PRINCIPLES OF MIMETIC DISCRETIZATIONS OF DIFFERENTIAL OPERATORS

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Abstract. Compatible discretizations transform partial differential equations to discrete algebraic problems that mimic fundamental properties of the continuum equations. We provide a common framework for mimetic discretizations using algebraic topology to guide our analysis. The framework and all attendant discrete structures are put together by using two basic mappings between differential forms and cochains. The key concept of the framework is a natural inner product on cochains which induces a combinatorial Hodge theory on the cochain complex. The framework supports mutually consistent operations of differentiation and integration, has a combinatorial Stokes theorem, and preserves the invariants of the De Rham cohomology groups. This allows, among other things, for an elementary calculation of the kernel of the discrete Laplacian. Our framework provides an abstraction that includes examples of compatible finite element, finite volume, and finite difference methods. We describe how these methods result from a choice of the reconstruction operator and explain when they are equivalent. We demonstrate how to apply the framework for compatible discretization for two scalar versions of the Hodge Laplacian.

Key words. Mimetic discretizations, compatible spatial discretizations, finite element methods, support operator methods, algebraic topology, De Rham complex, Hodge operator, Stokes theorem

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1. Introduction. Partial differential equations (PDEs) are ubiquitous in science and engineering. A key step in their numerical solution is the *discretization* that replaces the PDEs by a system of algebraic equations. Like any other model reduction, discretization is accompanied by losses of information about the original problem and its structure. One of the principal tasks in numerical analysis is to develop *compatible*, or *mimetic*, algebraic models that yield stable, accurate, and physically consistent approximate solutions. Historically, finite element (FE), finite volume (FV), and finite difference (FD) methods have achieved compatibility by following different paths that reflected their specific approaches to discretization.

Finite element methods begin by converting the PDEs into an equivalent variational equation and then restrict that equation to finite dimensional subspaces. Compatibility of the discrete problem is governed by variational inf-sup conditions, which imply existence of uniformly bounded discrete solution operators; see [6, 18, 46]. In finite volume methods the PDEs are first replaced by equivalent integral equations that express balance of global quantities valid on all subdomains of the problem domain. The algebraic equations are derived by sampling balance equations on a finite set of admissible subdomains (the finite volumes). Their compatibility is achieved by using the Stokes theorem to define the discrete differential operators [32, 42, 44, 57]. Finite difference methods approximate vector and scalar functions by discrete values on a grid and compatibility is realized by choosing the locations of these variables on the grid [28, 33, 34, 51, 60].

In spite of their differences, compatible FE, FV, and FD methods can result in discrete problems with remarkably similar properties. The observation that their compatibility is tantamount to having discrete structures that mimic vector calculus identities and theorems emerged independently and at about the same time in the FE, FV, and FD literature. For instance in [14, 15, 16, 37] Bossavit and Kotiuga demonstrated connections between stable finite elements for the Maxwell's equations and Whitney forms. In finite volume methods the idea of discrete field theory guided development of covolume methods [42, 43, 44], while support operator and mimetic methods [48, 50, 33, 34, 35, 36] combined the Stokes theorem with variational Green's

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identities to derive compatible finite differences. Algebraic topology was used to analyze mimetic discretizations by Hyman and Scovel in [31] and more recently by Mattiussi [39], Schwalm et al. [47] and Teixeira [53, 54]. Further research also revealed connections between some compatible methods. For instance, mimetic FD for the Poisson equation can be obtained from mixed FE by quadrature choice [12, 13, 19]. Another example is the equivalence between a covolume method and the classical Marker-and-Cell (MAC) scheme on uniform grids [43] and the analysis of [39] that relates finite volume and finite elements by using the concept of a "spread cell".

This research helped to evolve and clarify the notion of spatial compatibility to its present meaning of a discrete setting that provides mutually consistent operations for discrete integration and differentiation that obey the standard vector identities and theorems, such as the Stokes theorem. It also highlighted the role of differential forms and algebraic topology in the design and analysis of compatible discretizations. The recent work in [2, 8, 9, 10, 22, 29, 30, 39, 44, 47, 52, 53, 57] and the papers in this volume further affirm that these tools are gaining wider acceptance among mathematicians and engineers. For instance, FE methods that have traditionally relied upon nonconstructive variational [6, 18] stability criteria[1] now are being derived by topological approaches that reveal physically relevant degrees of freedom and their proper encoding. Of particular note are the papers by Arnold et al. [4, 2] which develop stable finite elements for mixed elasticity, and by Hiptmair [29], Demkowicz et al. [22] and Arnold et al. [3] which define canonical procedures for building piecewise polynomial differential complexes.

The key role played by differential forms and algebraic topology in compatible discretizations is not accidental. Exterior calculus provides powerful tools and concise formalism to encode the structure of many PDEs and to expose their local and global invariants. For instance, integration of differential forms is an abstraction of the measurement process, while the Stokes theorem connects differentiation and integration to reveal global equilibrium relations. Algebraic topology, on the other hand, supplies structures that mimic exterior calculus on finite grids and so is a natural discretization tool for differential forms. The application of algebraic topology in modeling dates back to 1923 when H. Weyl [58] used it to describe electrical networks. Other early works of note are Branin [17] and in particular Dodziuk [24] whose combinatorial Hodge theory has great similarity with mixed FE on simplices. However, these papers contained few applications to numerical analysis. The first deliberate application of algebraic topology to solve PDEs numerically is due to Hyman and Scovel [31] who, drawing upon some of the ideas in [24], used it to develop mimetic finite difference methods.

The present paper extends the approach originated in [31] to create a general framework for compatible discretizations that includes FE, FV, and FD methods as special cases. We first translate scalar and vector functions to their differential form equivalents and consider the computational grid to be an algebraic topological complex. The grid consists of 0-cells (nodes), 1-cells (edges), 2-cells (faces), and 3-cells (volumes) which combine to form k-chains; $k = 0, 1, 2, 3$. For simplicity we focus on simplicial grids; however, most of the developments easily carry over to general polyhedral domain partitions.

All necessary discrete structures in our framework are put together by two basic operations: a reduction map $\mathcal{R}$ and a reconstruction map $\mathcal{I}$, such that $\mathcal{I}$ is a right inverse of $\mathcal{R}$. We take $\mathcal{R}$ to be the De Rham map that reduces differential forms to linear functionals on chains, i.e., cochains. Therefore, discrete k-forms are encoded as k-cell quantities. For differential forms, the operators Div, Grad and Curl are generated by the exterior derivative d. Stokes theorem states that d is dual to the boundary operator ∂ with respect to

[1] One exception in FEM was the Grid Decomposition Property (GDP), formulated by Fix et al. [26], that gives a topological rather than variational stability condition for mixed discretizations of the Kelvin principle derived from the Hodge decomposition. The GDP is essentially equivalent to an inf-sup condition; see Bochev and Gunzburger [7].

the pairing between forms and chains. To define the discrete operators we mimic this property and use the duality between chains and cochains. Thus, the discrete Div, Grad and Curl are generated by the coboundary δ which is dual to ∂ with respect to this pairing.

The reconstruction map $\mathcal{I}$ translates cochains back to differential forms and induces the *natural* inner product that is central to our approach. This product gives rise to a *derived* adjoint δ^*, a discrete Laplacian $-\triangle = \delta\delta^* + \delta^*\delta$ and hence a combinatorial Hodge theory [25, 24]. By applying a discrete version of Hodge's theorem and De Rham's theorem, we can compute the size of the kernel of this Laplacian in an elementary way.

The global (combinatorial) and the local (metric) properties of the discrete models are determined by $\mathcal{R}$ and $\mathcal{I}$, respectively. The discrete derivative, induced by $\mathcal{R}$, is purely combinatorial and invariant under homeomorphisms. The adjoint δ^* is induced by the inner product and depends on the choice of $\mathcal{I}$.

The present work, based on mappings between differential forms and cochains, differs from other approaches that use differential forms and algebraic topology to provide common frameworks for compatible discretizations. Most notably, we make the inner product on cochains the key concept of our approach because it is sufficient to generate a combinatorial Hodge theory. As a result, distinctions between compatible FE, FV, and FD methods arise from the choice of $\mathcal{I}$ and so equivalence of different models can be established by comparing their reconstruction operators. In contrast, the primary concept in [30, 52, 54] is the discrete $\star$ operator. Different models are distinguished by their choice of the discrete $\star$ and its construction is the central problem.

As an aside, we point out that developments in the FE literature focus primarily on approximation of differential forms by piecewise polynomials of arbitrary degree [1, 3, 22] and less on the equivalence between the discrete models. Except in the lowest-order case, such spaces include degrees of freedom that are not cochains and result in differential operators that are not purely combinatorial. The main advantage of cochain encoding used in this work is seen in the possibility to maintain a clear distinction between the global and the local features in the discrete model. High-order formulations on cochains are also possible by using an appropriate reconstruction operator [32, 57]. Generally, reconstruction stencils for $\mathcal{I}$ grow, which is seen as the principal drawback of this approach. However, the number of degrees of freedom does not increase.

2. Differential forms. We review the basic concepts necessary for the numerical framework. Given an n-dimensional vector space E and an integer $0 \leq k \leq n$, we denote by Λ^k the vector space of algebraic k-forms, that is, all k-linear, antisymmetric maps[2] $\omega_k : E \times \ldots \times E \mapsto \mathbb{R}$; see [5]. The subscript k in ω_k will be used only when necessary to distinguish between different forms. Dimension of Λ^k is C^n_k and the unique element ω_n of Λ^n is a volume form. We recall the wedge product $\wedge : \Lambda^k \times \Lambda^l \mapsto \Lambda^{k+l}$ for $k + l \leq n$ with the property that $\omega_k \wedge \omega_l = (-1)^{kl}\omega_l \wedge \omega_k$. An inner product $(\cdot,\cdot)$ on $E \times E$ induces an inner product $(\cdot,\cdot)$ on $\Lambda^k \times \Lambda^k$. The latter gives rise to a unique metric conjugation operator $\star : \Lambda^k \mapsto \Lambda^{n-k}$, defined by the relation [23, 27]

$$\omega \wedge \star\xi = (\omega, \xi)\omega_n \quad \forall \omega, \xi \in \Lambda^k \, . \tag{2.1}$$

Let $T\Omega$ denote the tangent bundle of a differentiable manifold Ω. A differential k-form on Ω is a map $\Omega \ni x \mapsto \omega(x) \in \Lambda^k(T_x\Omega)$, where $T_x\Omega$ is the tangent space at x. In what follows the set of all smooth k-forms on Ω is denoted by $\Lambda^k(\Omega)$. The exterior derivative $d : \Lambda^k \mapsto \Lambda^{k+1}$; $k = 0, 1, \ldots, n-1$ satisfies

$$d(\omega_k \wedge \omega_l) = (d\omega_k) \wedge \omega_l + (-1)^k \omega_k \wedge (d\omega_l)\, ; k + l < n \tag{2.2}$$

[2] Equivalently, Λ^k can be defined as the dual of Λ_k - the space of all k-vectors; see [23, 27].

and $dd = 0$ and therefore gives rise to an exact sequence

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \xmapsto{d} \Lambda^1(\Omega) \xmapsto{d} \Lambda^2(\Omega) \xmapsto{d} \Lambda^3(\Omega) \longmapsto 0 \tag{2.3}$$

called De Rham complex.

