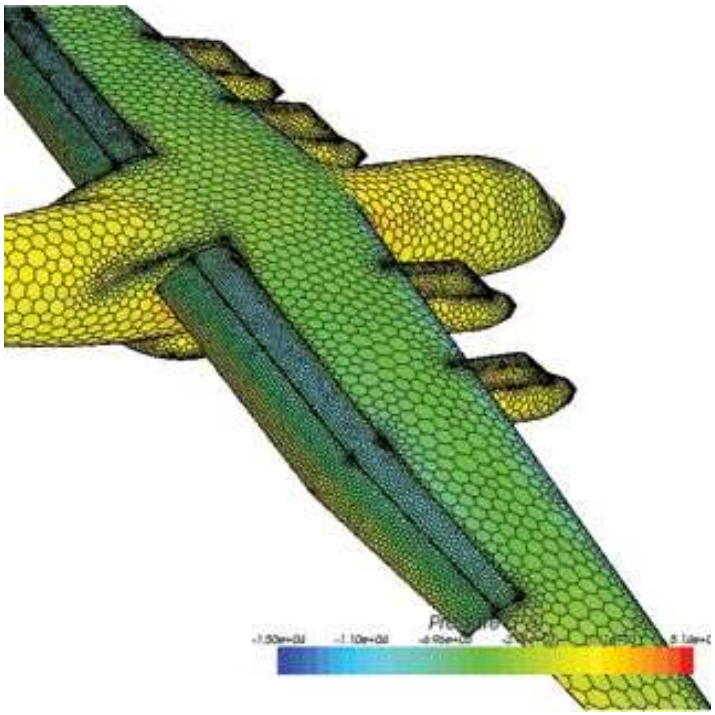


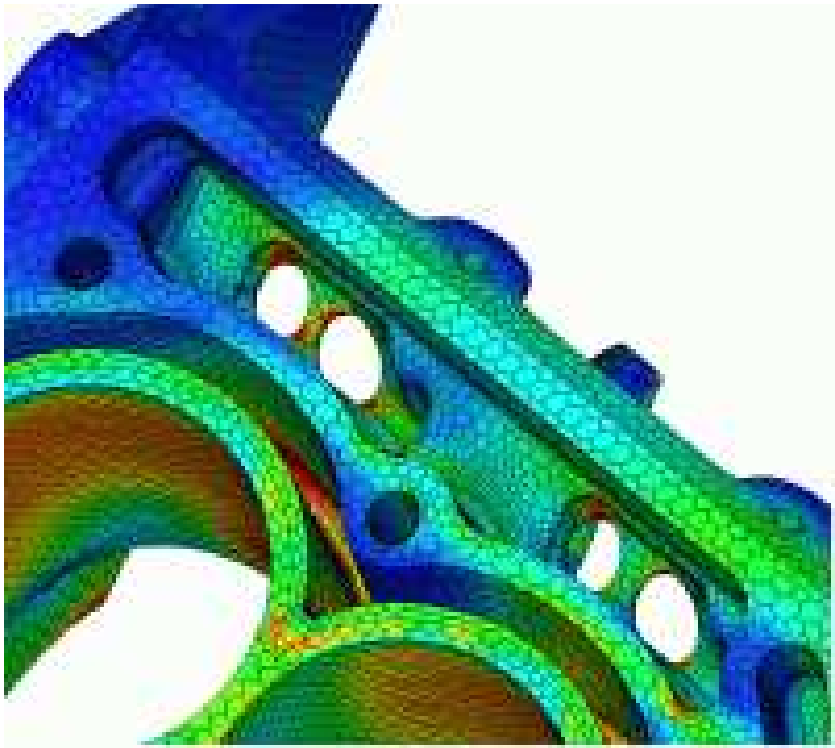
tetrakaidecahedron



Weaire-Phelan (1994)

Polyhedra in CFD

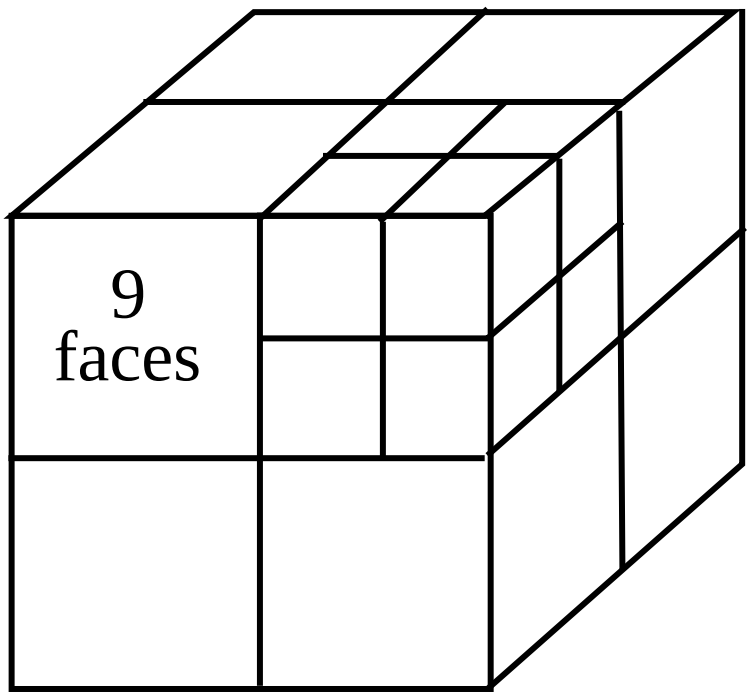


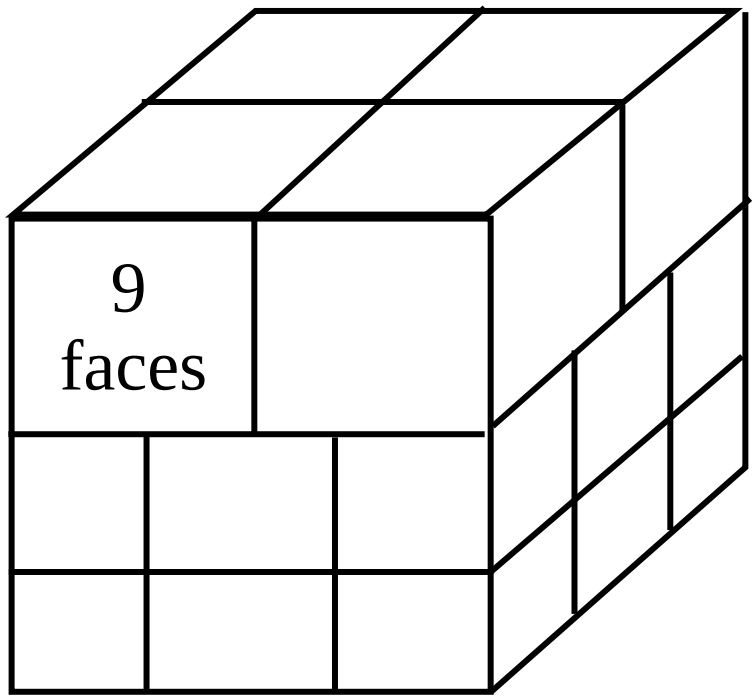


Images are courtesy of CD-Adapco

Simulations of **Polyhedral Meshes** can be **10 times faster** than on **Tetrahedral Meshes**

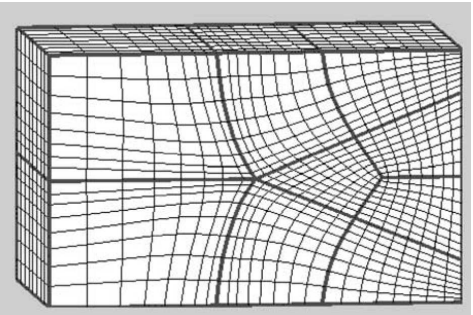
Polyhedra in AMR

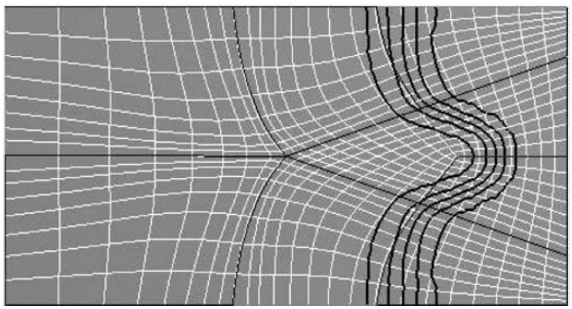


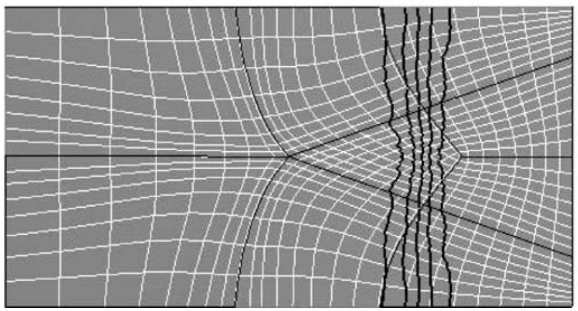


Locally refined and **Non-matching** meshes
are the **Polyhedral Meshes**

Problems: Mesh effects



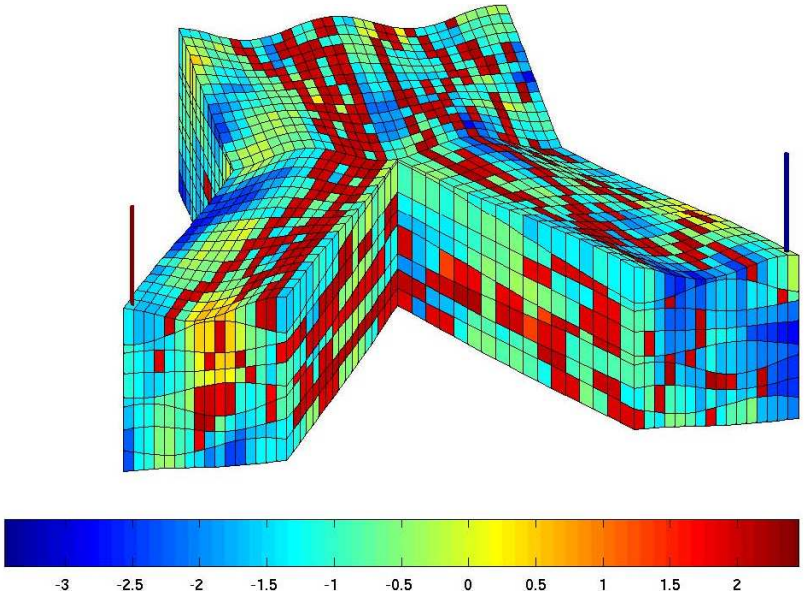




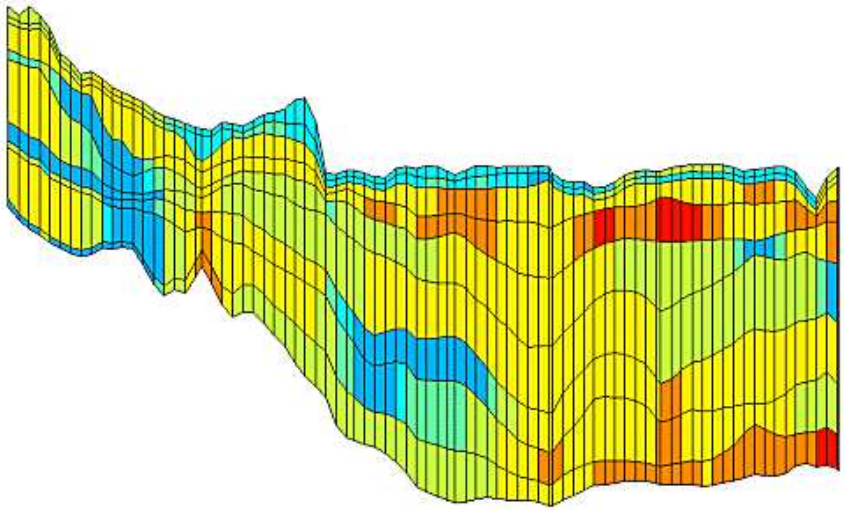
Lee, S.H., Jenny, P., Tchelepi, H.A., Comput. Geosciences, 6, 2001.

Mesh Imprint in the Solution is the Problem for **Standard Methods**

Problems: Curved Faces



Log scale of permeability



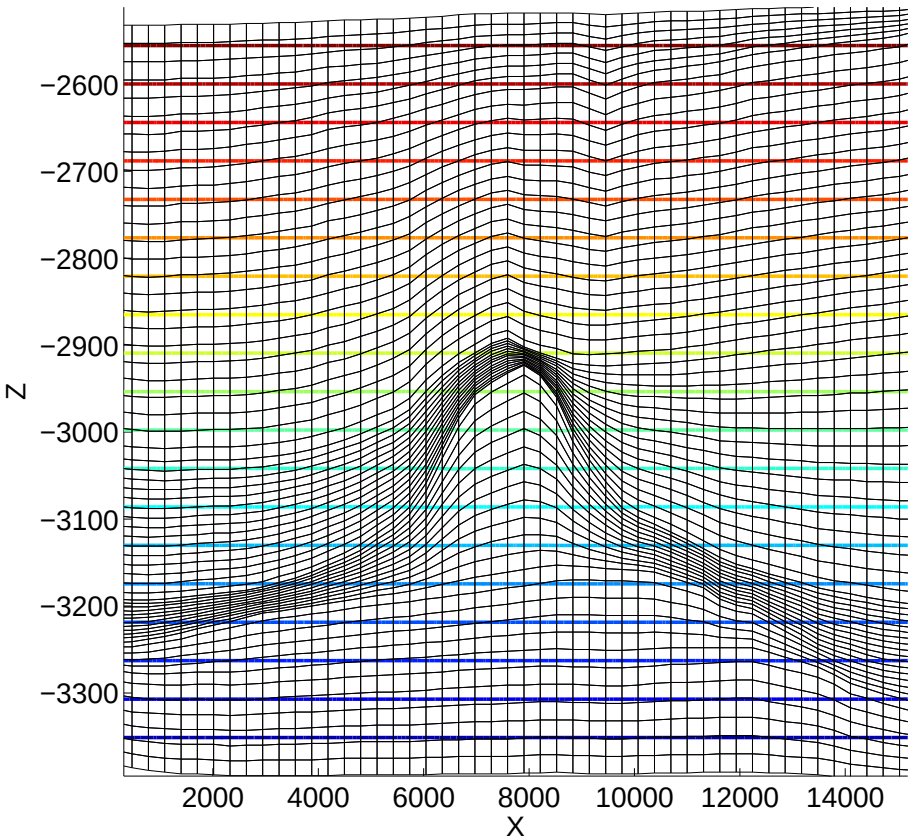
Vertical cut of a hexahedral mesh

Curved Faces Result in $O(1)$ **Error**
in the **Standard Methods**

Mimetic Methods

Mimic Important Properties of the Physical and Mathematical Models

Mimetic Methods

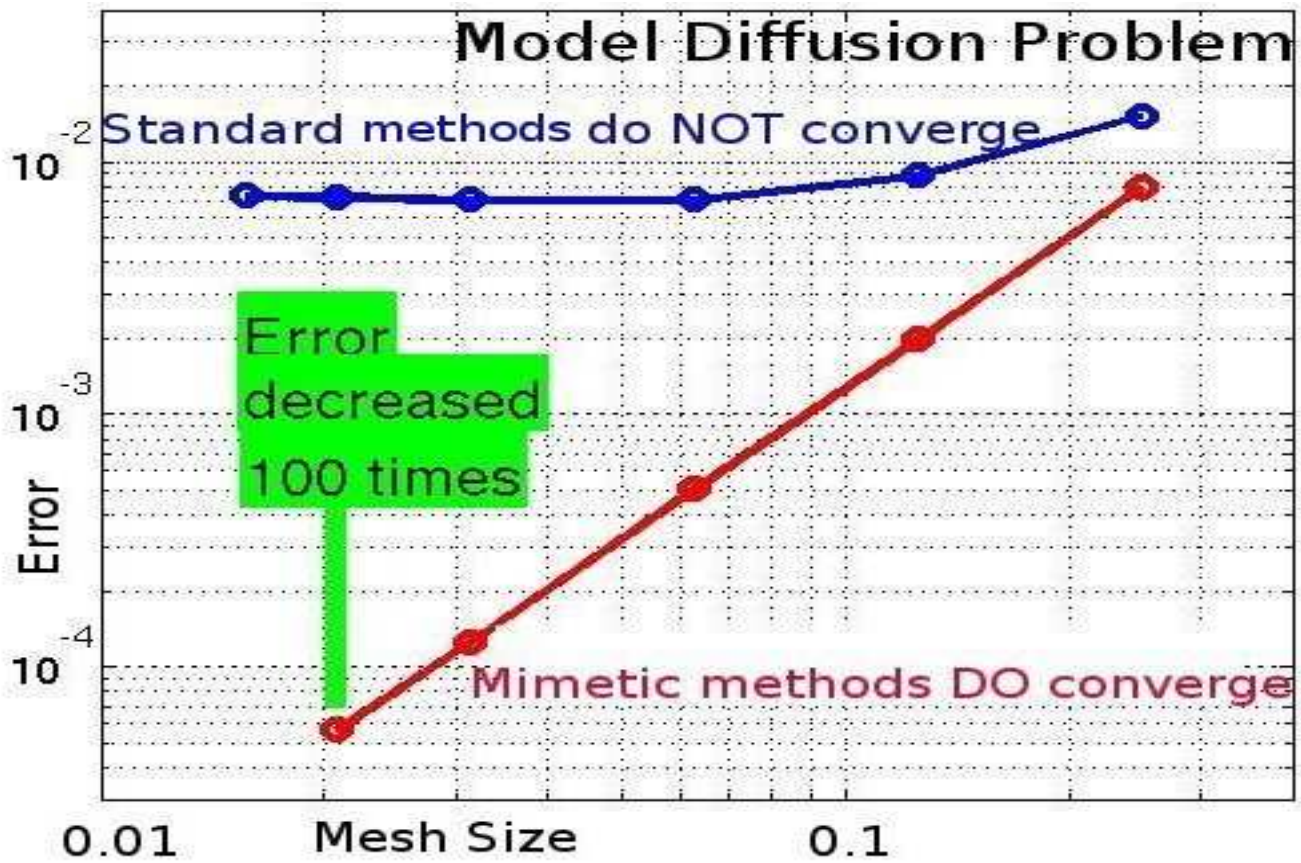


$$\mathbb{K} = \begin{bmatrix} 10 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\lambda_1 \approx 0.092$$
$$\lambda_2 \approx 10.91$$

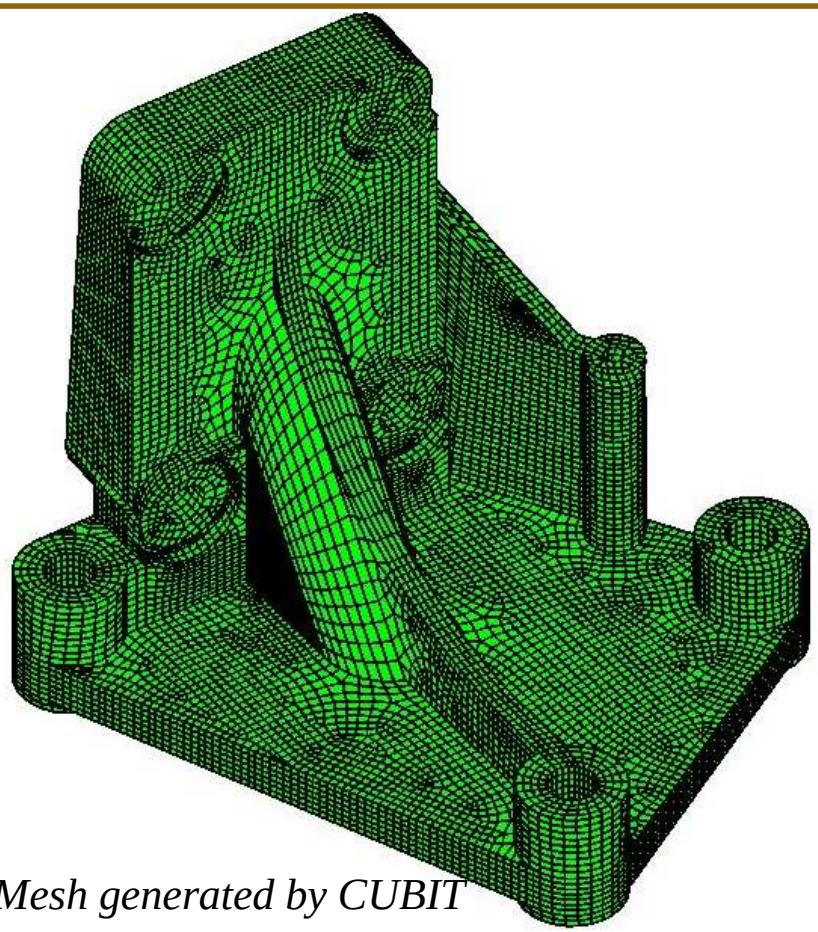
Mimetic Methods preserve Linear Solutions for Problems with **Tensor Coefficients** on **Non-smooth Meshes**.

Mimetic Methods



Mimetic Methods work on **Polyhedral Meshes** while Standard Methods Fail

Mimetic Methods



Mesh generated by CUBIT

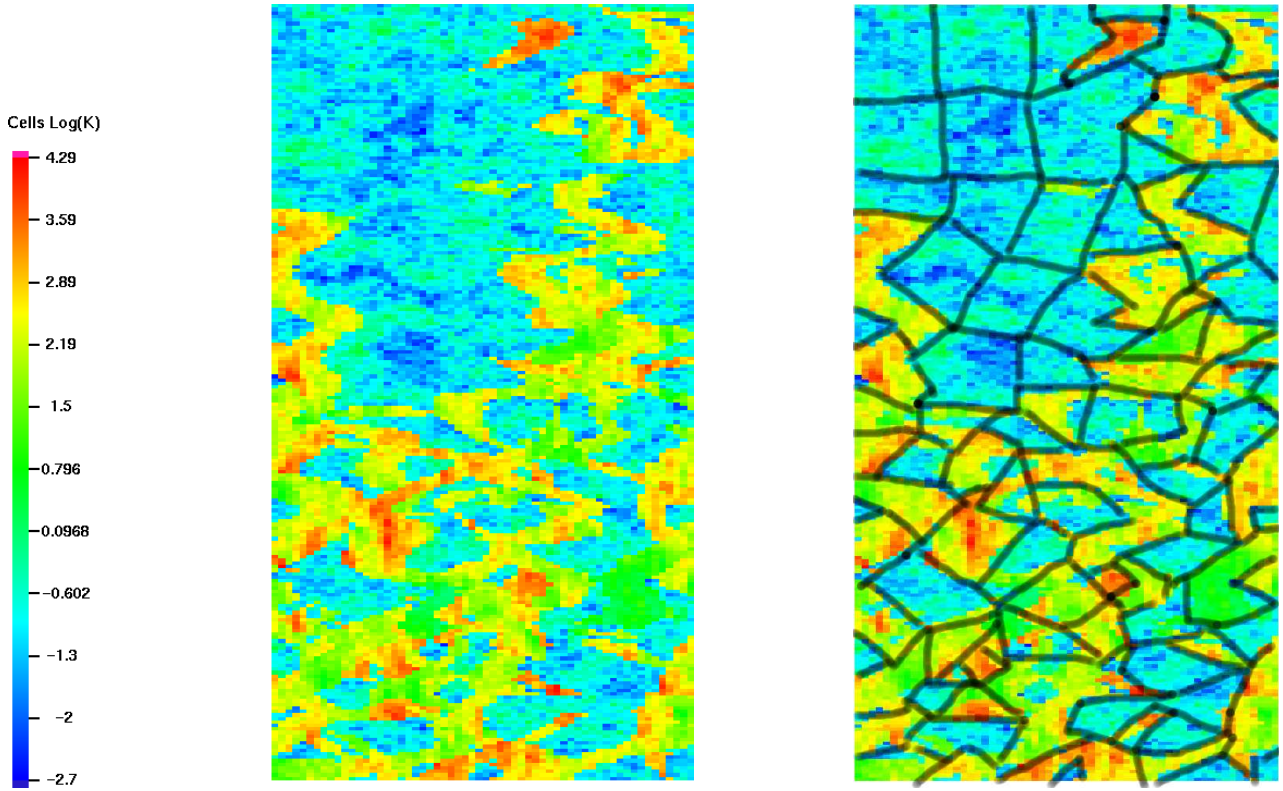
Model Diffusion Problem

% curved	$\|\|\boldsymbol{u}-\boldsymbol{u}^h\|\|_{L^2}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_\infty$
0.00	7.86e-4	1.79e-4
0.05	7.80e-4	1.62e-4
0.44	6.69e-4	1.54e-4
2.25	3.54e-4	1.59e-5

Mimetic Methods reduce Error in Velocity

10 times treating only **2% of Curved Faces**

Mimetic Methods



In Multi-scale Framework, **Mimetic Methods** allows to use **Arbitrary Coarse-Scale Meshes**.