Integration operation for differential k-forms can be defined on k-dimensional manifolds without any reference to a metric structure [5, 23]. The Stokes theorem

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega\,, \tag{2.4}$$

expresses the classical Newton-Leibnitz, Gauss divergence, and Stokes circulation theorems. As a corollary to this theorem and (2.2), we have, for $k + l + 1 = n$, the integration by parts formula

$$\int_{\partial\Omega} \omega_k \wedge \omega_l = \int_{\Omega} (d\omega_k) \wedge \omega_l + (-1)^k \int_{\Omega} \omega_k \wedge (d\omega_l)\,. \tag{2.5}$$

On a Riemannian manifold Ω the metric tensor g_{ij} induces Euclidean structure on $T_x\Omega$ and inner product $(\cdot,\cdot)$ on $\Lambda^k(T_x\Omega)$. The latter brings about an L^2 inner product on $\Lambda^k(\Omega)$ defined by

$$(\omega, \xi)_\Omega = \int_\Omega (\omega, \xi)\omega_n\,. \tag{2.6}$$

In view of (2.1), an equivalent definition is

$$(\omega, \xi)_\Omega = \int_\Omega \omega \wedge \star\xi\,. \tag{2.7}$$

The Hilbert spaces obtained by completion of smooth k-forms in the metric induced by (2.6) will be denoted by $\Lambda^k(L^2, \Omega)$.

It is also profitable to introduce the Sobolev spaces [3]

$$\Lambda^k(d, \Omega) = \{\omega \in \Lambda^k(L^2, \Omega) \;|\; d\omega \in \Lambda^{k+1}(L^2, \Omega)\}\,,$$

of square integrable k-forms whose exterior derivative is also square integrable.

The inner product (2.6) gives rise to an adjoint operator $d^* : \Lambda^k(\Omega) \mapsto \Lambda^{k-1}(\Omega)$. Assuming that Ω is the whole manifold, or that one of the forms has compact support, the adjoint is defined by

$$(d\omega, \xi)_\Omega = (\omega, d^*\xi)_\Omega \quad \text{for all } \omega \in \Lambda^{k-1}(\Omega)\,,\ \xi \in \Lambda^k(\Omega)\,.$$

The adjoint gives rise to the Hodge Laplacian $-\triangle_k = dd^* + d^*d$, which is a mapping $\Lambda^k(\Omega) \mapsto \Lambda^k(\Omega)$.

We assume that the boundary $\partial\Omega$ of domain Ω for the PDEs consists of two disjointed, smooth, possibly empty boundary components Γ_1 and Γ_2. At any boundary point a form can be decomposed into its tangential and normal components, $\omega = \omega_t + \omega_n$. If η is the inward pointing unit covector, then $\omega_n = g \wedge \eta$ where $\star g = \star\omega \wedge \eta$. The Green's formula

$$(d\omega, \xi)_\Omega - (\omega, d^*\xi)_\Omega = \int_{\partial\Omega} \omega \wedge \star\xi = \int_{\partial\Omega} \omega_t \wedge \star\xi_n \tag{2.8}$$

follows from (2.4) and (2.5).

Let $\Lambda_0^k(\Omega)$ be the smooth k-forms ω such that

$$\omega_t = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \omega_n = 0 \text{ on } \Gamma_2 \,. \tag{2.9}$$

The boundary conditions imposed on $\Lambda_0^k(\Omega)$ imply that $d^* = (-1)^k \star d\star$. Thus, the adjoint has the property that $d^*d^* = 0$. If the metric is the standard Euclidean metric, then the effect of d^* on scalar and vector functions is the same as that of d.

Using (2.8) we see that for $\omega, \xi \in \Lambda_0^k(\Omega)$

$$(-\triangle_k \omega, \xi)_\Omega = (d\omega, d\xi)_\Omega + (d^*\omega, d^*\xi)_\Omega \,.$$

The right-hand side in the above formula is the Dirichlet integral.

The relation between forms and vector and scalar functions in $\mathbb{R}^3$ is determined as follows. Let $\{x_i\}_{i=1}^3$ and $\{dx_i\}_{i=1}^3$ denote the local coordinates and their conjugates, respectively, that is, $dx_i(x_j) = \delta_{ij}$. A 0-form is dual to zero-dimensional manifolds (points) and so it is a scalar function. A 3-form is dual to three-dimensional manifolds (volumes) and so it has the form

$$\omega = \phi(\mathbf{x}) dx_1 \wedge dx_2 \wedge dx_3 \,.$$

This defines a relation $\omega \leftrightarrow \phi$ where ϕ is a scalar function. Therefore, 0- and 3-forms can be identified with scalar functions. A 1-form is dual to one-dimensional manifolds and can be written as

$$\omega = \mathbf{u}_1(\mathbf{x}) dx_1 + \mathbf{u}_2(\mathbf{x}) dx_2 + \mathbf{u}_3(\mathbf{x}) dx_3 \,,$$

while a 2-form is dual to two-dimensional manifolds and can be written as

$$\omega = \mathbf{u}_1(\mathbf{x}) dx_2 \wedge dx_3 + \mathbf{u}_2(\mathbf{x}) dx_3 \wedge dx_1 + \mathbf{u}_3(\mathbf{x}) dx_1 \wedge dx_2 \,.$$

This defines a relation $\omega \leftrightarrow \mathbf{u}$, between 1- and 2-forms and vector fields in $\mathbb{R}^3$.

To emphasize correspondences between forms and fields, sometimes we will write $\omega^{\mathbf{u}}$ or ω^ϕ so that

$$d\omega_0^\phi = \omega_1^{\nabla\phi}\,; \quad d\omega_1^{\mathbf{u}} = \omega_2^{\nabla\times\mathbf{u}}\,; \quad \text{and} \quad d\omega_2^{\mathbf{u}} = \omega_3^{\nabla\cdot\mathbf{u}}\,. \tag{2.10}$$

That is, exterior derivative of a $0-, 1-, 2-$form is equivalent to application of Grad, Curl, or Div, respectively, to the corresponding scalar or vector field.

Furthermore, if $\omega^{\mathbf{u}}$ and $\xi^{\mathbf{v}}$ are two 1-forms, then the wedge product $\omega^{\mathbf{u}} \wedge \xi^{\mathbf{v}}$ is a 2-form with corresponding vector function $\mathbf{u} \times \mathbf{v}$. If $\eta^{\mathbf{w}}$ is a 2-form, then the wedge $\omega^{\mathbf{u}} \wedge \eta^{\mathbf{w}}$ is a 3-form with scalar function $\mathbf{u} \cdot \mathbf{w}$.

For the Hilbert spaces $\Lambda^k(d,\Omega)$ boundary conditions are imposed for $k = 0, 1, 2$ either on Γ_1 or Γ_2 but not both at the same time. In this paper we consider the spaces $\Lambda_i^k(d,\Omega)$ with boundary conditions on Γ_i; $i = 1, 2$. The correspondence (2.10) allows us to identify $\Lambda^k(d,\Omega)$, $k = 0, 1, 2$ with the Sobolev spaces $H(\Omega, \mathrm{grad})$, $H(\Omega, \mathbf{curl}\,)$, and $H(\Omega, \mathrm{div}\,)$ of square integrable functions whose gradient, curl, and divergence are also square integrable. With $\Lambda^3(d,\Omega) \simeq L^2(\Omega)$ we have an L^2 version of the De Rham complex (2.3):

$$\mathbb{R} \hookrightarrow H(\Omega, \mathrm{grad}) \xmapsto{\nabla} H(\Omega, \mathbf{curl}\,) \xmapsto{\nabla\times} H(\Omega, \mathrm{div}\,) \xmapsto{\nabla\cdot} L^2(\Omega) \longmapsto 0. \tag{2.11}$$

The spaces $\Lambda_i^k(d,\Omega)$ correspond to Sobolev spaces constrained by boundary conditions on Γ_i:

$$\begin{aligned} H_i(\Omega, \mathrm{grad}) &= \{\phi \in H(\Omega, \mathrm{grad}) \mid \phi = 0 \quad \text{on } \Gamma_i\} \\ H_i(\Omega, \mathbf{curl}\,) &= \{\mathbf{w} \in H(\Omega, \mathbf{curl}\,) \mid \mathbf{w} \times \mathbf{n} = 0 \quad \text{on } \Gamma_i\} \\ H_i(\Omega, \mathrm{div}\,) &= \{\mathbf{w} \in H(\Omega, \mathrm{div}\,) \mid \mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_i\} \,. \end{aligned}$$

They form a De Rham complex relative to Γ_i.

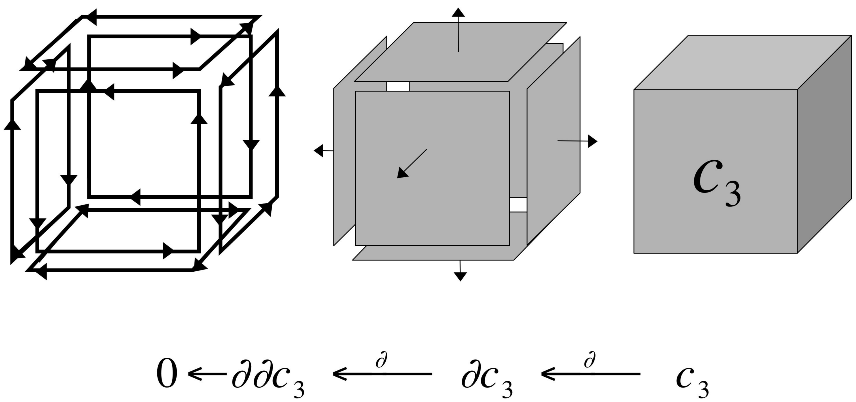

FIG. 1. *The boundary ∂ of a k-simplex is $(k-1)$-chain. The action of ∂ on a 3-cell illustrates that $\partial\partial c_3 = 0$.*

3. Algebraic topology. Our goal is to develop discrete structures that support mutually consistent, mimetic notions of integral, derivative, and inner product. The approach adopted in this paper is guided by algebraic topology and draws upon the ideas of [31]. This section reviews the necessary basic concepts. For further details we refer the reader to Cairns [21] or Flanders [27].

For brevity we restrict our attention to computational grids that are triangulations of Ω by a simplicial complex. All discrete structures developed in this paper and their mimetic properties can be extended to general polyhedral partitions of Ω such as considered in [38].

A k-simplex s_k is an ordered collection $[\mathbf{p}_0, \dots, \mathbf{p}_k]$ of $(k+1)$, $k \le n$ distinct points in $\mathbb{R}^n$ such that they span a k-plane. A k-chain is a formal linear combination

$$c_k = \sum_i a_i s_k^i$$

where a_i are real constants and s_k^i are k-simplices. A set of k-chains is denoted by C_k.

The boundary ∂ of a k-simplex is $(k-1)$-chain is defined by the formula

$$\partial[\mathbf{p}_0, \dots, \mathbf{p}_k] = \sum_{i=1}^{k} (-1)^i [\mathbf{p}_0, \dots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \dots, \mathbf{p}_k]\,. \tag{3.1}$$

A direct calculation shows that $\partial\partial = 0$. Boundary of a chain is defined by linearity; see Fig. 1

$$\partial c = \sum_i a_i \partial c_k^i\,. \tag{3.2}$$

The collection $\{C_0, C_1, C_2, C_3\}$ is called *complex* if for any $c \in C_k$, $\partial c \in C_{k-1}$. This gives rise to an exact sequence

$$0 \longleftarrow C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \longleftarrow 0 \tag{3.3}$$

where $\partial_k : C_{k+1} \mapsto C_k$ is the boundary operator on k-chains. The sequence (3.3) is called *exact* since $\mathrm{Range}\,\partial_k \subset \mathrm{Ker}\,\partial_{k-1}$, which follows from $\partial\partial = 0$.