Pressure equation

- second order stationary PDE:

$$\begin{aligned} -\mathrm{div}\,(\mathbb{K}\nabla p) &= b \quad \text{in} \quad \Omega \\ p &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

- mixed formulation:

$$\begin{aligned} \mathrm{div}\vec{u} &= b \\ \vec{u} &= -\mathbb{K}\nabla p \end{aligned}$$

- integrating 1st equation over mesh element E, we get

$$\left(\mathcal{DIV}\boldsymbol{u}^h\right)_E = b_E^h, \qquad \left(\mathcal{DIV}\,\boldsymbol{u}^h\right)_E = \frac{1}{|E|}\sum_{f\in\partial E} u_f^h\,|f|.$$

How to discretize the gradient?

Mimetic philosophy

Continuum Problem	Mimetic Method
$\begin{aligned} \mathrm{div}\vec{u} &= b \\ \vec{u} &= -\mathbb{K}\nabla p \end{aligned}$	$\begin{aligned} \mathcal{DIV}\boldsymbol{u}^h &= \boldsymbol{b}^h \\ \boldsymbol{u}^h &= -\mathcal{GRAD}\boldsymbol{p}^h \end{aligned}$
■ $\mathbb{K}\nabla = -\mathrm{div}^*$ ■ $\ker(\mathbb{K}\nabla) = constants$	■ $\mathcal{GRAD} = -\mathcal{DIV}^*$ ■ $\ker(\mathcal{GRAD}) = constants$

Mimetic finite difference method

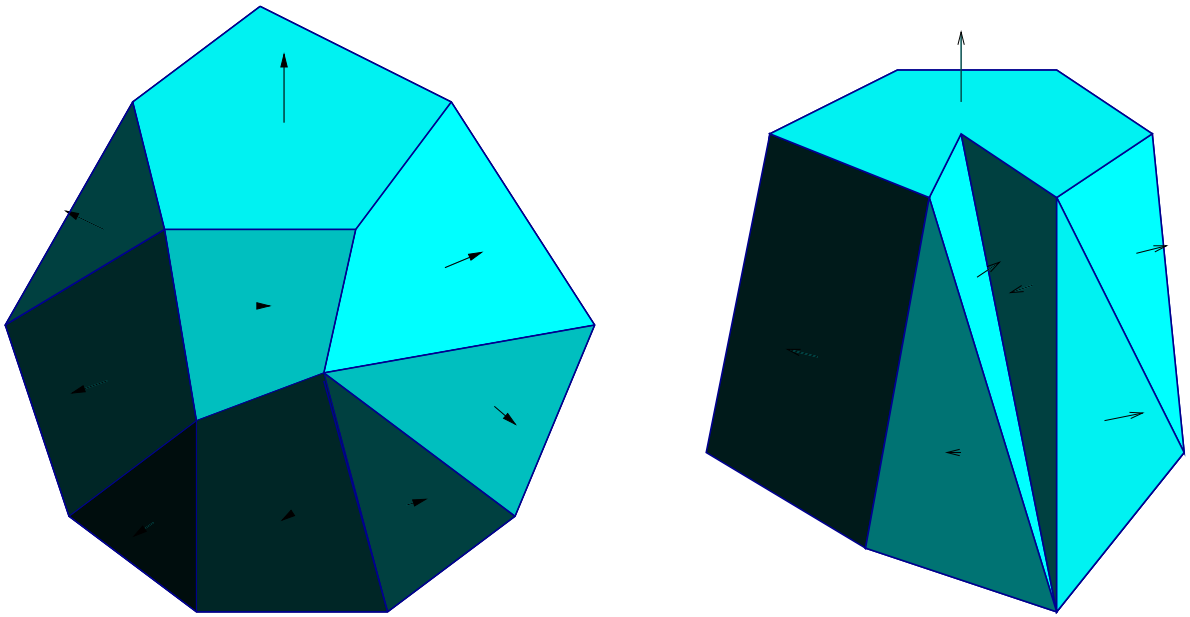
$$\mathbb{K}\nabla = -\mathrm{div}^* \qquad \ker(\mathbb{K}\nabla) = constants$$

Four-step methodology:

1. Define degrees of freedom for $\boldsymbol{p}^h \in Q_h$ and $\boldsymbol{u}^h \in X_h$
2. Discretize the divergence operator, $\mathcal{DIV}$
3. Equip discrete spaces with inner products $[\cdot\,,\,\cdot]_Q$ and $[\cdot\,,\,\cdot]_X$
4. Derive the gradient operator, $\mathcal{GRAD}$, from discrete Green's formula

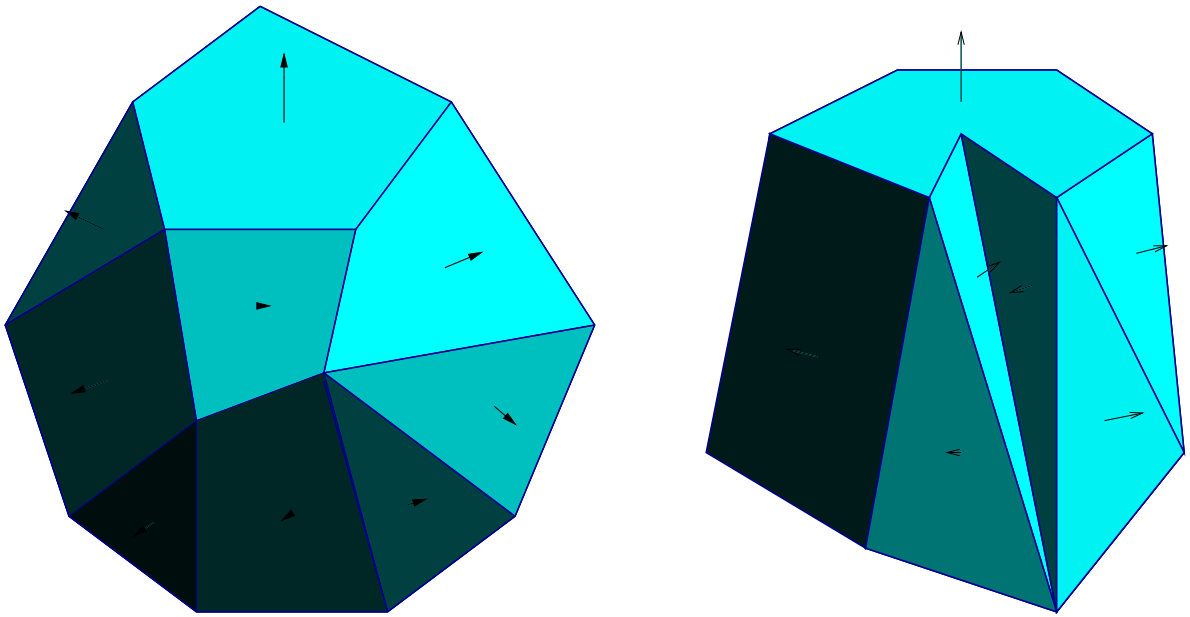
$$\mathcal{GRAD} = -\mathcal{DIV}^* \qquad \ker(\mathcal{GRAD}) = constants$$

Step 1: Degrees of freedom



- $\boldsymbol{p}^h$ is constant on element E
  - p_E^h be the degree of freedom associated with E

Step 1: Degrees of freedom

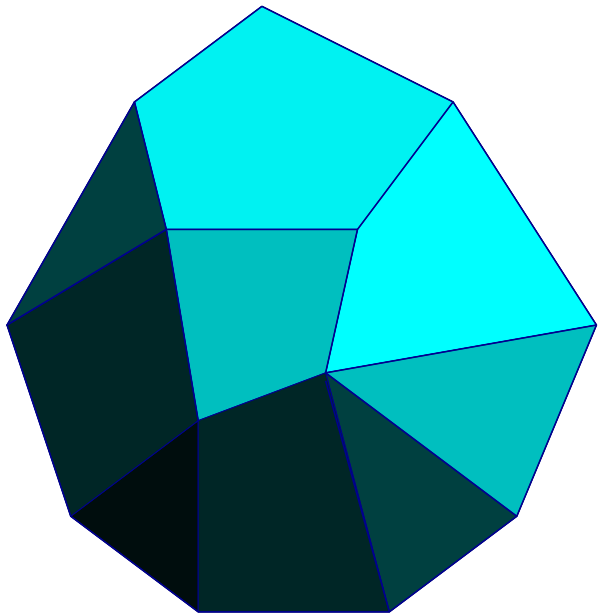


- $\boldsymbol{u}^h$ is constant on faces of element E
  - u_f^h be the flux associated with face f

Step 2: Primary operator

Divergence theorem:

$$\int_E \operatorname{div} \vec{u} = \oint_{\partial E} \vec{u} \cdot \vec{n}$$



implies

$$\left(\mathcal{DIV}\, \boldsymbol{u}^h\right)_E = \frac{1}{|E|} \sum_{f \in \partial E} u_f^h \, |f|$$

Step 3: Discrete Green formula

- $$[\boldsymbol{p}^h, \boldsymbol{q}^h]_Q = \sum_{E \in \Omega_h} p_E^h \, q_E^h \, |E| \approx \int_\Omega pq$$

Step 3: Discrete Green formula

- $$[\boldsymbol{p}^h, \boldsymbol{q}^h]_Q = \sum_{E \in \Omega_h} p_E^h \, q_E^h \, |E| \approx \int_\Omega pq$$

- $$[\boldsymbol{u}^h, \boldsymbol{v}^h]_X = \sum_{E \in \Omega_h} [\boldsymbol{u}^h, \boldsymbol{v}^h]_E \approx \sum_{E \in \Omega_h} \int_E (\mathbb{K}^{-1} \vec{u}) \cdot \vec{v}$$

Step 3: Discrete Green formula

- $$[\boldsymbol{p}^h,\, \boldsymbol{q}^h]_Q = \sum_{E\in\Omega_h} p_E^h\, q_E^h\, |E| \approx \int_\Omega pq$$

- $$[\boldsymbol{u}^h,\, \boldsymbol{v}^h]_X = \sum_{E\in\Omega_h} [\boldsymbol{u}^h,\, \boldsymbol{v}^h]_E \approx \sum_{E\in\Omega_h} \int_E (\mathbb{K}^{-1}\vec{u})\cdot\vec{v}$$

$$[\boldsymbol{u}^h,\, \boldsymbol{v}^h]_E = \sum_{i,j=1}^{k_E} \mathbb{M}_{E,i,j}\, u_{f_i}^h\, v_{f_j}^h$$

where $\mathbb{M}_E = \mathbb{M}_E{}^T > 0$ and k_E is the number of faces of E ($k_E = 6$ for a cube)

Step 4: Derived operator

- The Green formula

$$\int_\Omega \vec{u} \cdot \mathbb{K}^{-1}(\mathbb{K}\nabla)p = -\int_\Omega p \operatorname{div} \vec{u}$$

- The discrete Green formula

$$[\boldsymbol{u}^h, \mathcal{GRAD}\, \boldsymbol{p}^h]_X = -[\boldsymbol{p}^h, \mathcal{DIV}\, \boldsymbol{u}^h]_Q$$

defines the gradient operator.

Step 4: Derived operator

The discrete Green formula

$$[\boldsymbol{u}^h, \mathcal{GRAD}\, \boldsymbol{p}^h]_X = -[\boldsymbol{p}^h, \mathcal{DIV}\, \boldsymbol{u}^h]_Q$$

defines the gradient operator such that we always have

$$\mathcal{GRAD} = -\mathcal{DIV}^*$$

with respect to inner products in Q and X.