The geometric realization of a k-simplex $[\mathbf{p}_0, \dots \mathbf{p}_k]$ is the map

$$t_i \mapsto \sum_{i=0}^{k} t_i \mathbf{p}_i\,, \quad \text{where } t_i \geq 0 \text{ and } \quad \sum_{i=1}^{k} t_i = 1\,.$$

This map returns the convex hull of the points $[\mathbf{p}_0, \dots \mathbf{p}_k]$. The numbers t_i are called *barycentric* coordinates, and they turn the complex $\{C_0, C_1, C_2, C_3\}$ into a metric space K. A triangulation of Ω is a homeomorphism $K \mapsto \Omega$. Given K, we denote by $L_1 \subset K$ and $L_2 \subset K$ the triangulations of Γ_1 and Γ_2.

The chain C_0 is a collection of zero simplices, i.e., points. We require that these points be given an ordering. This ordering determines an orientation for each k-simplex in K. A simplex $[\mathbf{p}_{i_0}, \dots, \mathbf{p}_{i_k}]$ has positive orientation if $\pi = \{i_0, \dots, i_k\}$ is an even permutation of the symbols $\{0, \dots, k\}$ and negative orientation otherwise. The subsets

$$Z_k = \{c_k \in C_k \,|\, \partial_{k-1} c_k = 0\} \quad \text{and} \quad B_k = \{b_k \in C_k \,|\, b_k = \partial_k c_{k+1} \text{ for } c_{k+1} \in C_{k+1}\}$$

of C_k are called k-cycles and k-boundaries, respectively. Because $\partial\partial = 0$, B_k is a subgroup of Z_k. The k^{th} homology group of K over $\mathbb{R}$, $\mathcal{H}_k(K, \mathbb{R}) = Z_k/B_k$ contains all cycles that are not boundary chains.

The dual C^k is the collection of all linear functionals on C_k. The elements of C^k are called k-cochains. We use the bracket notation $\langle \cdot, \cdot \rangle$ to denote the duality pairing of chains and cochains. The adjoint of ∂, $\delta : C^k \mapsto C^{k+1}$, defined by

$$\langle a, \partial c \rangle = \langle \delta a, c \rangle \tag{3.4}$$

satisfies $\delta\delta = 0$ and forms an exact sequence

$$0 \longrightarrow C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} C^3 \longrightarrow 0\,, \tag{3.5}$$

dual to (3.3). As before, we define the k-cocycles Z^k, the k-coboundaries B^k of C^k, and the k^{th} cohomology group $\mathcal{H}^k(K, \mathbb{R}) = Z^k/B^k$.

The collection $\{\sigma_k^i\}$, $i = 1, 2, \dots$ of positively oriented k-chains forms a basis for the chain complex. Since K is finite, C_k is finite dimensional and isomorphic to C^k. The isomorphism $J : C^k \mapsto C_k$ is given by

$$Ja = \sum_i \langle a, \sigma_k^i \rangle \sigma_k^i\,. \tag{3.6}$$

We identify σ_k^i with its dual so that $\langle \sigma_k^i, \sigma_k^j \rangle = \delta_{ij}$. Then a cochain can be written as $a = \sum a_i \sigma_k^i$ and its action on a chain $c = \sum c_i \sigma_k^i$ is given by

$$\langle a, c \rangle = \sum_i a_i c_i\,.$$

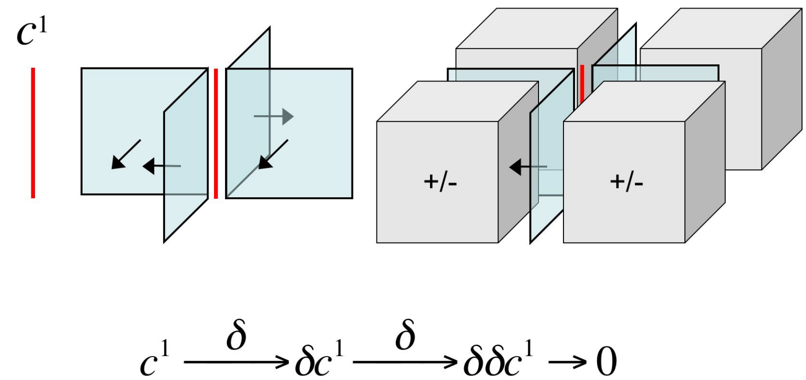

FIG. 2. *The coboundary operator is defined by $\delta[\mathbf{p}_0, \ldots, \mathbf{p}_k] = \sum_{\mathbf{p}} [\mathbf{p}, \mathbf{p}_0, \ldots, \mathbf{p}_k]$, where the points $[\mathbf{p}, \mathbf{p}_0, \ldots, \mathbf{p}_k]$ form a $(k+1)$-simplex, returns a cochain that contains all $(k+1)$ simplices that have $[\mathbf{p}_0, \ldots, \mathbf{p}_k]$ as part of their boundary. The action of δ on a 1-cell illustrates that $\delta\delta c^1 = 0$.*

From this, the coboundary operator is computed to be

$$\delta[\mathbf{p}_0, \ldots, \mathbf{p}_k] = \sum_{\mathbf{p}} [\mathbf{p}, \mathbf{p}_0, \ldots, \mathbf{p}_k]$$

where the sum is over all points $\mathbf{p}$ such that $[\mathbf{p}, \mathbf{p}_0, \ldots, \mathbf{p}_k]$ is a $(k+1)$-simplex. In other words, the coboundary returns a cochain that contains all $(k+1)$ simplices that have $[\mathbf{p}_0, \ldots, \mathbf{p}_k]$ as part of their boundary; see Fig. 2.

To accommodate boundary conditions, define the subspace $C^k_i \subset C^k$ to be the set of all k-cochains that vanish on L_i, the triangulation of Γ_i:

$$C^k_i = \{a \in C^k \mid \langle a, c_k \rangle = 0 \ \forall c_k \in L_i\},$$

and C^k_0 to be the cochains that vanish on $L_1 \cup L_2$. In a similar way we construct the groups Z^k_i, B^k_i and the k^{th} relative cohomology group $\mathcal{H}^k_i = \mathcal{H}^k(K, L_i, \mathbb{R})$.

We stress that geometrically C^k and C_k are distinct despite the isomporphism J. An element of C_k is a formal sum of k-simplices, whereas an element of C^k is a linear function that maps elements of C_k into real numbers. This distinction also extends to the role of chains and cochains in the discretization. The k-chains represent subsets of the nodes, edges, faces, and cells in the grid. The k-cochains are the collections of real numbers $\{a_i\}$ associated with these subsets. Therefore, the chains are the physical objects that make the computational grid, while the cochains are the discrete functions that live on that grid. In particular, the proper way to store scalar functions on the grid is as 0- or 3-cochains, while the proper way to store vector fields is as 1- or 2-cochains.

4. Framework for mimetic discretizations. This section develops structures for mimetic discretization of PDEs by using algebraic topology and two basic operations. A reduction operator maps forms to cochains and gives rise to combinatorial operations of differentiation and integration that satisfy a Stokes theorem. A reconstruction operator translates cochains to differential forms and is used to obtain the *natural* inner and wedge product operations. The natural operations provide the *derived* analogues of the adjoint d^* and the Hodge Laplacian.

4.1. Basic operations.

Reduction. Information about physical quantities is obtained by measuring. Integration of differential forms is an abstraction of this process and motivates our choice of the De Rham map $\Lambda^k(\Omega) \mapsto C^k$ for the reduction operation. This map is defined by

$$\langle \mathcal{R}\omega, c\rangle = \int_c \omega \tag{4.1}$$

where $c \in C^k$ is a k-chain and $\omega \in \Lambda^k(\Omega)$ is a k-form. The mapping $\omega \mapsto \mathcal{R}\omega$ establishes discrete representation of k-forms in terms of global quantities associated with a chain complex. Thus, we encode discrete k-forms as k-cell quantities. The following property of $\mathcal{R}$ will prove useful in the sequel.

LEMMA 4.1. *The De Rham map has the commuting diagram property $\mathcal{R}d = \delta\mathcal{R}$.*

Proof. Using the Stokes formula (2.4) and the duality of ∂ and δ gives

$$\langle \mathcal{R}d\omega, c\rangle = \int_c d\omega = \int_{\partial c} \omega = \langle \mathcal{R}\omega, \partial c\rangle = \langle \delta\mathcal{R}\omega, c\rangle \,. \tag{4.2}$$

□

In what follows we refer to this property as *CDP1*, the first commuting diagram.

Reconstruction. Central to our approach is the notion of an inner product on cochains. Its *natural* definition requires an operation $\mathcal{I}$ that serves as an approximate inverse to $\mathcal{R}$ and translates the global information stored in C^k back to local representations. In contrast to $\mathcal{R}$, where the De Rham map (4.1) is the obvious candidate, the choice of $\mathcal{I}$ is flexible because of the many possible ways in which global data from C^k can be combined in a local field representation.

The operator $\mathcal{I}$ must satisfy two basic conditions. We will call a bounded linear mapping $\mathcal{I} : C^k \mapsto \Lambda^k(L^2, \Omega)$ an L^2 *mimetic reconstruction operator* if $\mathcal{I}$ is a right inverse of $\mathcal{R}$ (*consistency property*)

$$\mathcal{R}\mathcal{I} = id \tag{4.3}$$

and an approximate left inverse of that operator (*approximation property*)

$$\mathcal{I}\mathcal{R} = id + O(h^s)\,, \tag{4.4}$$

where s and h are positive real numbers that give the approximation order and the partition size in K, respectively.

From (4.3) it follows that $\mathcal{I}$ is *unisolvent* in the sense that

$$\operatorname{Ker}\mathcal{I} = \{0\}\,. \tag{4.5}$$

We require the range of $\mathcal{I}$ to contain square integrable k-forms and (4.3) implies that these forms are continuous on the k-chains of the complex K. However, they may be discontinuities along the $m \neq k$-cells of the complex, or even within the k-cells of K, and so they may not belong to $\Lambda^k(d,\Omega)$. For mimetic reconstruction operators $\mathcal{I}$ whose range is a subspace of the Sobolev space $\Lambda^k(d,\Omega)$ we impose an additional condition that serves to coordinate the action of the exterior derivative and the coboundary operator. This condition takes the form of a second commuting diagram property, *CDP2*,

$$d\mathcal{I} = \mathcal{I}\delta\,. \tag{4.6}$$

We will call such mappings *conforming* mimetic reconstruction operators. The Whitney map [59, 24, 31] is an example of a regular mimetic reconstruction operator.

4.2. Discrete structures. For a mimetic reconstruction operator $\mathcal{I}$, the range of $\mathcal{I}\mathcal{R}$, considered as an operator $\Lambda^k(d,\Omega) \mapsto \Lambda^k(L^2,\Omega)$, is a subspace of $\Lambda^k(L^2,\Omega)$ given by

$$\Lambda^k(L^2,K) = \{\omega_h \in \Lambda^k(L^2,\Omega)\,|\,\omega_h = \mathcal{I}\mathcal{R}\omega \ \text{ for some } \omega \in \Lambda^k(d,\Omega)\}\,. \tag{4.7}$$

When $\mathcal{I}$ is a conforming operator, the range of $\mathcal{I}\mathcal{R}$ is a subspace of $\Lambda^k(d,\Omega)$ given by

$$\Lambda^k(d,K) = \{\omega_h \in \Lambda^k(d,\Omega)\,|\,\omega_h = \mathcal{I}\mathcal{R}\omega \ \text{ for some } \omega \in \Lambda^k(d,\Omega)\}\,. \tag{4.8}$$

The spaces $\Lambda^k_i(L^2,K)$ and $\Lambda^k_i(d,K)$ are defined similarly using $\Lambda^k_i(d,\Omega)$.

4.2.1. Combinatorial operations. These operations are induced by the action of $\mathcal{R}$ and are completely independent of any metric structures.

Exterior derivative. Formula (2.10) shows that Grad, Curl and Div are generated by the action of d on 0-, 1-, and 2-forms. Therefore, their discrete versions will be generated by a discrete counterpart of d acting on 0-,1-, and 2-cochains. To find the discrete version of d on K we note that forms are dual to manifolds with respect to the pairing induced by integration and that according to the Stokes theorem (2.4), d is the adjoint of ∂. To define a discrete derivative we mimic this by using the duality of C^k and C_k and formula (3.4) which states that δ is dual to ∂. Thus, the discrete Grad, Curl and Div are generated by the coboundary. The CDP1 property asserts the consistency of this definition: The action of d on ω followed by a reduction to cochain equals the reduction of ω to cochain followed by the action of δ.

Integration. The integral of $a \in C^k$ is defined on chains C_k by duality:

$$\int_\sigma a = \langle a, \sigma\rangle \quad \forall a \in C^k;\ \sigma \in C_k\,. \tag{4.9}$$

4.2.2. Natural operations. These are defined by composition of $\mathcal{I}$ and the desired analytic operation. Natural operations are the best imitation of the analytic operations on cochains.

Inner product. The L^2 inner product (2.6) on $\Lambda^k(\Omega)$ is the integral of the inner product on $\Lambda^k(T_x\Omega)$. We mimic this relationship by setting up the *local* inner product

$$(a,b) \stackrel{\text{def}}{=} (\mathcal{I}a, \mathcal{I}b) \qquad \forall\, a,b \in C^k\,. \tag{4.10}$$

The discrete L^2 inner product on C^k is the integral of (4.10):

$$(a,b)_\Omega \stackrel{\text{def}}{=} \int_\Omega (a,b)\omega_n \qquad \forall\, a,b \in C^k\,. \tag{4.11}$$

Unisolvency (4.5) of $\mathcal{I}$ guarantees that (4.10) and (4.11) are nondegenerate and are indeed inner products.

Wedge product. The operation $\wedge : C^k \times C^p \mapsto C^{p+k}$ is introduced by using the wedge product of differential forms. Specifically, we set

$$a \wedge b = \mathcal{R}(\mathcal{I}a \wedge \mathcal{I}b) \quad \forall\, a \in C^k\,;\, b \in C^p\,. \tag{4.12}$$

4.2.3. Derived operations. These operations are induced by the existing natural operations.

The discrete adjoint. The inner product on C_0^k induces an adjoint δ^* of δ characterized by the identity

$$(\delta a, b)_\Omega = (a, \delta^* b)_\Omega \qquad \forall\, a \in C_0^k;\ b \in C_0^{k+1}\,. \tag{4.13}$$

The adjoint is a mapping $C_0^{k+1} \mapsto C_0^k$, has the property that $\delta^*\delta^* = 0$, and provides a second set of discrete Grad, Curl, and Div operations. In PDEs modeling physical problems, often a vector function is associated naturally with a 1-form or a 2-form, while a scalar function can be associated with a 0-form or a 3-form. This identification determines whether the vector function should be encoded in C_0^1 or C_0^2 and the scalar function in C_0^0 or C^3. This in turn determines the discrete version of Div, Curl and Grad to use.

Hodge Laplacian. We define the discrete Laplacian $\mathcal{D} : C_0^k \mapsto C_0^k$ with δ and its adjoint δ^* as

$$-\mathcal{D} = \delta^*\delta + \delta\delta^* \tag{4.14}$$

to mimic $-\triangle = d^*d + dd^*$.

Remark 4.1. *Derived operations are needed to avoid internal inconsistencies between the discrete operations. Because $\mathcal{I}$ is only an approximate left inverse of $\mathcal{R}$, some natural definitions with clash with each other. For example, a natural counterpart of (4.13) mimics $d^* = (-1)^k \star d\star$ and defines $\delta^* = (-1)^k\mathcal{R}\star d \star \mathcal{I}$. Besides the fact that this requires $\mathcal{I}$ to be conforming, the real problem is that the* natural *δ^* is not the adjoint of δ with respect to the* natural *inner product (4.11). Indeed, from (4.4) and (4.6)*

$$\begin{aligned}(\delta^* a, b)_\Omega &= (-1)^k(\mathcal{I}\mathcal{R}\star d \star \mathcal{I}a, \mathcal{I}b)_\Omega = (-1)^k(\star d \star \mathcal{I}a, \mathcal{I}b)_\Omega + O(h^s)\\ &= (\mathcal{I}a, d\mathcal{I}b)_\Omega + O(h^s) = (\mathcal{I}a, \mathcal{I}\delta b)_\Omega + O(h^s) = (a, \delta b)_\Omega + O(h^s)\,.\end{aligned}$$

4.3. Mimetic properties. We now establish the mimetic properties of the discrete operations.

Derivative and integral. In addition to $\delta\delta = \delta^*\delta^* = 0$, derivatives have the following mimetic property.

Lemma 4.2. *Assume that $\mathcal{I}$ is conforming and let $a_h = \mathcal{I}a$, $b_h = \mathcal{I}b$ for $a \in C^k$, $b \in C^{k+1}$. Then*

$$(da_h, b_h)_\Omega = (\delta a, b)_\Omega \quad \textit{and} \quad (a_h, d^* b_h)_\Omega = (a, \delta^* b)_\Omega\,. \tag{4.15}$$

Proof. The first identity follows directly from CDP2 (4.6) and the definition of the mimetic inner product. To prove the second identity we use (4.6), (4.11), and that d^* is the adjoint of d:

$$(a_h, d^* b_h)_\Omega = (da_h, b_h)_\Omega = (d\mathcal{I}a, \mathcal{I}b)_\Omega = (\mathcal{I}\delta a, \mathcal{I}b)_\Omega = (\delta a, b)_\Omega = (a, \delta^* b)_\Omega\,.$$

□

The discrete Stokes theorem is a consequence of the identity

$$\langle \delta a, \sigma\rangle = \langle a, \partial\sigma\rangle \quad \forall a \in C^k;\, \sigma \in C_{k+1}\,.$$

From (4.1), (4.3), and (4.9) we have the property

$$\int_\sigma a = \langle a, \sigma\rangle = \langle \mathcal{R}\mathcal{I}a, \sigma\rangle = \int_\sigma \mathcal{I}a\,. \tag{4.16}$$

Combinatorial Hodge theory. We recall the relative singular cohomology of Ω over $\mathbb{R}$:

$$\bar{\mathcal{H}}_0^k = \mathrm{Ker}\,\delta/\mathrm{Range}\,\delta \quad \text{on singular } k\text{-cochains that vanish on } \Gamma_1\,,$$

the De Rham cohomology:

$$\bar{\mathcal{H}}^k = \mathrm{Ker}\,d/\mathrm{Range}\,d \quad \text{on } \Lambda_1^k$$

and the De Rham theorem

$$\bar{\mathcal{H}}^k \simeq \bar{\mathcal{H}}_0^k\,.$$

Let $H^k(\Omega) = \{h \in \Lambda_0^k(\Omega)\,|\,\triangle h = 0\}$, the space of smooth harmonic k-forms. The Hodge decomposition[3] theorem [23] states that $\dim(\mathrm{Ker}\triangle_k) = \dim(\bar{\mathcal{H}}^k)$ and every $\omega \in \Lambda_0^k(\Omega)$ has a decomposition

$$\omega = df + h + d^*g \tag{4.17}$$

where $f \in \Lambda_0^{k-1}(\Omega)$, $g \in \Lambda_0^{k+1}(\Omega)$, and $h \in H^k(\Omega)$. In the vector calculus this theorem implies that any vector function $\mathbf{u}$ has a decomposition $\mathbf{u} = \nabla\times\mathbf{w} + \nabla\phi + h$ where h is harmonic and ϕ is a scalar. It also implies that any real function has the decomposition $f = g + \nabla\cdot\mathbf{v}$, where g is harmonic.

The kernel of the discrete Laplacian $H^k(K) = \{h \in C_0^k\,|\,\mathcal{D}h = 0\}$ is the set of all harmonic cochains in C_0^k. Its characterization mimics that of $H^k(\Omega)$:

$$H^k(K) = \{c \in C_0^k\,|\,\delta c = \delta^* c = 0\}\,. \tag{4.18}$$

THEOREM 4.1. *Every $a \in C_0^k$ has a decomposition*

$$a = \delta b + h + \delta^* c\,, \tag{4.19}$$

where $b \in C_0^{k-1}$, $c \in C_0^{k+1}$ and $h \in H^k(K)$.

Theorem 4.1 is a consequence of $\delta\delta = 0$ and the definition of δ^* as the adjoint to δ. This is another important reason to choose the derived definition (4.13) of δ^* instead of the natural one in Remark 4.1.

To compute $\dim(\mathrm{Ker}\mathcal{D})$ we need the following result.

LEMMA 4.3. *The kernel of $\mathcal{D}$ is isomorphic to the k^{th} relative cohomology group $\mathcal{H}_0^k$.*

Proof. Note that if $a = \delta b + h + \delta^* c$ is in $\mathrm{Ker}(\delta_k)$, then from $\delta\delta = 0$ and (4.18)

$$0 = \delta a = \delta\delta b + \delta h + \delta\delta^* c = \delta\delta^* c\,.$$

This identity implies that $(\delta^* c, \delta^* c) = 0$ and hence $\delta^* c = 0$. Thus, if $\delta a = 0$, then $a = h + \delta b$, and the correspondence $a \leftrightarrow h$ provides an isomorphism $\mathrm{Ker}\,\delta/\mathrm{Range}\,\delta \mapsto \mathrm{Ker}\,\mathcal{D}$. $\square$

COROLLARY 4.1. *The size of the kernels of the analytic and discrete Laplacians is the same.*

[3]This theorem is primarily a consequence of the fact that if $T : V \mapsto V$ is a bounded linear operator on a Hilbert space V such that $T^2 = 0$, then

$$V = \mathrm{Range}\,T \oplus \mathrm{Range}\,T^* \oplus H\,,$$

where $H = \{x \in V\,|\,Tx = T^*x = 0\}$. A simple proof is as follows. Define $V' = (\mathrm{Range}\,T \oplus \mathrm{Range}\,T^*)^\perp$ and let $x \in V'$. Then $\langle Ty, x\rangle = 0$ and $\langle T^*y, x\rangle = 0$ for all $y \in V$ imply that $Tx = T^*x = 0$ and $x \in H$. For $T = d$ the proof is complicated by the fact that d is an unbounded operator on a domain in L^2.