More about mimetic philosophy

- In electromagnetics we mimic:

$$\text{div curl} = 0, \qquad \text{curl grad} = 0$$

$$\int_\Omega \text{curl}\vec{E} \cdot \vec{H} = \int_\Omega \text{curl}\vec{H} \cdot \vec{E}$$

- In CFD we mimic:

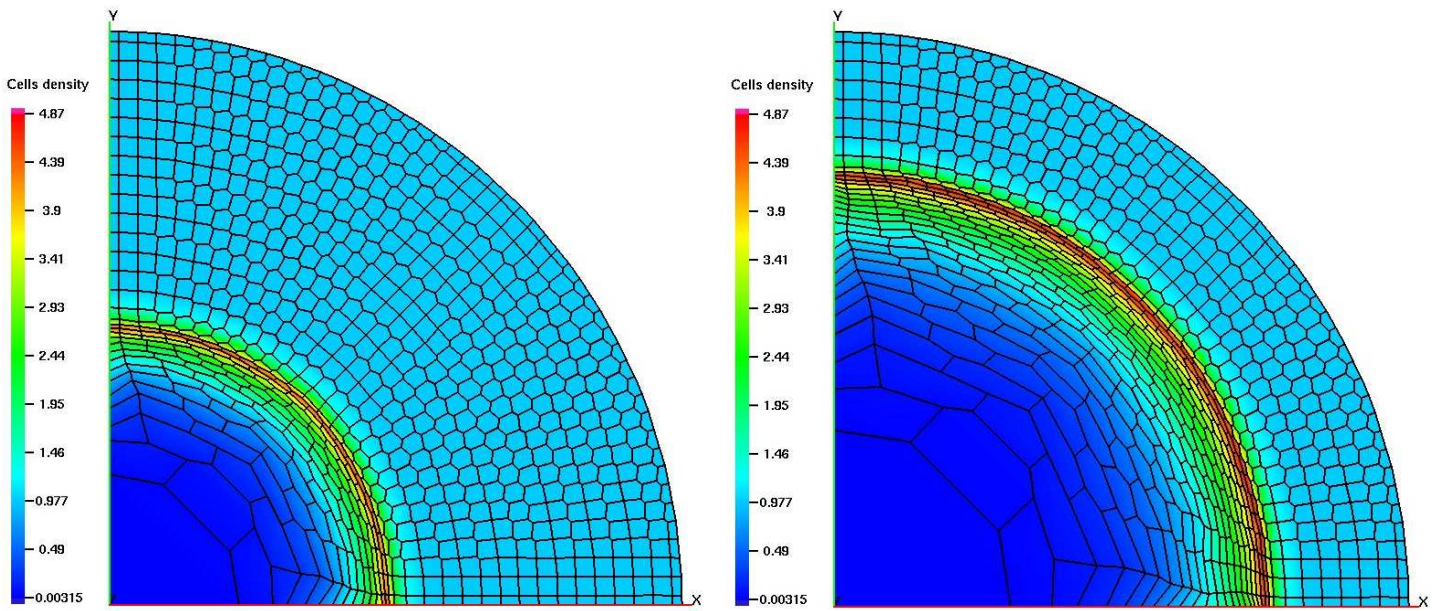
$$\int_\Omega \text{grad}\,\vec{u} : \underline{\underline{T}} = -\int_\Omega \text{div}\,\underline{\underline{T}} \cdot \vec{u}$$

More about mimetic philosophy

- Tensor viscosity methods:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p + \mathrm{div}(\mu \nabla \boldsymbol{u})$$

- The primary operator is gradient.



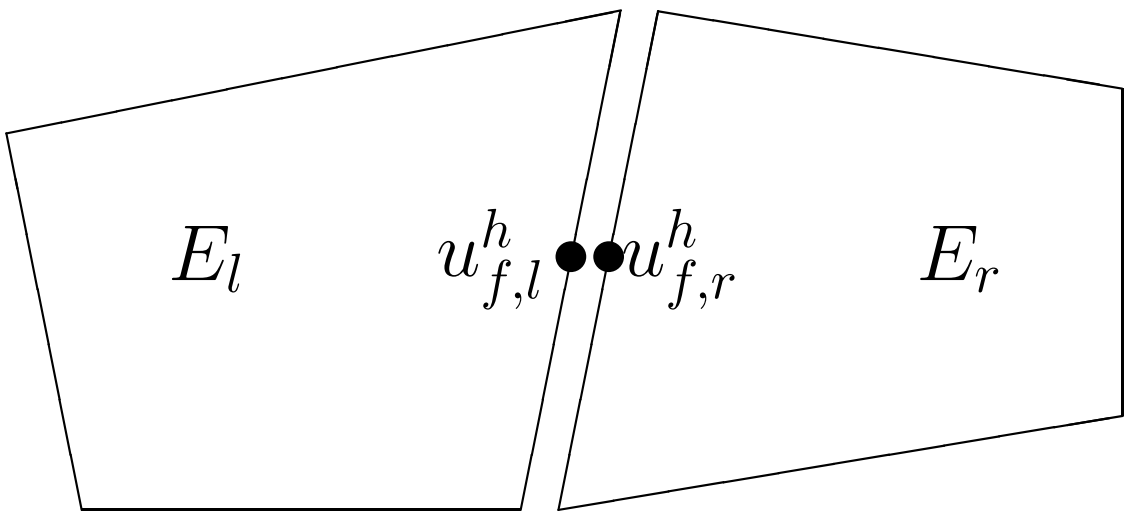
Solution method

The discrete problem is

$$\begin{aligned} \boldsymbol{u}^h &= -\mathcal{GRAD}\,\boldsymbol{p}^h \\ \mathcal{DIV}\boldsymbol{u}^h &= \boldsymbol{b}^h \end{aligned}$$

Solution method is to introduce copies of face-fluxes and obvious constraints:

$$u^h_{f,l} = -u^h_{f,r}$$



Solution method

Formulas for $\mathcal{GRAD}$, $\mathcal{DIV}$, and the KKT theory give

$$\begin{bmatrix} \mathbb{M}_X & \mathbb{B}^T & \mathbb{C}^T \\ \mathbb{B} & 0 & 0 \\ \mathbb{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{b} \\ 0 \end{bmatrix}$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers and

$$\begin{bmatrix} \mathbb{M}_X & \mathbb{B}^T \\ \mathbb{B} & 0 \end{bmatrix}$$

is block diagonal. Elimination of $\boldsymbol{u}$ and $\boldsymbol{p}$ gives

$$\mathbb{S}\boldsymbol{\lambda} = \boldsymbol{g}, \qquad \mathbb{S} = \mathbb{S}^T > 0.$$

Solution method

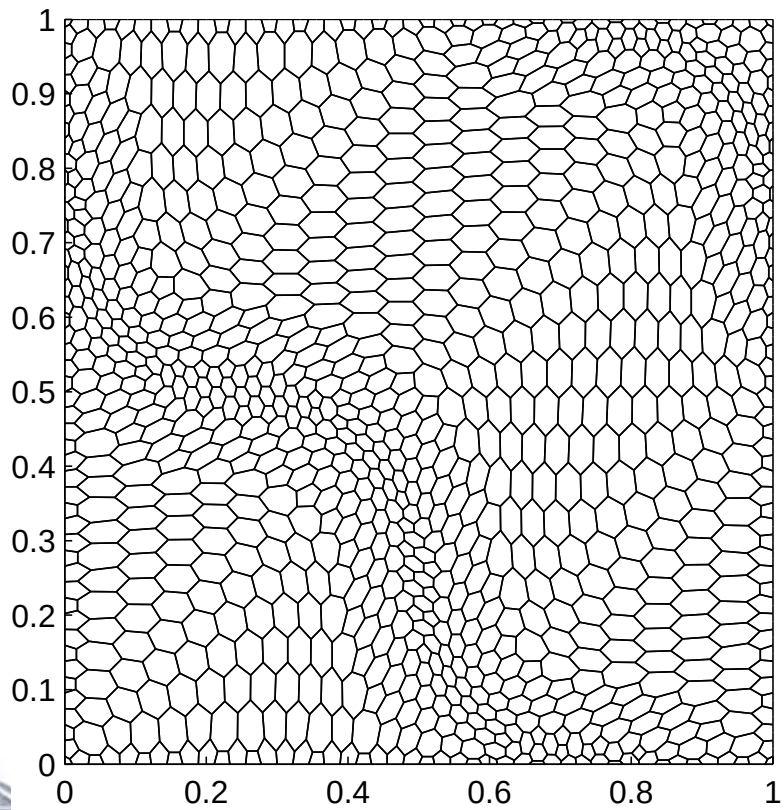
$$\mathbb{S}\boldsymbol{\lambda} = \boldsymbol{g}, \qquad \mathbb{S} = \mathbb{S}^T > 0$$

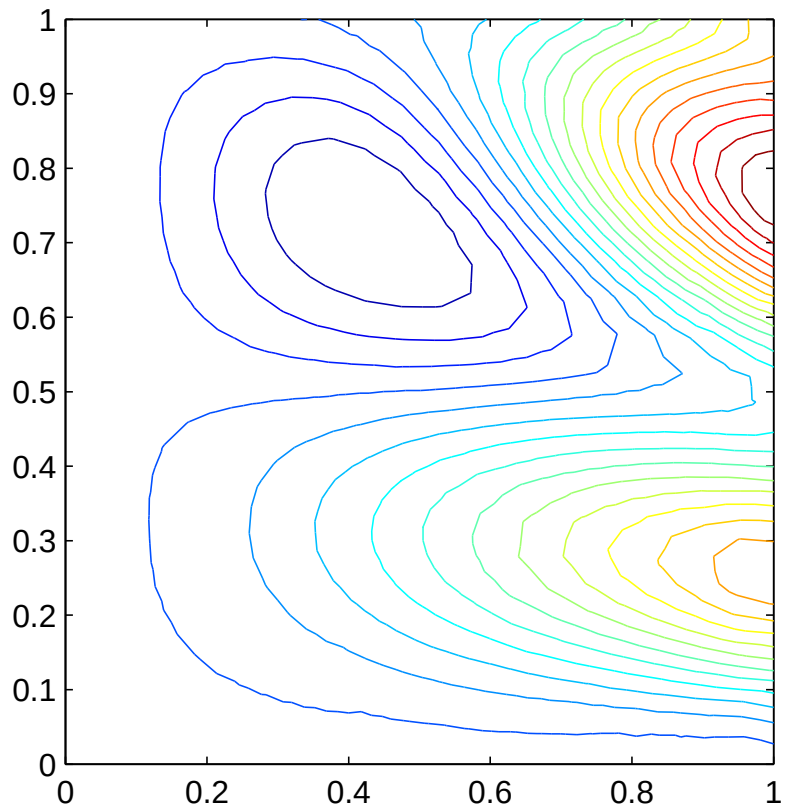
"Multigrid is ready for real problems!"

(D. Moulton, T-7)

Example 1: Polygonal meshes

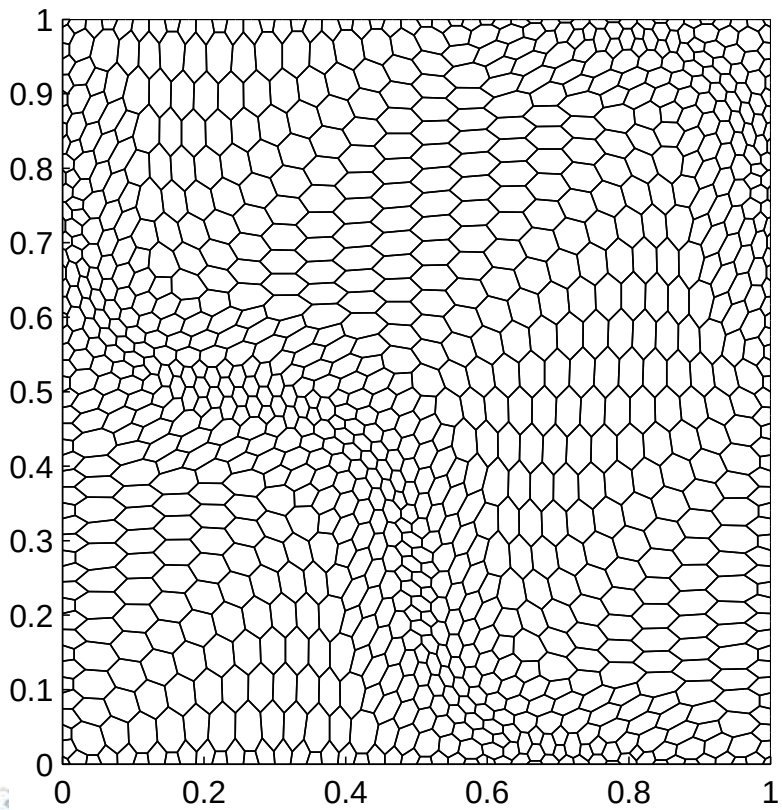
$$p(x,\, y) = x^3 y^2 + x \sin(2\pi xy) \sin(2\pi y), \qquad \mathbb{K} = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}$$

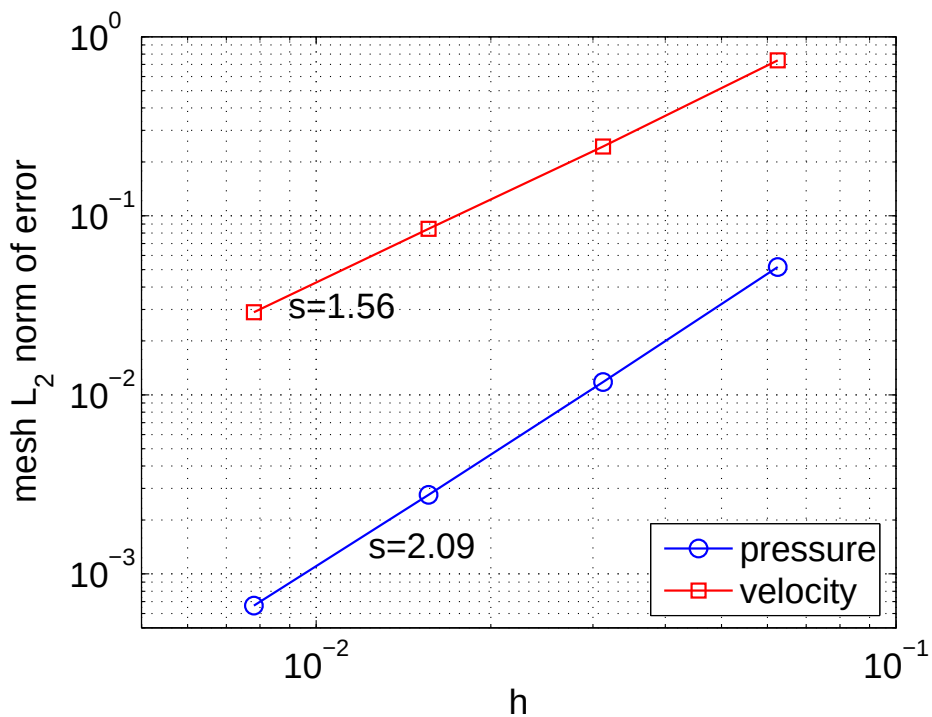




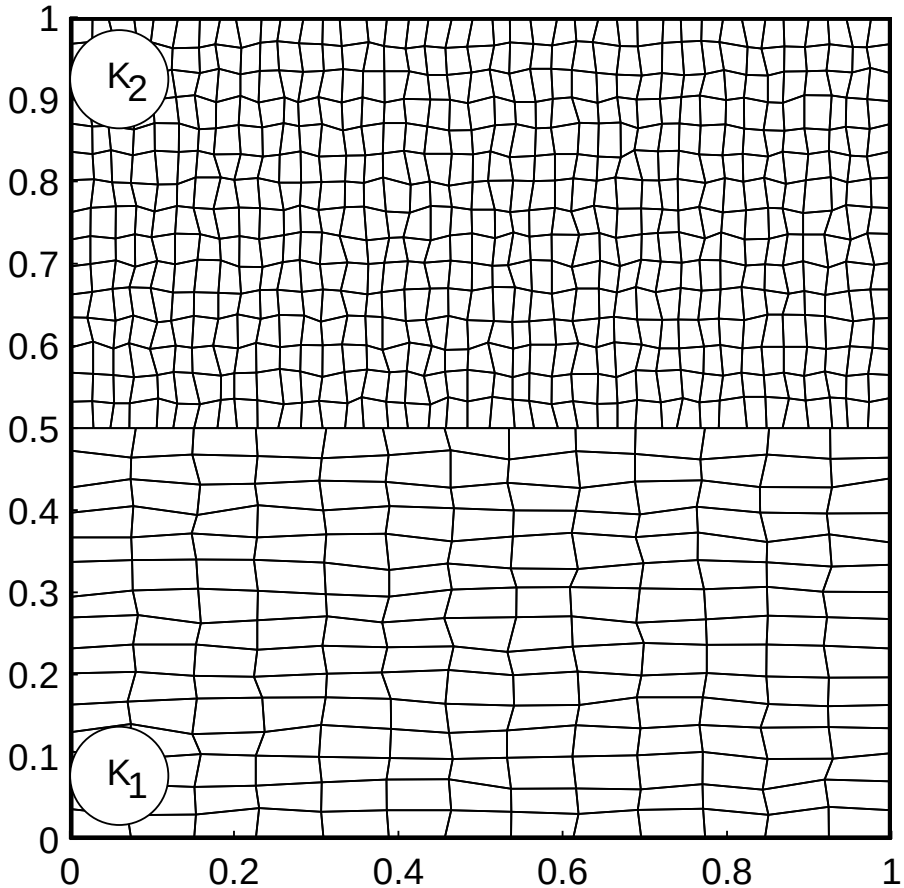
Example 1: Polygonal meshes

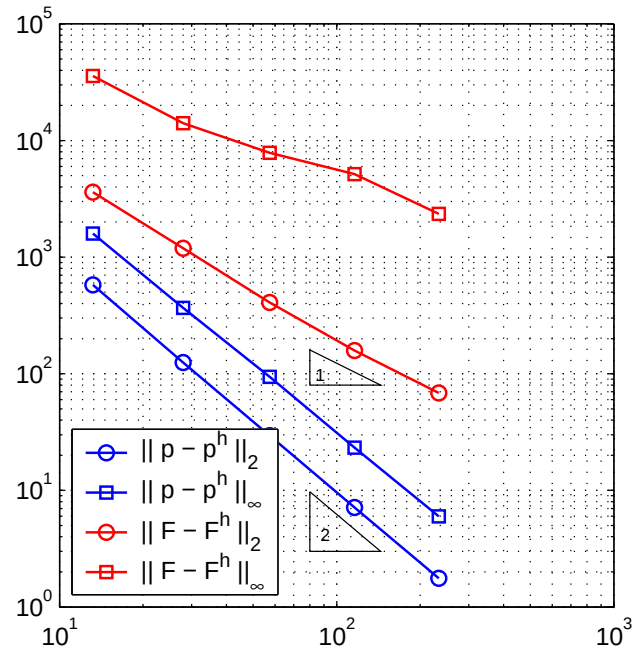
$$p(x,\, y) = x^3 y^2 + x \sin(2\pi x y) \sin(2\pi y), \qquad \mathbb{K} = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}$$





Example 2: Non-matching meshes





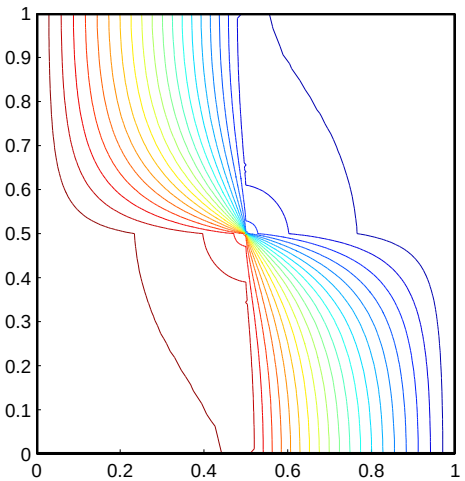
- $K_1 = 1, K_2 = 10^6$
- aspect ratio variations:

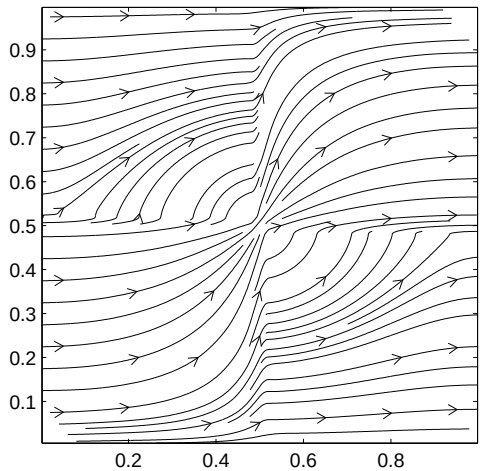
$$167 < \max_{cells} \frac{\max_k \ell^k}{\min_k \ell^k} < 2024.$$

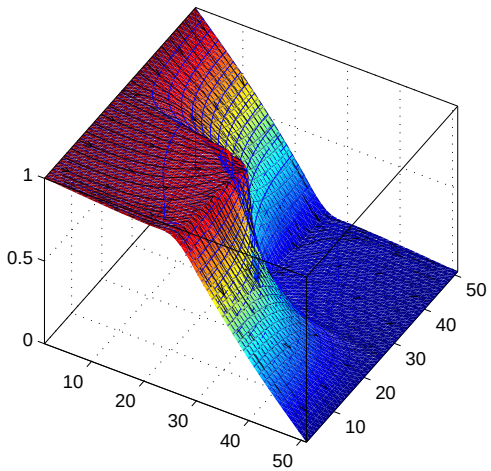
- exact solution is

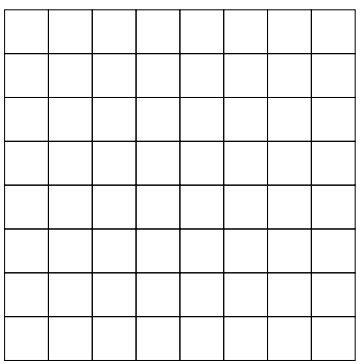
$$p(x,\, y) = \begin{cases} \dfrac{7}{16} - \dfrac{K_2}{12K_1} + \dfrac{2K_2}{3K_1}\, y^3, & y < 0.5, \\ y - y^4, & y \geq 0.5. \end{cases}$$

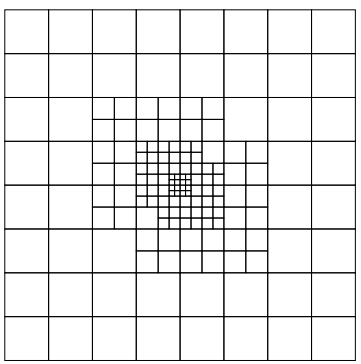
Example 3: AMR meshes

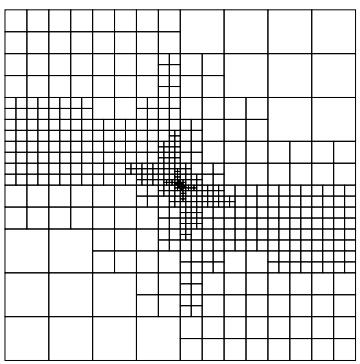


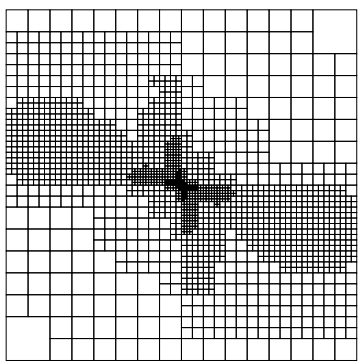












- we have degenerate polygons with 5, 6 and 7 edges.

Mimetic Methods

2-nd order accurate for pressure and **1-st** order accurate for velocity on **arbitrary meshes** and for problems with **discontinuous tensor** coefficients

Key element of the MFD method is the inner product

$$[\boldsymbol{u}^h, \boldsymbol{v}^h]_E \approx \int_E (\mathbb{K}^{-1}\vec{u}) \cdot \vec{v}$$

Mimic octopus

Key element of the MFD method is the inner product

$$[\boldsymbol{u}^h, \boldsymbol{v}^h]_E \approx \int_E (\mathbb{K}^{-1}\vec{u}) \cdot \vec{v}$$

Stability

There exist positive constants c and C such that

$$c \int_E |\vec{v}|^2 \leq \int_E (\mathbb{K}^{-1}\vec{v}) \cdot \vec{v} \leq C \int_E |\vec{v}|^2$$

We mimic it by requiring that

$$c|E|\boldsymbol{v}^T \boldsymbol{v} \leq \boldsymbol{v}^T \mathbb{M}_E \boldsymbol{v} \leq C|E|\boldsymbol{v}^T \boldsymbol{v} \qquad \forall \boldsymbol{v}$$

Consistency

Integration by parts formula is

$$\int_E \mathbb{K}_E^{-1}(\mathbb{K}_E \nabla p^1) \cdot \vec{v} = -\int_E p^1 \mathrm{div}\vec{v} + \sum_{f \in \partial E} \int_f p^1 (\vec{v} \cdot \vec{n}_f)$$

We mimic it by requiring that for any linear p^1:

$$(\boldsymbol{v}^h)^T \mathbb{M}_E \boldsymbol{u}^1 \equiv -(\mathcal{DIV}\boldsymbol{v}^h)_E \int_E p^1 + \sum_{f \in \partial E} v_f^h \int_f p^1$$

where $\boldsymbol{u}^1 = (\mathbb{K}_E \nabla p^1)^I$ and $\boldsymbol{v}^h$ is arbitrary.