Proof. From Lemma 4.3 it follows that

$$\dim(\operatorname{Ker}\mathcal{D}_k) = \dim\mathcal{H}_0^k\,.$$

Furthermore, $\dim(\mathcal{H}_0^k) = \dim(\bar{\mathcal{H}}_0^k)$ (Cairns [21]) and $\dim(\bar{\mathcal{H}}_0^k) = \dim(\bar{\mathcal{H}}^k)$ (De Rham's theorem). The assertion follows from $\dim(\bar{\mathcal{H}}^k) = \dim(\ker\triangle_k)$. ▯

It is remarkable that the size of the kernel of the analytic and discrete Laplacians depends only upon the topology of the domain and not the specific nature of these Laplacians.

Natural inner product. The definition of the discrete L^2 product (4.11) mimics definition (2.6). Using (4.10) we find that this inner product has the property that

$$(a,b)_\Omega = \int_\Omega (a,b)\omega_n = \int_\Omega (\mathcal{I}a,\mathcal{I}b)\omega_n = \int_\Omega \mathcal{I}a\wedge\star\mathcal{I}b\,,$$

which mimics the property (2.7) of the analytic inner product.

Vector calculus. The discrete versions of the vector calculus identities hold exactly for the discrete operators defined by δ and δ^*

LEMMA 4.4. *The discrete versions of Grad, Curl, and Div satisfy Curl Grad $\equiv 0$ and Div Curl $\equiv 0$*

Proof. For the two discrete derivatives the identities are $\delta\delta = 0$ and $\delta^*\delta^* = 0$. The first follows by duality of chains and cochains:

$$\langle\delta\delta a, b\rangle = \langle\delta a,\partial b\rangle = \langle a,\partial\partial b\rangle = 0$$

The second follows by the duality of δ and δ^* with respect to the discrete inner product:

$$(\delta^*\delta^* a, b)_\Omega = (\delta^* a,\delta b)_\Omega = (a,\delta\delta b)_\Omega = 0\,.$$

▯

As a corollary to this Lemma we also have a discrete version of Poincaré's lemma which states that on a contractable domain every closed form is a differential. The discrete version of this lemma is that every cocycle is a coboundary. Therefore, on contractable domains we have existence of discrete potentials. This mimetic property can be used to transfer solenoidal fields between two different cell complexes [11] and gauge discrete problems [15].

The wedge product. We show that (4.12) has the same commutation property as the true wedge product. If $\mathcal{I}$ is also conforming, then the effect of δ on (4.12) is algebraically the same as that of the exterior derivative on forms, and so properties of the discrete wedge and the discrete derivative are properly coordinated.

LEMMA 4.5. *Let $\wedge : C^k\times C^p \mapsto C^{k+p}$ be defined by (4.12). Then*

$$a\wedge b = (-1)^{kp} b\wedge a\,, \tag{4.20}$$

and if $\mathcal{I}$ is conforming mimetic reconstruction,

$$\delta(a\wedge b) = \delta a\wedge b + (-1)^k a\wedge\delta b \tag{4.21}$$

for all $a\in C^k$ and $b\in C^p$.

Proof. The commutation identity (4.20) follows directly from (4.12) and the like property of forms. The second identity is a consequence of the CDP1 property of $\mathcal{R}$ and the CDP2 property of $\mathcal{I}$:

$$\begin{aligned}\delta(a\wedge b) &= \delta\mathcal{R}(\mathcal{I}a\wedge\mathcal{I}b) \overset{CDP1}{=} \mathcal{R}d(\mathcal{I}a\wedge\mathcal{I}b)\\ &= \mathcal{R}(d\mathcal{I}a\wedge\mathcal{I}b)+(-1)^k\mathcal{R}(\mathcal{I}a\wedge d\mathcal{I}b) \overset{CDP2}{=} \mathcal{R}(\mathcal{I}\delta a\wedge\mathcal{I}b)+(-1)^k\mathcal{R}(\mathcal{I}a\wedge\mathcal{I}\delta b)\\ &= \delta a\wedge b+(-1)^k a\wedge\delta b\,.\end{aligned}$$

□

The wedge product is nonassociative: $(a\wedge b)\wedge c\neq a\wedge(b\wedge c)$.

4.4. Discrete $\star$. In this section we discuss complications arising in the construction of a discrete $\star$ operation and explain why it is not among the discrete operations that comprise our mimetic framework.

A *natural* discrete $\star$ operation uses $\mathcal{I}$ to translate cochains to forms, applies the analytic $\star$ and then reduces the result back to cochains. Thus, a natural operator $\overset{N}{\star}\colon C^k\mapsto C^{n-k}$ is defined by

$$\overset{N}{\star}=\mathcal{R}\star\mathcal{I}\,. \tag{4.22}$$

Tarhasaari *et al* [52] proposed this formula for a primal-dual cell complex.

The *derived* discrete $\star$ is defined in terms of the existing natural operations. We use the inner product (4.11) and the wedge product (4.12) to mimic[4] (2.7) and define $\overset{D}{\star}\colon C^k\mapsto C^{n-k}$ by the formula

$$\int_\Omega a\wedge\overset{D}{\star}b=(a,b)_\Omega\quad\forall\,a,b\in C^k\,. \tag{4.23}$$

In Section 5 we show that the derived $\star$ is related to an algebraic definition proposed by Hiptmair [30].

Lemma 4.6. *The operator $\overset{N}{\star}$ has a commuting diagram property on the range of $\mathcal{IR}$, that is*

$$\overset{N}{\star}\mathcal{R}\omega_h=\mathcal{R}\star\omega_h\quad\forall\omega_h\in\Lambda^k(L^2,K)\,. \tag{4.24}$$

Proof. From (4.7) we know that any $\omega_h\in\Lambda^k(L^2,K)$ has the form $\omega_h=\mathcal{IR}\omega$ for some $\omega\in\Lambda^k(d,\Omega)$. Using this characterization and the fact that $\mathcal{RI}=id$ gives

$$(\overset{N}{\star}\mathcal{R})\omega_h=(\overset{N}{\star}\mathcal{R})(\mathcal{IR}\omega)=(\mathcal{R}\star\mathcal{I})(\mathcal{RI})(\mathcal{R}\omega)=(\mathcal{R}\star)(\mathcal{IR}\omega)=(\mathcal{R}\star)\omega_h\,.$$

□

Lemma 4.7. *The operator $\overset{D}{\star}$ has a weak commuting diagram property on C^k:*

$$\int_\Omega\mathcal{IR}(\mathcal{I}a\wedge\mathcal{I}\overset{D}{\star}a)=\int_\Omega\mathcal{I}a\wedge\star\mathcal{I}a\,. \tag{4.25}$$

Proof. Using (4.16) and (4.12)

$$\int_\Omega a\wedge\overset{D}{\star}a=\int_\Omega\mathcal{I}(a\wedge\overset{D}{\star}a)=\int_\Omega\mathcal{IR}(\mathcal{I}a\wedge\mathcal{I}\overset{D}{\star}a)\,,$$

[4] The discrete $\star$ acts on cochains and is a global operation. Thus, we mimic the global relation (2.7) instead of the local formula (2.1) which defines the analytic $\star$ locally.

which is the left-hand side in (4.25). Using (4.11) and (2.7)

$$(a, a)_\Omega = (\mathcal{I}a, \mathcal{I}a)_\Omega = \int_\Omega \mathcal{I}a \wedge \star\mathcal{I}a\,,$$

which is the right-hand side in (4.25). □

Similar arguments can be used to show that

$$\int_\Omega a \wedge \overset{N}{\star}\, a = (a, a)_\Omega + O(h^s)\,, \tag{4.26}$$

which implies that $\overset{D}{\star} - \overset{N}{\star} = O(h^s)$. Formula (4.26) also means that the natural operator $\overset{N}{\star}$ is not compatible with the natural inner and wedge product definitions, while (4.24) means that it is compatible with the reduction map $\mathcal{R}$. Exactly the opposite is true for the derived operator $\overset{D}{\star}$. By construction this operator is compatible with the natural inner product and the natural wedge product but is incompatible with $\mathcal{R}$ and $\mathcal{I}$. Finally, neither $\overset{N}{\star}$, nor $\overset{D}{\star}$ is compatible with the derived adjoint δ^* defined in (4.13).

The problems with the discrete $\star$ operation arise from the fact that its action must be coordinated with two natural operations. The natural definition fails to accomplish this, while forcing the discrete $\star$ into compliance with the two natural operations leads to other incompatibilities. In contrast, an operation like δ^* requires a single natural operation for its definition and has a "built-in" compatibility with that operation.

These observations show that if a discrete $\star$ operation is required, then it *must* be made the primary object of the discrete framework and then used to define all other necessary structures. However, construction of a good discrete $\star$ is nontrivial and more difficult than the construction of a good inner product. For instance, the analytic $\star$ is local and invertible. To mimic this in finite dimensions the discrete $\star$ must be given by a diagonal matrix with positive entries. This is impossible unless K has a dual complex $\tilde{K}$ such that C^k is isomorphic[5] to $\tilde{C}^{n-k}$. In all other cases, the discrete $\star$ will be a rectangular matrix.

As a rule, the need for a discrete $\star$ arises from discretization of material laws. Because of the difficulties with this operator, we prefer to either incorporate these laws in the inner product or to enforce them in a weak, L^2 sense. In the first case we work with δ^* and in the second we solve a constrained optimization problem. These alternatives to a discrete $\star$ offer several valuable advantages. Besides being sufficient for a combinatorial Hodge theory, the inner product gives rise to a symmetric and positive semidefinite Laplacian. In contrast, direct discretization of material laws by an independently defined discrete $\star$ and the subsequent formation of the Laplacian through this operation may lead to operators that have imaginary and/or negative eigenvalues with the attendant stability problems; see [49] for examples in computational electromagnetism. On the other hand, the weak enforcement of the material laws is justified by their approximate nature as summaries of complex interactions.

In summary, the natural inner product leads to well-behaved discrete structures and is much easier to construct than a good discrete $\star$ operator. Choosing the inner product to be the primary discrete operation will also mimic the analytic case where the $\star$ operator is induced by the inner product, but not vice versa.

5. Algebraic realizations. Let $m_k = \dim C_0^k$. The map

$$a = \sum_{i=1}^{m_k} a_i \sigma_k^i \mapsto \mathbf{a} = (a_1, \ldots, a_{m_k})$$

[5] It is worth pointing out that when $\tilde{C}^{n-k}$ and C^k have the same dimension, the *covolume* reconstruction operator gives rise to an inner product that is compatible with a diagonal discrete $\star$; see [44, 45, 57]. Thus, in this case, explicit definition of a discrete $\star$ operation can also be avoided.

establishes an isomorphism $C_0^k \mapsto \mathbb{R}^{m_k}$. Then $\mathcal{R}$ can be viewed as a map $\Lambda_0^k(\Omega) \mapsto \mathbb{R}^{m_k}$, defined by

$$\mathbf{a} = \mathcal{R}\omega \quad \text{if and only if} \quad a_i = \int_{\sigma_k^i} \omega \,,$$

while $\mathcal{I}$ is an approximate inverse of this map. As a result, all mimetic operations on cochains can be realized by matrices acting on their coefficient vectors. The action of $\delta : C_0^k \mapsto C_0^{k+1}$ is given by a matrix $\mathbb{D}_k \in \mathbb{R}^{m_{k+1} \times m_k}$ with the property that $\mathbb{D}_{k+1}\mathbb{D}_k = 0$. This matrix has elements -1, 0, and 1 which reflect the combinatorial nature of the discrete derivative δ

The local and the L^2 inner products on C_0^k are associated with the symmetric and positive definite matrices $\mathbb{M}_k(x), \mathbb{M}_k \in \mathbb{R}^{m_k \times m_k}$ such that

$$(a,b) = \mathbf{a}^T \mathbb{M}_k(x) \mathbf{b} \quad \text{and} \quad (a,b)_\Omega = \mathbf{a}^T \mathbb{M}_k \mathbf{b} \,, \tag{5.1}$$

respectively.