Consistency

$$(\boldsymbol{v}^h)^T \, \mathbb{M}_E \, \boldsymbol{u}^1 \equiv -(\mathcal{DIV}\boldsymbol{v}^h)_E \int_E p^1 + \sum_{f \in \partial E} v_f^h \int_f p^1$$

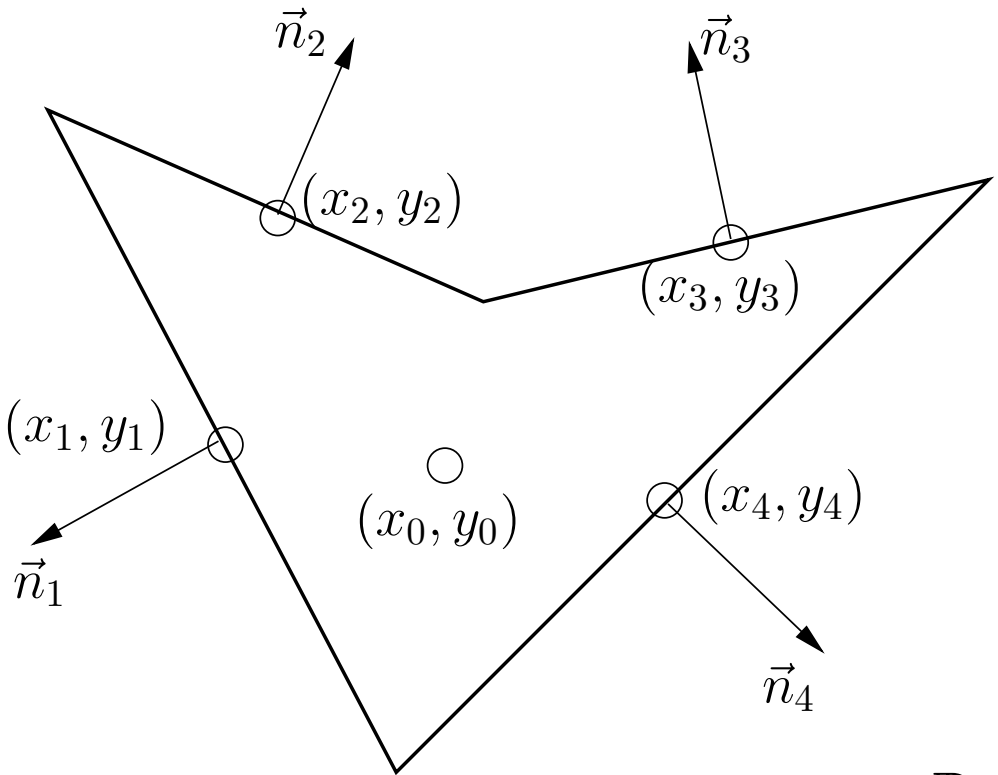
Since $\boldsymbol{v}^h$ is arbitrary, we get the **matrix equation**:

$$\mathbb{M}_E \, \boldsymbol{u}^1(p^1) = \boldsymbol{r}(p^1) \qquad \forall p^1,$$

or, equivalently, $\mathbb{M}_E \mathbb{N} = \mathbb{R}$, where

$$\mathbb{N} = [\boldsymbol{u}^1(x), \boldsymbol{u}^1(y)], \qquad \mathbb{R} = [\boldsymbol{r}(x), \boldsymbol{r}(y)].$$

Illustration of $\mathbb{N}$ and $\mathbb{R}$



$$\mathbb{N} = \begin{bmatrix} n_{1,x} & n_{1,y} \\ n_{2,x} & n_{2,y} \\ n_{3,x} & n_{3,y} \\ n_{4,x} & n_{4,y} \end{bmatrix} \mathbb{K}_E$$

$$\mathbb{R} = \begin{bmatrix} |\ell_1|(x_1 - x_0) & |\ell_1|(y_1 - y_0) \\ |\ell_2|(x_2 - x_0) & |\ell_2|(y_2 - y_0) \\ |\ell_3|(x_3 - x_0) & |\ell_3|(y_3 - y_0) \\ |\ell_4|(x_4 - x_0) & |\ell_4|(y_4 - y_0) \end{bmatrix}$$

Cute lemma

Lemma.

$$\mathbb{N}^T\,\mathbb{R} = \mathbb{R}^T\,\mathbb{N} = |E|\,\mathbb{K}_E.$$

Proof.

$$\begin{aligned}
|E|(\mathbb{K}_E)_{1,1} &= \int_E (\mathbb{K}_E \nabla x)\cdot\nabla(x-x_0) \\
&= \int_{\partial E} (\mathbb{K}_E \nabla x\cdot\vec{n})(x-x_0) = \sum_i (u^1(x))_i \int_{f_i} (x-x_0) \\
&= \sum_i (u^1(x))_i |\ell_i|(x_i - x_0) = \sum_i \mathbb{N}_{i,1}\mathbb{R}_{i,1} = (\mathbb{N}^T\,\mathbb{R})_{1,1}
\end{aligned}$$

Similarly,

$$|E|(\mathbb{K}_E)_{2,1} = \int_E \mathbb{K}_E\nabla y\cdot\nabla(x-x_0) = \int_{\partial E} (\mathbb{K}_E\nabla y\cdot\vec{n})(x-x_0) = (\mathbb{N}^T\,\mathbb{R})_{2,1}$$

Family of solutions

Theorem. Let columns of $\mathbb{C}$ span $\ker(\mathbb{N}^T)$, i.e.

$$\mathbb{N}^T\,\mathbb{C} = 0 \qquad \text{and} \qquad \mathbb{C}^T\,\mathbb{N} = 0.$$

Let $\mathbb{U}$ be an SPD matrix. Then,

$$\mathbb{M}_E = \frac{1}{|E|}\mathbb{R}\,\mathbb{K}_E^{-1}\,\mathbb{R}^T + \mathbb{C}\,\mathbb{U}\,\mathbb{C}^T$$

is the SPD matrix and solves

$$\mathbb{M}_E\mathbb{N} = \mathbb{R}.$$

KKT theory requires only $\mathbb{M}_E^{-1}$

Derivation of $\mathbb{M}_E^{-1}$

Theorem. Let columns of $\mathbb{D}$ span $\ker(\mathbb{R}^T)$, i.e.

$$\mathbb{R}^T\,\mathbb{D} = 0 \qquad \text{and} \qquad \mathbb{D}^T\,\mathbb{R} = 0.$$

Let $\tilde{\mathbb{U}}$ be an SPD matrix. Then,

$$\mathbb{M}_E^{-1} = \frac{1}{|E|}\mathbb{N}\,\mathbb{K}_E^{-1}\,\mathbb{N}^T + \mathbb{D}\,\tilde{\mathbb{U}}\,\mathbb{D}^T$$

is the SPD matrix and solves

$$\mathbb{M}_E^{-1}\mathbb{R} = \mathbb{N}.$$

Implementation issues

$$\mathbb{M}_E^{-1} = \frac{1}{|E|}\mathbb{N}\,\mathbb{K}_E^{-1}\,\mathbb{N}^T + \mathbb{D}\,\tilde{\mathbb{U}}\,\mathbb{D}^T$$

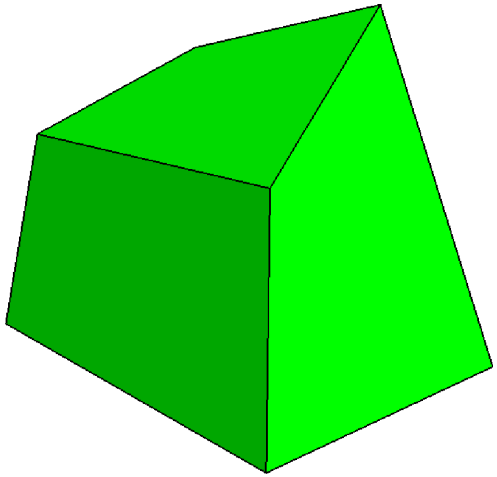
Let $\tilde{\mathbb{R}}$ obtained by orthonormalizing columns of $\mathbb{R}$,

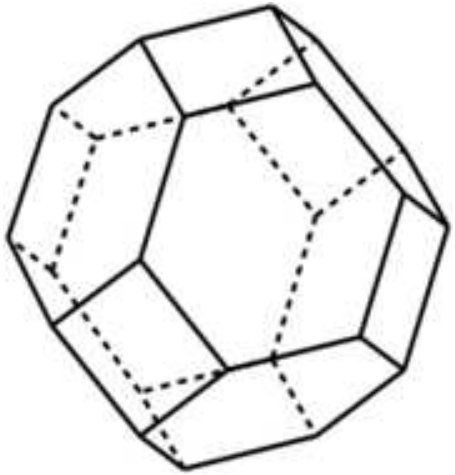
$$\tilde{\mathbb{U}} = \tilde{u}\,\mathbb{I}, \qquad \text{and} \qquad \tilde{u} = \frac{\mathrm{trace}(\mathbb{K}_E^{-1})}{|E|}.$$

Then,

$$\boxed{\mathbb{M}_E^{-1} = \frac{1}{|E|}\mathbb{N}\,\mathbb{K}_E^{-1}\,\mathbb{N}^T + \tilde{u}(\mathbb{I} - \tilde{\mathbb{R}}\tilde{\mathbb{R}}^T)}$$

- computing of $\mathbb{M}_E^{-1}$ requires $(2d+1)k_E^2 + 4d^2 k_E$ ops.



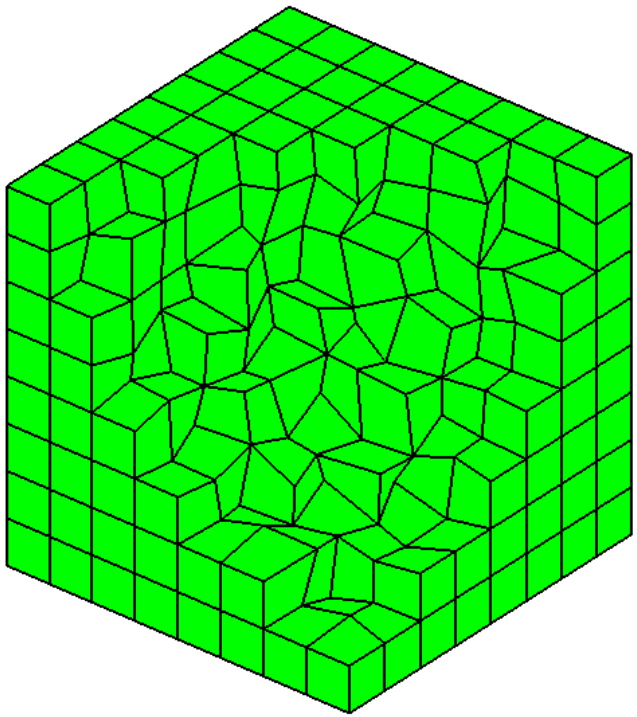


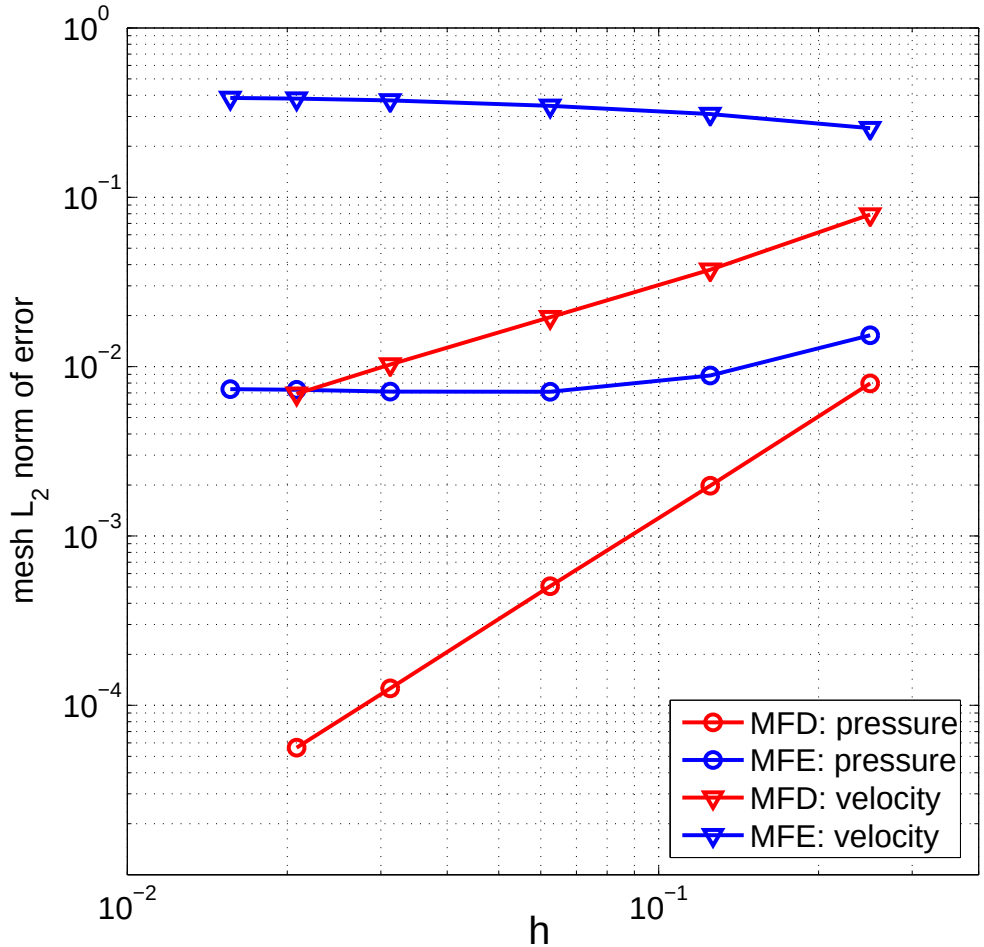
Mimetic Methods

work on

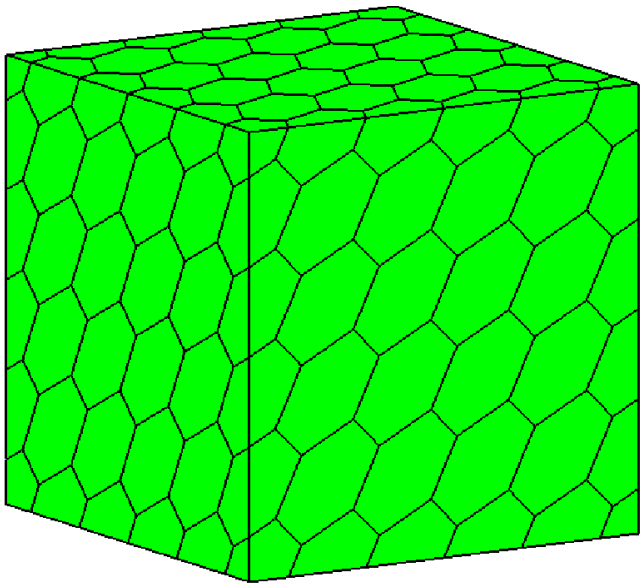
Generalized Polyhedral Meshes

Example 1: Standard method fail

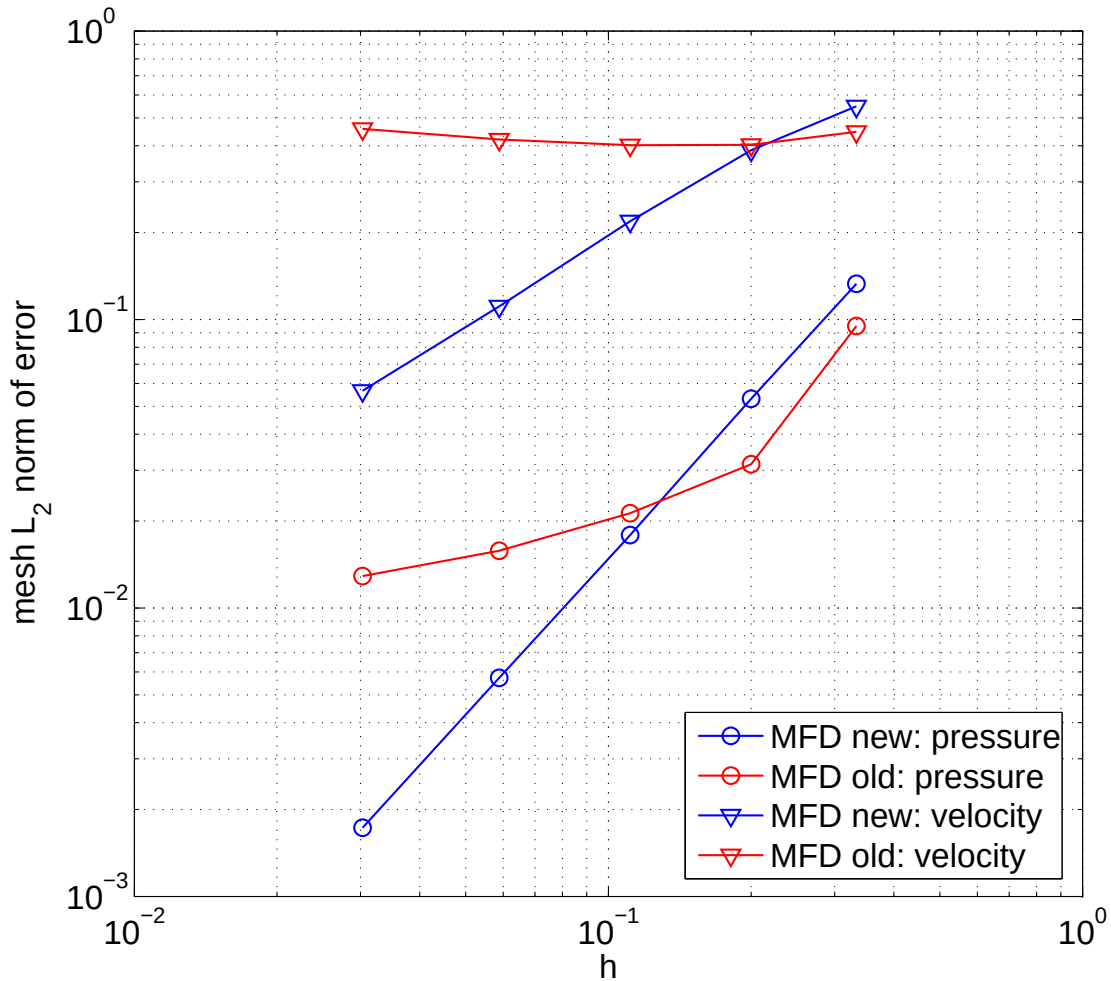




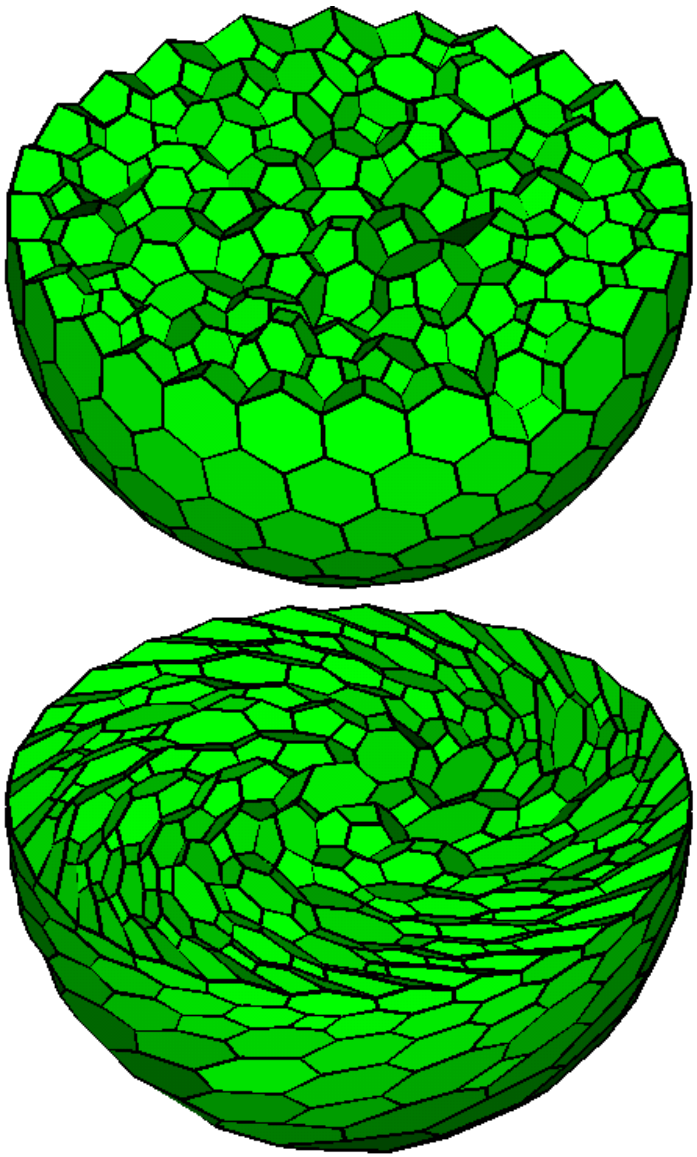
Example 2: Treating curved faces



- 68% of interior mesh faces are non-planar



Examples 3: Twisted meshes



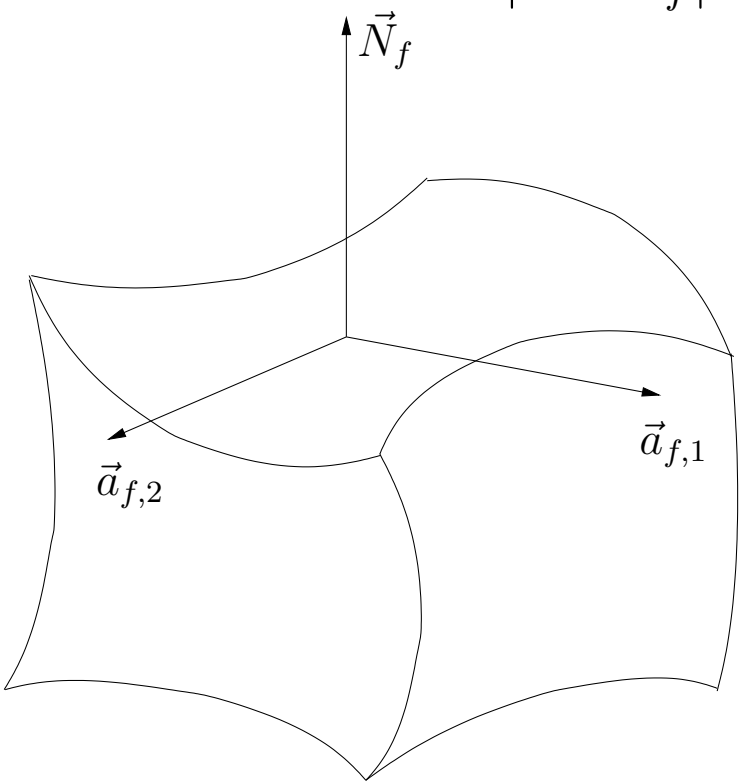
- 663 polyhedrons with upto 23 faces
- numerical results:

feature	top	bottom
CN (min)	1.002	1.009
CN (max)	2.500	115.0
CN (avg)	1.316	2.194
$\|\boldsymbol{p} - \boldsymbol{p}^h\|_Q$	8.39e-3	1.76e-2
$\|\boldsymbol{u} - \boldsymbol{u}^h\|_X$	9.20e-2	2.20e-1
$\|\boldsymbol{p} - \boldsymbol{p}^h\|_\infty$	1.68e-2	2.95e-2
$\|\boldsymbol{u} - \boldsymbol{u}^h\|_\infty$	2.43e-1	5.70e-1

Implementation issues

- the mesh face f is called *strongly curved* if

$$|\vec{n} - \vec{N}_f| > \sigma |f|^{1/2} \qquad \vec{N}_f = \frac{1}{|f|} \int_f \vec{n}.$$



- On curved faces, we use 3 d.o.f.:

$$\boldsymbol{u}_f^h \cdot \vec{N}_f = \frac{1}{|f|} \int_f \vec{u} \cdot \vec{n}$$

and

$$\boldsymbol{u}_f^h \cdot \vec{a}_{f,i} = \frac{1}{|f|} \int_f \vec{u} \cdot \vec{a}_{f,i} \quad i = 1, 2.$$

on moderately curved faces:	on strongly curved faces:
$\boldsymbol{u}_{E_1,f}^h \cdot \vec{N}_f = \boldsymbol{u}_{E_2,f}^h \cdot \vec{N}_f$	$\boldsymbol{u}_{E_1,f}^h = \boldsymbol{u}_{E_2,f}^h$

Implementation issues

- matrices $\mathbb{M}_E$ and $\mathbb{M}_E^{-1}$ are generated as before using extension of the cute lemma to curved faces.

- each moderately curved face introduce 2 interface velocity unknowns:

$$\mathbb{M}_E = \begin{bmatrix} M_E^{11} & M_E^{12} \\ M_E^{21} & M_E^{22} \end{bmatrix} \quad \text{(internal d.o.f.)}$$

- solver requires only $\left(M_E^{11} - M_E^{12}\left[M_E^{22}\right]^{-1} M_E^{21}\right)^{-1}$ which is the first diagonal block of $\mathbb{M}_E^{-1}$!