The action of δ^* is given by a matrix $\mathbb{D}_k^* \in \mathbb{R}^{m_k \times m_{k+1}}$. Since δ^* is derived from δ and the natural inner product, it follows that $\mathbb{D}_k^*$ can be expressed in terms of the matrices that represent these operations. From

$$\mathbf{a}^T (\mathbb{D}_{k+1}^*)^T \mathbb{M}_k \mathbf{b} = (\delta_{k+1}^* a, b)_\Omega = (a, \delta_k b)_\Omega = \mathbf{a}^T \mathbb{M}_{k+1} \mathbb{D}_k \mathbf{b}$$

we see that $\mathbb{D}_{k+1}^* = \mathbb{M}_k^{-1} \mathbb{D}_k^T \mathbb{M}_{k+1}$ and

$$\mathbb{D}_k^* \mathbb{D}_{k+1}^* = \mathbb{M}_{k-1}^{-1} \mathbb{D}_{k-1}^T \mathbb{M}_k \mathbb{M}_k^{-1} \mathbb{D}_k^T \mathbb{M}_{k+1} = \mathbb{M}_{k-1}^{-1} \mathbb{D}_{k-1}^T \mathbb{D}_k^T \mathbb{M}_{k+1} = 0 \,,$$

as expected from a derivative.

The discrete Laplacian Δ_k is also a derived operation and its action is given by the matrix

$$\mathbb{L}_k = (\mathbb{M}_k^{-1} \mathbb{D}_k^T \mathbb{M}_{k+1} \mathbb{D}_k + \mathbb{D}_{k-1} \mathbb{M}_{k-1}^{-1} \mathbb{D}_{k-1}^T \mathbb{M}_k) \in \mathbb{R}^{m_k \times m_k} \,.$$

We have the formula

$$(\delta_{k+1}^* \delta_k a, b) = \mathbf{a}^T \mathbb{D}_k^T (\mathbb{M}_k^{-1} \mathbb{D}_k^T \mathbb{M}_{k+1})^T \mathbb{M}_k \mathbf{b} = \mathbf{a}^T \mathbb{D}_k^T \mathbb{M}_{k+1} \mathbb{D}_k \mathbf{b} = (\delta_k a, \delta_k b)$$

and a similar formula for $(\delta_{k-1} \delta_k^* a, b)$.

To find a matrix expression for the wedge product $\wedge : C_0^1 \times C_0^1 \mapsto C_0^2$ we use the formula

$$a_1 \wedge b_1 = \mathcal{R}(\mathcal{I} a_1 \wedge \mathcal{I} b_1) = \sum_{i=1}^{m_2} c_i \sigma_2^i$$

and the commutation property (4.20) to conclude that each coefficient c_i is a skew-symmetric bilinear form of the coefficient vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore, c_i is given by a skew-symmetric matrix $\mathbb{W}_{11}^i \in \mathbb{R}^{m_1 \times m_1}$ and

$$a_1 \wedge b_1 = \sum_{i=1}^{m_2} (\mathbf{a}^T \mathbb{W}_{11}^i \mathbf{b}) \sigma_2^i \,.$$

For $\wedge : C_0^1 \times C_0^2 \mapsto C_0^3$ and $\wedge : C_0^2 \times C_0^1 \mapsto C_0^3$ we have the formulas

$$a_1 \wedge b_2 = \mathcal{R}(\mathcal{I} a_1 \wedge \mathcal{I} b_2) = \sum_{i=1}^{m_3} c_i^{12} \sigma_3^i \quad \text{and} \quad b_2 \wedge a_1 = \mathcal{R}(\mathcal{I} b_2 \wedge \mathcal{I} a_1) = \sum_{i=1}^{m_3} c_i^{21} \sigma_3^i \,,$$

respectively. The coefficients c_i^{12} and c_i^{21} are bilinear functions of $\mathbf{a}$ and $\mathbf{b}$ and so they are given by matrices $\mathbb{W}_{12}^i \in \mathbb{R}^{m_1\times m_2}$ and $\mathbb{W}_{21}^i \in \mathbb{R}^{m_2\times m_1}$, respectively. From (4.20) it follows that $\mathbb{W}_{12}^i = (\mathbb{W}_{12}^i)^T$ and

$$a_1 \wedge b_2 = \sum_{i=1}^{m_3} (\mathbf{a}^T \mathbb{W}_{12}^i \mathbf{b})\sigma_3^i \quad \text{and} \quad b_2 \wedge a_1 = \sum_{i=1}^{m_3} (\mathbf{b}^T (\mathbb{W}_{12}^i)^T \mathbf{a})\sigma_3^i\,. \tag{5.2}$$

Matrix representations for the remaining two wedge products follow in a similar fashion. From (5.1) and (5.2) we can obtain a matrix representation for $\overset{D}{\star}\colon C_0^1 \mapsto C_0^2$. Using (5.2) and definition (4.9) the matrix form of the left hand side in (4.23) is

$$\int_\Omega a\wedge \overset{D}{\star}\, a = \langle a\wedge \overset{D}{\star}\, a, \sum_{i=1}^{m_3} \sigma_3^i\rangle = \sum_{i=1}^{m_3} \mathbf{a}^T \mathbb{W}_{12}^i (\overset{D}{\star}\,\mathbf{a})\langle \sigma_3^i, \sigma_3^i\rangle = \sum_{i=1}^{m_3} \mathbf{a}^T \mathbb{W}_{12}^i (\overset{D}{\star}\,\mathbf{a})\mu_i$$

where $\overset{D}{\star}\,\mathbf{a} \in \mathbb{R}^{m_2}$ is the coefficient vector of $\overset{D}{\star}\, a$ and $\mu_i = \langle \sigma_3^i, \sigma_3^i\rangle$ is the volume of the ith basis 3-cell. The matrix form of the right hand side in (4.23) is

$$(a,a)_\Omega = \sum_{i=1}^{m_3} \mathbf{a}^T \mathbb{M}_3 \mathbf{a}\,.$$

Let $\mathbb{W}_{12} = \sum_{i=1}^{m_3} \mu_i \mathbb{W}_{12}^i$. Then, the matrix form of (4.23) is

$$\mathbb{W}_{12}(\overset{D}{\star}\,\mathbf{a}) = \mathbb{M}_3\mathbf{a}\,. \tag{5.3}$$

This formula reflects the fact that the derived operator $\overset{D}{\star}$ relies on two natural operations and so is associated with a *pair* of matrices related to these operations. A formula similar to (5.3) was used in [30] for an axiomatic definition of a discrete $\star$ operation.

Algebraic realizations of the mimetic operations are summarized in Table 1.

Table 1
Algebraic realizations of mimetic operations

Operation	Matrix form	Type
δ	$\mathbb{D}_k$	incidence matrix
$(\cdot,\cdot)$	$\mathbb{M}_k$	SPD
$a^1 \wedge b^1$	$\sum \mathbb{W}_{11}$	skew symmetric
$a^1 \wedge b^2$	$\sum \mathbb{W}_{12}$	$\mathbb{W}_{12} =$ $\mathbb{W}_{21}^T$
$b^2 \wedge a^1$	$\sum \mathbb{W}_{21}$	
δ^*	$\mathbb{M}_k^{-1}\mathbb{D}_k^T\mathbb{M}_{k+1}$	rectangular
$\mathcal{D}$	$\mathbb{M}_k^{-1}\mathbb{D}_k^T\mathbb{M}_{k+1}\mathbb{D}_k + \mathbb{D}_{k-1}\mathbb{M}_{k-1}^{-1}\mathbb{D}_{k-1}^T\mathbb{M}_k$	square
$\overset{D}{\star}\colon C^1 \mapsto C^2$	$(\mathbb{W}_{12}, \mathbb{M}_3)$	pair

6. Examples of reconstruction operators. For simplicity we present examples of reconstruction operators in two-dimensions and restrict attention to operators that translate 1-cochains to 1-forms. We will consider three operators $\mathcal{I} : C^1 \mapsto \Lambda^1(L^2, \Omega)$, one of which will be conforming. To explain the action of these operators it suffices to consider a space C_1 consisting of a single 1-chain $c_1 = \sum_{i=1}^{3} c_1^i$ which is a boundary of a 2-simplex c_2 that forms the space C_2. In two dimensions c_2 is a triangle and the 1-cells $\{c_1^i\}$ are its edges. Two edges, c_1^i and c_1^j, intersect at a vertex c_0^k, $k \neq i, j$. The set $\{c_0^k\}$ forms the space C_0.

Using the isomorphism (3.6), the elements of C^1 can be written as $c^1 = \sum_{i=1}^{3} a_i c_1^i$ where $a_i = \int_{c_1^i} \omega$ for some $\omega \in \Lambda^1(d, \Omega)$. The value of a_i gives the circulation of the vector field $\mathbf{u}$, associated with ω, along the edge c_1^i.

Covolume reconstruction. To define the covolume reconstruction operator [57] the simplex c_2 is divided into three subsimplices c_2^i by connecting the circumcenter of c_2 with its vertices c_0^i as shown in Fig. 3. Each subsimplex is bordered by exactly one of the edges c_1^i; we denote that subsimplex by c_2^i.

The covolume reconstruction operator maps the 1-cochain c^1 into a 1-form $\omega^{\mathbf{u}}$ whose associated vector field $\mathbf{u}$ is piecewise constant on each subsimplex, determined according to the rule

$$\mathbf{u}|_{c_2^i} = a_i c_1^i; \quad i = 1, 2, 3\,. \tag{6.1}$$

The range of the operator defined in (6.1) is in the Hilbert space $\Lambda^1(L^2, \Omega)$ but not in the Sobolev space $\Lambda^1(d, \Omega)$. Therefore, covolume reconstruction is not conforming. A unique property of covolume reconstruction is that derived operators have local stencils and that there is a discrete $\star$ star operation that is compatible with the natural inner product [57]. As a result, the matrix $\mathbb{M}$ that gives the action of the natural inner product is diagonal

$$\mathbb{M} = \begin{pmatrix} h_1 h_1^{\perp} & 0 & 0 \\ 0 & h_2 h_2^{\perp} & 0 \\ 0 & 0 & h_3 h_3^{\perp} \end{pmatrix}. \tag{6.2}$$

In (6.2) h_i is the length of c_1^i and $h_i^{\perp}$ is the length of the perpendicular from the circumcenter to c_1^i.

These properties follow from the fact that covolume reconstruction can be associated with cochains on a Voronoi-Delaunay grid complex; see [42, 44, 57]. This association also implies that existence of the covolume reconstruction is contingent upon the existence of the Voronoi regions and so the simplexes must satisfy an angle condition [44].

Mimetic reconstruction. Mimetic reconstruction [33] acts in a similar way to recover a form $\omega^{\mathbf{u}}$ whose associated vector field $\mathbf{u}$ is a piecewise constant on c_2. As a result, the reconstructed form is in the Hilbert space $\Lambda^1(L^2, \Omega)$ but not in the Sobolev space $\Lambda^1(d, \Omega)$. The main difference between covolume and mimetic reconstruction is in the choice of the subregions. In the mimetic approach, the subregions are associated with the vertices c_0^k of c_2, have quadrilateral shapes, and are bordered by the edges c_1^i and c_1^j, $i, j \neq k$; see Fig. 3. Each subregion is determined by connecting the midpoint of c_1^i with an arbitrary but fixed point inside the triangle. We denote the subregion associated with the vertex c_0^k by q_2^k and its area by V_k. The mimetic reconstruction operator builds on c_2 the following piecewise constant field:

$$\mathbf{u}|_{q_2^k} = a_i c_1^i + a_j c_1^j; \quad k = 1, 2, 3;\; i, j \neq k\,. \tag{6.3}$$

Mimetic reconstruction is less restrictive than the covolume $\mathcal{I}$ because existence of the subregions is not contingent upon the circumcenter being inside the triangle. However, mimetic reconstruction gives rise to

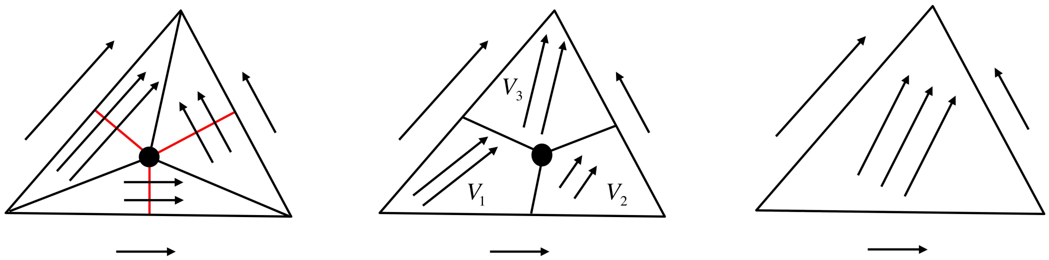

FIG. 3. *The reconstruction operators are shown for the 1-cochains: covolume, mimetic, and Whitney, respectively. In the first figure, the covolume reconstruction operator divides the simplex into three subsimplices by connecting the circumcenter of with its vertices. Each subsimplex is bordered by exactly one of the edges. The covolume reconstruction operator maps the 1-cochain into a 1-form whose associated vector field is piecewise constant on each subsimplex. In the second figure, mimetic reconstruction acts in a similar way to recover a form with a piecewise constant vector field. In the mimetic approach, the subregions are associated with the vertices, have quadrilateral shapes, and are bordered by the edges adjacent to each vertex. The third figure of the Whitney map is an example of a regular mimetic reconstruction operator. In contrast to the previous two reconstruction operators, the Whitney map builds a polynomial 1-form from the cochain using a basis of polynomial 1-forms associated with the edges.*

nonlocal derived operators [34]. If ϕ_k is the angle associated with the vertex c_0^k, the inner product matrix on c_2 is given by

$$\mathbb{M} = \begin{pmatrix} \dfrac{V_2}{\sin^2\phi_2} + \dfrac{V_3}{\sin^2\phi_3} & \dfrac{V_3\cos\phi_3}{\sin^2\phi_3} & \dfrac{V_2\cos\phi_2}{\sin^2\phi_2} \\ \dfrac{V_3\cos\phi_3}{\sin^2\phi_3} & \dfrac{V_1}{\sin^2\phi_1} + \dfrac{V_3}{\sin^2\phi_3} & \dfrac{V_1\cos\phi_1}{\sin^2\phi_1} \\ \dfrac{V_2\cos\phi_2}{\sin^2\phi_2} & \dfrac{V_1\cos\phi_1}{\sin^2\phi_1} & \dfrac{V_1}{\sin^2\phi_1} + \dfrac{V_2}{\sin^2\phi_2} \end{pmatrix}. \tag{6.4}$$

Whitney reconstruction. The Whitney map [24, 59] is an example of a conforming reconstruction operator whose range is in the Sobolev space $\Lambda^1(d,\Omega)$. In contrast to the previous two reconstruction operators, the Whitney map builds a polynomial 1-form on c_2 from the cochain c^1 using a *basis* of polynomial 1-forms associated with the edges c_1^i. The basis 1-forms are defined by the formula

$$\omega_k = t_i dt_j - t_j dt_i; \quad i, j \neq k,\ i < j\,, \tag{6.5}$$

where t_i are the barycentric coordinates. The vector field corresponding to the basis 1-form is given by

$$\mathbf{u}_k = t_i \nabla t_j - t_j \nabla t_i; \quad i, j \neq k,\ i < j\,.$$

Therefore, the Whitney reconstruction map translates the cochain c^1 to the 1-form $\omega^1 = \sum_{k=1}^3 a_k\omega_k$ with a vector field

$$\mathbf{u} = \sum_{k=1}^{3} a_k (t_i \nabla t_j - t_j \nabla t_i)\,. \tag{6.6}$$

The reconstructed image of c^1 is in the smooth space $\Lambda^1(\Omega)$. When K consists of more than one 2-simplex, the range of the Whitney map contains piecewise polynomial 1-forms obtained by gluing together the reconstructed images from the individual triangles. It is possible to show [24] that the resulting 1-forms are in the Sobolev space $\Lambda^1(d,\Omega)$.

7. Application to PDEs. We consider mimetic discretizations of the elliptic boundary value problems

$$\left\{\begin{array}{rcll} -\triangle_0\phi & = & f & \\ \phi & = & 0 & \text{on } \Gamma_1 \\ \mathbf{n}\cdot\nabla\phi & = & 0 & \text{on } \Gamma_2 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} -\triangle_3\psi & = & f & \\ \mathbf{n}\cdot\nabla\psi & = & 0 & \text{on } \Gamma_1 \\ \psi & = & 0 & \text{on } \Gamma_2 \end{array}\right. \tag{7.1}$$

respectively. Note that $-\triangle_0 = d^*d$ and $-\triangle_3 = dd^*$. To better illustrate the formation of the discrete mimetic equations we use equivalent first-order formulations of (7.1):

$$\left\{\begin{array}{rcll} d\phi-\mathbf{u} & = & 0 & \\ d^*\mathbf{u} & = & f & \\ \phi & = & 0 & \text{on } \Gamma_1 \\ \mathbf{n}\cdot\mathbf{u} & = & 0 & \text{on } \Gamma_2 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} d^*\psi-\mathbf{v} & = & 0 & \\ d\mathbf{v} & = & f & \\ \mathbf{n}\cdot\mathbf{v} & = & 0 & \text{on } \Gamma_1 \\ \psi & = & 0 & \text{on } \Gamma_2 \end{array}\right. \tag{7.2}$$

. In (7.2) the variables acted upon by d, their boundary conditions, and the equations involving d are called *primal*. The other variables, boundary conditions and equations, are called *dual*.

7.1. Direct mimetic discretization. In the direct approach we use that d and $\mathcal{R}$ commute and apply $\mathcal{R}$ to translate the primal equation and boundary condition to combinatorial cochain equations. Reduction of the primal equation fixes the type of cochains in the discrete model. The discrete primal equation is uniquely determined by the mesh topology of the triangulation K and does not require a reconstruction operator $\mathcal{I}$. However, this operator is needed for the discretization of the dual equation. Because $\mathcal{R}$ and d^* do not commute, the discrete dual equation cannot be obtained by an application of $\mathcal{R}$. Instead, we derive it by using the discrete adjoint δ^* to mimic the analytic dual. Therefore, the discrete dual equation depends on the choice of the reconstruction map $\mathcal{I}$ and is not unique. Note that $\mathcal{I}$ is only needed to induce the adjoint and does not have to be a conforming reconstruction operator.

For $\triangle_0$ the primal variable ϕ is 0-form and the dual variable $\mathbf{u}$ is 1-form. We approximate them by $\phi_0 = \mathcal{R}\phi \in C^0_1$ and $\mathbf{u}_1 = \mathcal{R}\mathbf{u} \in C^1$. For $\triangle_3$ the primal variable $\mathbf{v}$ is a 2-form $\mathbf{v}$ and the dual variable is a 3-form ψ. They are approximated by $\mathbf{v}_2 = \mathcal{R}\mathbf{v} \in C^2_1$ and $\psi_3 = \mathcal{R}\psi \in C^3$, respectively. Applying $\mathcal{R}$ to the primal equations in (7.2) and using CDP1 gives

$$0 = \mathcal{R}(d\phi - \mathbf{u}) = \delta\mathcal{R}\phi - \mathcal{R}\mathbf{u} = \delta\phi_0 - \mathbf{u}_1 \quad \text{and} \quad 0 = \mathcal{R}(d\mathbf{v} - f) = \delta\mathcal{R}\mathbf{v} - \mathcal{R}f = \delta\mathbf{v}_2 - f_3\,,$$

respectively. Hence, the direct mimetic models for $\triangle_0$ and $\triangle_3$ are

$$\left\{\begin{array}{rcl} \delta\phi_0-\mathbf{u}_1 & = & 0 \\ \delta^*\mathbf{u}_1 & = & f_0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \delta^*\psi_3-\mathbf{v}_2 & = & 0 \\ \delta\mathbf{v}_2 & = & f_3 \end{array}\right. \tag{7.3}$$

, respectively. In (7.3) the primal boundary conditions on Γ_1 constrain the spaces for the primal variables. The boundary conditions on Γ_2 are enforced weakly through the definition of δ^* as adjoint to δ.

The methods in (7.3) can be realized using any one of the three reconstruction operators (6.1), (6.3), or (6.6). With the covolume reconstruction the derived adjoint δ^* has local stencil and (7.3) is equivalent to a finite volume method on Delaunay-Voronoi grid complex. With the mimetic and Whitney reconstructions the stencil of δ^* is not local. For these two operators (7.3) is a conservative finite difference scheme on an unstructured grid.

If $\mathbf{u}_1$ and $\mathbf{v}_2$ are eliminated from (7.3) we obtain the equations

$$\delta^*\delta\phi_0 = f_0 \quad \text{and} \quad \delta\delta^*\phi_3 = f_3 \tag{7.4}$$

that represent direct discretizations of the equations in (7.1) by the discrete Laplace operators $\mathcal{D}_0 = \delta^*\delta$ and $\mathcal{D}_3 = \delta\delta^*$.

7.2. Conforming mimetic discretization. In the conforming approach, the analytic equations are restricted to finite dimensional spaces in the range of $\mathcal{IR}$. In contrast to the direct approach, where only discrete derivatives are used, this requires $\mathcal{I}$ to be conforming. Assuming that such $\mathcal{I}$ is given, we approximate ϕ and $\mathbf{u}$ by $\phi_0^h \in \Lambda_1^0(d,K)$ and $\mathbf{u}_1^h \in \Lambda^1(d,K)$, respectively. For ψ and $\mathbf{v}$ the approximations are $\psi_3^h \in \Lambda^3(K)$ and $\mathbf{v}_2^h \in \Lambda_1^2(d,K)$. The conforming discretizations of (7.2) are given by

$$\left\{ \begin{array}{rcl} d\phi_0^h - \mathbf{u}_1^h & = & 0 \\ d^*\mathbf{u}_1^h & = & f_0^h \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} d^*\psi_3^h - \mathbf{v}_2^h & = & 0 \\ d\mathbf{v}_2^h & = & f_3^h \end{array} \right. \tag{7.5}$$

, respectively, where $f_0^h = \mathcal{IR}f$ and $f_h^3 = \mathcal{IR}f$.