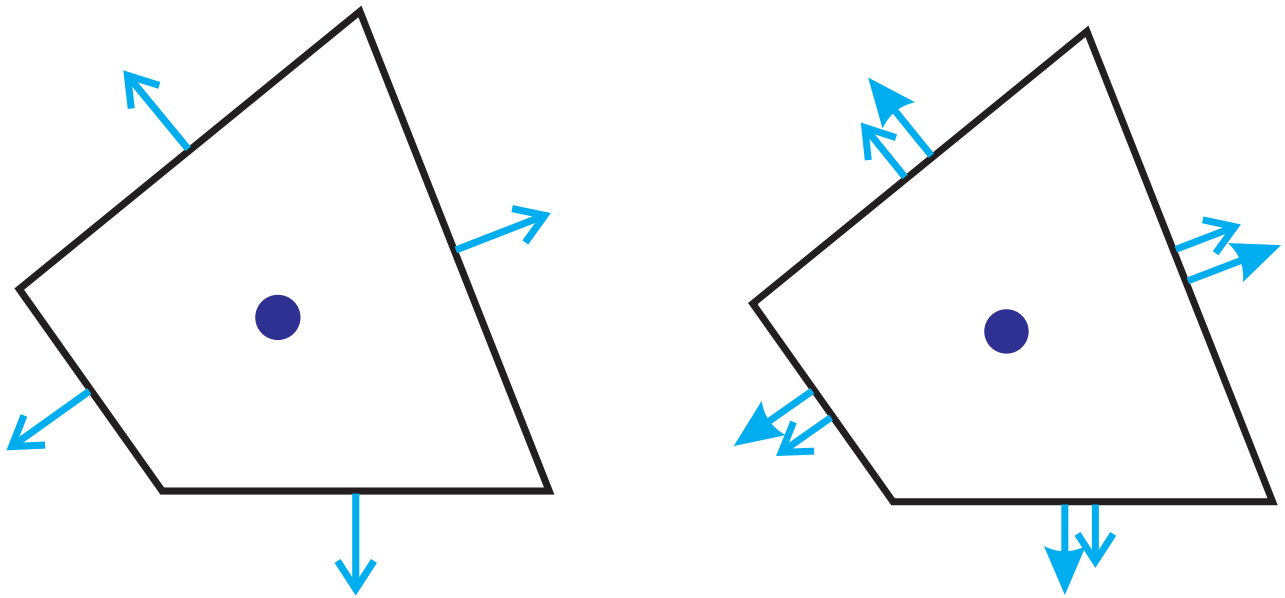
Concluding remarks

- Mimetic methods mimic important properties of the physical and mathematical models.
- MMs are much less sensitive to mesh non-smoothness than the standard methods.
- MMs work on polyhedral meshes while standard methods fails.
- MMs are inexpensive and easy to implement.
- Solid mathematical theory of MMs does exists and grows.

High-order Mimetic Methods

(with PhD student **Vitaliy Gyrya**, PennState)

Introducing velocity moments



$$\vec{u} \quad \dashrightarrow \quad \{\boldsymbol{u}_e^0,\ \boldsymbol{u}_e^1\} \approx \left\{ \int_e (\vec{u}\cdot\vec{n}_e)\, ds,\ \int_e s\,(\vec{u}\cdot\vec{n}_e)\, ds \right\}$$

Mimicking again...

Integration by parts formula is

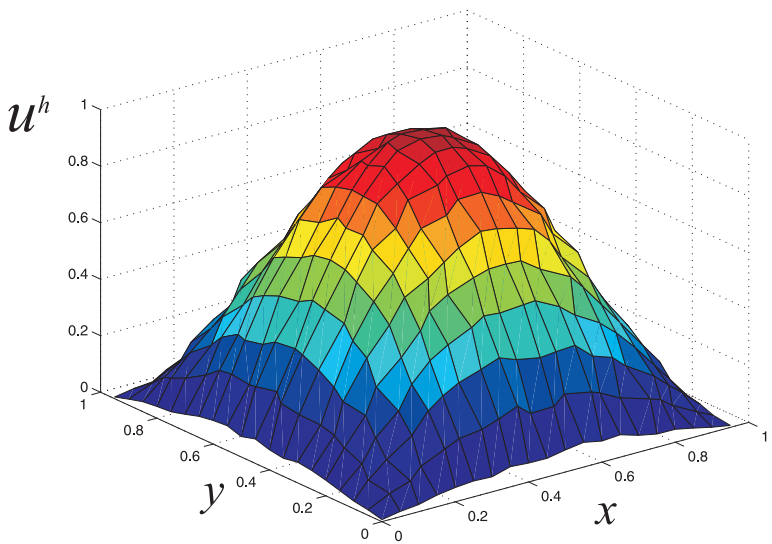
$$\int_E \nabla p^2 \cdot \vec{v} = -\int_E p^2 \mathrm{div}\vec{v} + \sum_{f \in \partial E} \int_f p^2 (\vec{v} \cdot \vec{n}_f)$$

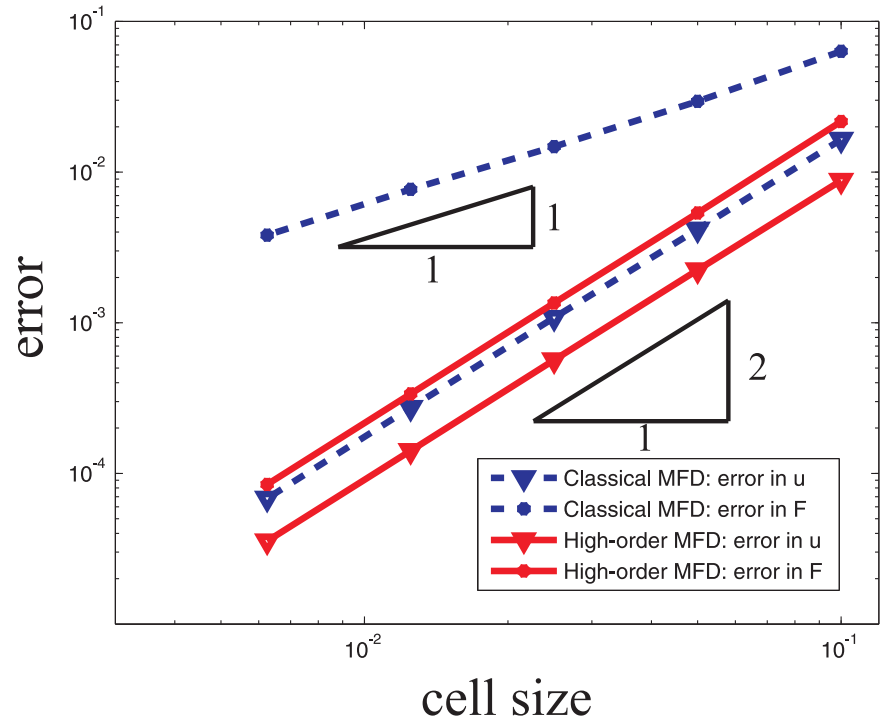
We mimic it by requiring that for any quadratic p^2:

$$(\boldsymbol{v}^h)^T \, \mathbb{M}_E \, \boldsymbol{u}^2 \equiv -(\mathcal{DIV}\boldsymbol{v}^h)_E \int_E p^2 + \sum_{f \in \partial E} \mathcal{Q}_f(p^2, \, v_f^h)$$

where $\boldsymbol{u}^2 = (\mathbb{K}_E \nabla p^2)^I$ and $\mathcal{Q}_f$ is a quadrature.

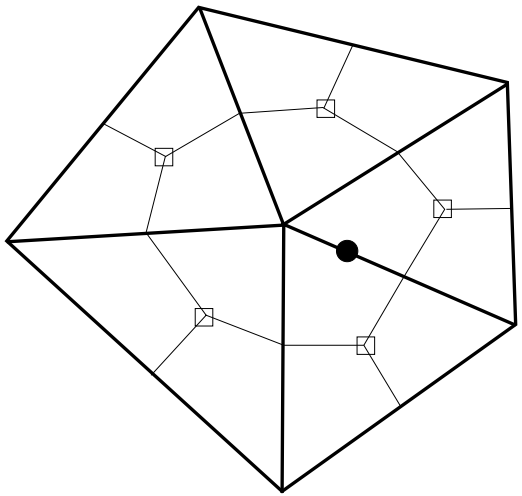
Illustrating example





Low-order MFD: $||\boldsymbol{p} - \boldsymbol{p}_h||_{L^2} \sim h^2, \qquad ||\boldsymbol{u} - \boldsymbol{u}_h||_{L^2} \sim h^1$

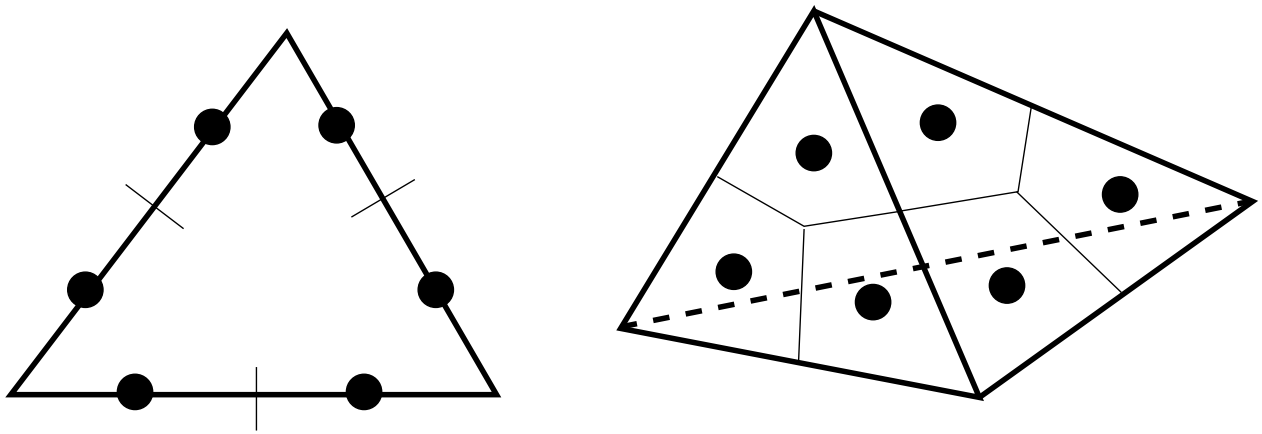
High-order MFD: $||\boldsymbol{p} - \boldsymbol{p}_h||_{L^2} \sim h^2, \qquad ||\boldsymbol{u} - \boldsymbol{u}_h||_{L^2} \sim h^2$



MPFA-type Mimetic Methods

(with **Ivan Yotov**, University of Pittsburgh)

Local flux formula



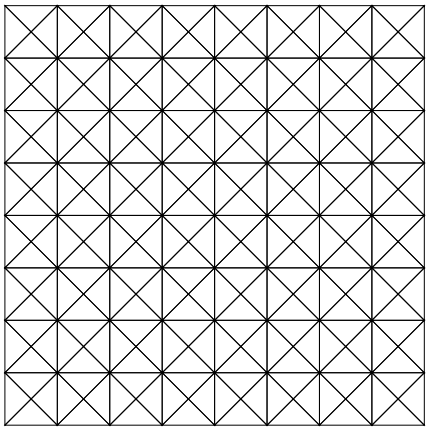
$$[\boldsymbol{u}^h,\, \boldsymbol{v}^h]_E = \sum_{c \in E} [\boldsymbol{u}^h,\, \boldsymbol{v}^h]_{E,c},$$

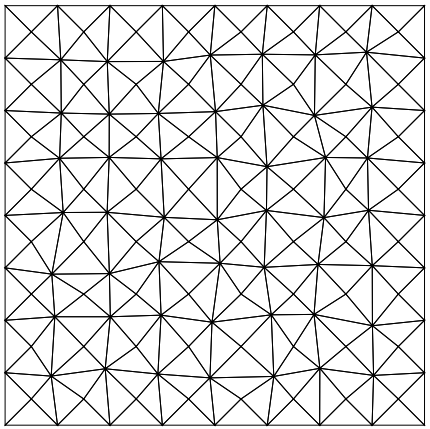
The Key Assumption is that
only velocities unknowns on facets of corner c are
involved in $[\cdot,\, \cdot]_{E,c}$

MPFA-MFD vs MPFA

- MPFA-MFD results in symmetric discretizations on simplicial meshes for full permeability tensors.
- MPFA-MFD produces a family of discretization methods.
- Theoretical analysis becomes possible with previously developed techniques.
- On general non-smooth and non-simplicial meshes both MPFA-MFD and MPFA result in non-symmetric schemes.

Illustrative example





$1/h$	$\|\|\boldsymbol{p}-\boldsymbol{p}^h\|\|_{L^2}$	$\|\|\boldsymbol{p}-\boldsymbol{p}^h\|\|_{\infty}$	$\|\|\boldsymbol{u}-\boldsymbol{u}^h\|\|_{L^2}$	$\|\|\boldsymbol{u}-\boldsymbol{u}^h\|\|_{\infty}$
8	1.14e-2	4.06e-2	2.94e-1	2.67e-0
16	2.93e-3	1.18e-2	1.24e-1	1.14e-0
32	7.13e-4	3.23e-3	5.97e-2	5.12e-1
64	1.77e-4	9.49e-4	3.01e-2	3.56e-1
128	4.48e-5	2.58e-4	1.52e-2	1.98e-1
Rate	2.00	1.82	1.06	0.92

Concluding remarks

- Mimetic methods mimic important properties of the physical and mathematical models.
- MMs are much less sensitive to mesh non-smoothness than the standard methods.
- MMs work on polyhedral meshes while standard methods fails.
- MMs are inexpensive and easy to implement.
- Solid mathematical theory of MMs does exist and grows.