In contrast to the direct methods in (7.3), the methods in (7.5) cannot be realized by the covolume or the mimetic reconstruction operators because they are not conforming. However, for (7.5) we can use the Whitney map (6.6). In this case, the scheme where the scalar is the primal variable reduces to the familiar Galerkin finite element method in which the scalar is approximated by continuous, piecewise linear polynomial finite elements on simplices. The second scheme, where the scalar is the dual variable, reduces to a mixed Galerkin method in which the scalar is approximated by a piecewise constant and the vector is approximated by the lowest order Raviart-Thomas spaces [18, 46]. For this reason we will call the schemes in (7.5) *Galerkin* and *mixed Galerkin*, respectively. The Whitney map has been extensively used in computational electromagnetism where it gives rise to the lowest-order Nedelec edge elements [14, 29, 40, 41].

THEOREM 7.1. *Assume that $\mathcal{I}$ is a conforming reconstruction operator, then the direct and the conforming mimetic models are equivalent.*

Proof. We give the details for $\triangle_0$; the proofs for $\triangle_3$ are very similar. For $\phi_0^h \in \Lambda_1^0(d,K)$ and $\mathbf{u}_1^h \in \Lambda^1(d,K)$ there exist $\phi \in \Lambda_1^0(d,\Omega)$ and $\mathbf{u} \in \Lambda^1(d,\Omega)$, such that $\phi_0^h = \mathcal{IR}\phi$ and $\mathbf{u}_1^h = \mathcal{IR}\mathbf{u}$, respectively. Using (4.6)

$$0 = d\phi_0^h - \mathbf{u}_1^h = d(\mathcal{IR}\phi) - \mathcal{IR}\mathbf{u} = \mathcal{I}\delta\mathcal{R}\phi - \mathcal{IR}\mathbf{u} = \mathcal{I}(\delta\mathcal{R}\phi - \mathcal{R}\mathbf{u}) = \mathcal{I}(\delta\phi_0 - \mathbf{u}_1)\,,$$

where $\phi_0 = \mathcal{R}\phi$ and $\mathbf{u}_1 = \mathcal{R}\mathbf{u}$. From (4.5) we conclude that $\delta\phi_0 - \mathbf{u}_1 = 0$, that is, the degrees of freedom of ϕ_0^h and $\mathbf{u}_1^h$ solve the direct equation. To prove equivalence of the dual equations note that for $\xi_0 \in C_1^0$ - arbitrary, and $\xi_0^h = \mathcal{I}\xi_0$ formula (4.15) implies the identity

$$(d^*\mathbf{u}_1^h, \xi_h^0)_\Omega = (\delta^*\mathbf{u}_1, \xi_0)_\Omega$$

while definition of f_0^h and the L^2 inner product give that

$$(f_h^0, \xi_h^0)_\Omega = (\mathcal{IR}f, \mathcal{I}\xi_0)_\Omega = (\mathcal{R}f, \xi_0)_\Omega = (f_0, \xi_0)\,.$$

Combining the two equations shows that

$$(\delta^*\mathbf{u}_1, \xi_0)_\Omega = (f_0, \xi_0) \quad \forall \xi_0 \in C_1^0 \quad \text{or} \quad \delta^*\mathbf{u}_1 = f_0\,.$$

Therefore, $\mathbf{u}_1^h$ solves $d^*\mathbf{u}_1^h = f_0^h$ if and only if $\mathbf{u}_1$ solves the direct dual equation $\delta^*\mathbf{u}_1 = f_0$. ◻

From this theorem we can conclude that realizations of the direct scheme (7.3) and the conforming scheme (7.5) by the Whitney map lead to two completely equivalent discretizations of the PDEs (7.2). Further connections between direct and conforming methods can be established by choosing specific quadrature points to compute the integrals in the conforming method [12, 13, 19]. Note that quadrature selection can be interpreted as yet another choice for the reconstruction operator.

7.3. Mimetic discretization with weak material laws. The first-order systems in (7.2) can be combined into a single problem by keeping the two primal equations and adding the constitutive laws

$$\mathbf{u} = \star\mathbf{v} \quad \text{and} \quad \psi = \star\phi \tag{7.6}$$

that express the dual variables in terms of the primal variables. We write the new system as

$$\left\{ \begin{array}{rcl} d\phi - \mathbf{u} & = & 0 \\ d\mathbf{v} + g\psi & = & f \end{array} \right. \quad \text{and} \quad \begin{array}{l} \mathbf{u} = \star\mathbf{v} \\ \psi = \star\phi \end{array} \tag{7.7}$$

where g is a function that can be identically zero; see [15, 14, 30, 55, 56] for discussions of such *factorization* diagrams. Instead of trying to approximate (7.7), which would require us to deal with the material laws and a discrete $\star$ operation, we first transform this system into an equivalent constrained optimization problem and then discretize that problem. Let

$$\mathcal{J}(\phi, \mathbf{u}; \psi, \mathbf{v}) = \frac{1}{2}\left(\|\psi - \star\phi\|^2 + \|\mathbf{u} - \star\mathbf{v}\|^2\right).$$

The optimization problem: *find* $(\phi, \mathbf{u}) \in \Lambda_1^0(\Omega) \times \Lambda^1(\Omega)$ *and* $(\psi, \mathbf{v}) \in \Lambda^3(\Omega) \times \Lambda_1^2(\Omega)$ *such that for all* $(\hat{\phi}, \hat{\mathbf{u}}) \in \Lambda_1^0(\Omega) \times \Lambda^1(\Omega)$ *and* $(\hat{\psi}, \hat{\mathbf{v}}) \in \Lambda^3(\Omega) \times \Lambda_1^2(\Omega)$

$$\mathcal{J}(\phi, \mathbf{u}; \psi, \mathbf{v}) \le \mathcal{J}(\hat{\phi}, \hat{\mathbf{u}}; \hat{\psi}, \hat{\mathbf{v}}) \quad \textit{subject to } d\hat{\phi} - \hat{\mathbf{u}} = 0 \textit{ and } d\hat{\mathbf{v}} + g\hat{\psi} = f \tag{7.8}$$

is an equivalent to (7.7). We use this optimization problem to devise direct and conforming mimetic methods in which material laws are enforced weakly and no explicit construction of a discrete $\star$ operation is required.

The idea is to approximate the four variables in (7.8) by the same cochains as in (7.3) or by the same conforming spaces as in (7.5). In the first case we have the constrained optimization problem *find* $(\phi_0, \mathbf{u}_1) \in C_1^0 \times C^1$ *and* $(\psi_3, \mathbf{v}_2) \in C^3 \times C_1^2$ *such that for all* $(\hat{\phi}_0, \hat{\mathbf{u}}_1) \in C_1^0 \times C^1$ *and* $(\hat{\psi}_3, \hat{\mathbf{v}}_2) \in C^3 \times C_1^2$

$$\mathcal{J}(\phi_0, \mathbf{u}_1; \psi_3, \mathbf{v}_2) \le \mathcal{J}(\hat{\phi}_0, \hat{\mathbf{u}}_1; \hat{\psi}_3, \hat{\mathbf{v}}_2) \quad \textit{subject to } \delta\hat{\phi}_0 - \hat{\mathbf{u}}_1 = 0 \textit{ and } \delta\hat{\mathbf{v}}_2 + g\hat{\psi}_3 = f_3 \tag{7.9}$$

which gives a direct mimetic method. If, instead, we use the conforming spaces, the optimization problem is *find* $(\phi^h, \mathbf{u}^h) \in \Lambda_1^0(d, K) \times \Lambda^1(d, K)$ *and* $(\psi^h, \mathbf{v}^h) \in \Lambda^3(d, K) \times \Lambda_1^2(d, K)$ *such that for all* $(\hat{\phi}^h, \hat{\mathbf{u}}^h) \in \Lambda_1^0(d, K) \times \Lambda^1(d, K)$ *and* $(\hat{\psi}^h, \hat{\mathbf{v}}^h) \in \Lambda^3(d, K) \times \Lambda_1^2(d, K)$

$$\mathcal{J}(\phi^h, \mathbf{u}^h; \psi^h, \mathbf{v}^h) \le \mathcal{J}(\hat{\phi}^h, \hat{\mathbf{u}}^h; \hat{\psi}^h, \hat{\mathbf{v}}^h) \quad \textit{subject to } d\hat{\phi}^h - \hat{\mathbf{u}}^h = 0 \textit{ and } d\hat{\mathbf{v}}^h + g\hat{\psi}^h = f^h \tag{7.10}$$

and we have a conforming mimetic method.

Because C^{n-k} and C^k and $\Lambda^k(d, K)$ and $\Lambda^{n-k}(d, K)$ have different dimensions, the primal and the dual variables cannot be related by a one-to-one map. Instead, we minimize their discrepancy in L^2 sense and so the material laws are imposed in a weak sense.

To realize (7.9) we can use any one of the three reconstruction operators (6.1), (6.3), or (6.6) and obtain a finite-difference like scheme. For the conforming method (7.10) we cannot use the covolume or the mimetic reconstruction, but we can use the Whitney map (6.6) to obtain a finite element-like scheme. We note that with the Whitney map realizations of (7.9) and (7.5) are completely equivalent.

For further details on mimetic discretizations with weak constitutive laws and their connection to least-squares minimization principles we refer to [7, 8, 9]. Examples of this idea in magnetostatics can be found in [14] and [20].

8. Conclusions. We described a general framework for mimetic discretizations that uses two basic operators to define all discrete structures. Scalars and vectors are translated to differential forms and then reduced to cochains. *Combinatorial* differentiation and integration operations are induced by the De Rham map which effects the reduction to cochains. The *natural* inner product and wedge product are defined by using a reconstruction operator that translates cochains back to forms. The inner product induces an adjoint derivative and a discrete Laplacian. Together with the combinatorial and natural operations these *derived* operations comprise the core of the mimetic framework.

The choice of the natural and derived operations is determined by the internal consistency of the framework. The natural definitions of the inner product and the wedge product are not compatible with a natural definition of the discrete $\star$. As a result, a consistent discrete framework requires a choice of its primary operation. We choose the primary operation to be the natural inner product on real cochain spaces. It would be equally valid to choose the primary operation to be the discrete $\star$ and its construction to be the principal computational task.

We choose to base our mimetic framework on the natural inner product instead of the $\star$ operation because of the complications that arise in the construction of the latter and because the inner product is sufficient to induce a combinatorial Hodge theory on cochains. For problems that require approximations of material laws we propose to consider constrained optimization formulations that enforce the laws weakly, instead of using their explicit discretization. In all other cases, our framework offers the choice of direct and conforming methods. Direct methods are representative of the type of discretizations that arise in FV and FD methods while conforming methods are typical of FE. We demonstrated that for regular reconstruction operators direct and conforming methods are equivalent. This opens up a possibility to carry out error analysis of direct mimetic methods by using variational tools from FE. Some recent examples are the analyses in [12, 13, 19].

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