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PERIODIC SOLUTIONS OF A LOGISTIC DIFFERENCE EQUATION*

F. C. HOPPENSTEADT AND J. M. HYMAN†

Abstract. Periodic solutions of the difference equation $x_{n+1}=mx_n(1-x_n)$ are studied for values of m, $0\leqq m\leqq 4$. It is shown that as m increases from zero, solutions having successively higher periods branch from old ones until the value $\mathbf{m}\doteq 3.57$ is reached, after which there is an infinity of periodic solutions. The solution set is said to be chaotic if there is an infinity of periodic solutions.

This investigation focuses on solution behavior in the chaotic regime. It is shown how as m increases from $\mathbf{m}$, solutions having various other periods are added to the solution set until at $m\doteq 3.83$, solutions of period three, and hence all periods are present. Finally, density functions are calculated numerically to describe the dynamics of solutions in portions of the chaotic regime.

1. Introduction.

The difference equation

$$x_{n+1}=mx_n(1-x_n) \tag{1}$$

arises in models of population dynamics as a discrete-time version of the logistic equation and as an approximation to a nonlinear renewal equation. For example, a nonlinear renewal equation for birth rate,

$$B(t)=(m/\mu)\int_{-1}^{\mu-1} B(t+s)[1-B(t+s)]_+\,ds,$$

was derived in [8] as a model of a population having a density dependent maternity function $((m/\mu)[1-B]_+)$ and having a high fertility (m/μ) over a short reproductive window (i.e., for rescaled ages $1-\mu\leqq a\leqq 1$). The equation (1) results on passing to the limit $\mu=0$. Equation (1) and similar equations have appeared in many different contexts, and their solutions have been the subject of many investigations. Even though (1) is one of the simplest nonlinear difference equations, its solutions exhibit a wide range of interesting dynamic behavior. In this note, we will describe the periodic solutions of (1) by means of simple geometric arguments and numerical computations.

Attention is restricted here to solutions of (1) lying in the unit interval. In fact, for any initial value lying in the unit interval, the corresponding solution of (1) will satisfy $0\leqq x_n\leqq 1$ provided $0\leqq m\leqq 4$. Therefore, we consider (1) only for values of m satisfying $0\leqq m\leqq 4$.

Instead of studying equation (1), we consider an equivalent problem of analyzing iterates of the function $f(x)=mx(1-x)$. For $n=2, 3, 4, \cdots$, we set $f^n(x)=f[f^{n-1}(x)]$. Thus, (1) is equivalent to $x_{n+1}=f(x_n)=\cdots=f^{n+1}(x_0)$.

A point $\mathbf{x}$ will be a *j-period solution* (or point) of (1) if it has least period j; i.e., if $f^j(\mathbf{x})=\mathbf{x}$, but $f^k(\mathbf{x})\neq\mathbf{x}$ for $k=0,\cdots,j-1$. The *orbit* of a j-period point is the set

$$O(\mathbf{x})=\{\mathbf{x}, f(\mathbf{x}),\cdots, f^{j-1}(\mathbf{x})\}.$$

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The solution set of (1) is called *chaotic* if there is an infinity of periodic solutions having distinct periods.

2. Analysis of the solution set of (1).

2.1. $\mathbf{0 \leqq m < m_8 \doteq 3.54}$. Solutions of (1) for these values of m are easily described by explicit calculation. For $0 \leqq m \leqq 1$, $x = 0$ is the only periodic solution, and all points in $[0, 1]$ approach zero under iterations of f. For $1 < m \leqq 3$, there is another one-period solution, $x = y_1 \equiv (m-1)/m$. It is stable in that all points in $(0, 1)$ approach y_1 under iterations of f. These stability properties are easily established by a geometric argument described, for example, in [1].

There is a two-period point y_2 which branches from y_1 at $m = 3$. It is stable in that all but countably many points in $[0, 1]$ evolve into $O(y_2)$. The solutions $x = 0$ and $x = y_1$ are unstable for $m > 3$. As m passes through a value $m_4 \doteq 3.44$, a solution of period four branches from y_2. Again an exchange of stabilities occurs, and y_4 is stable on an interval $m_4 \leqq m < m_8$. These results are described in Fig. 1. At this point, explicit calculations become less interesting, and we proceed by other methods to describe the solutions.

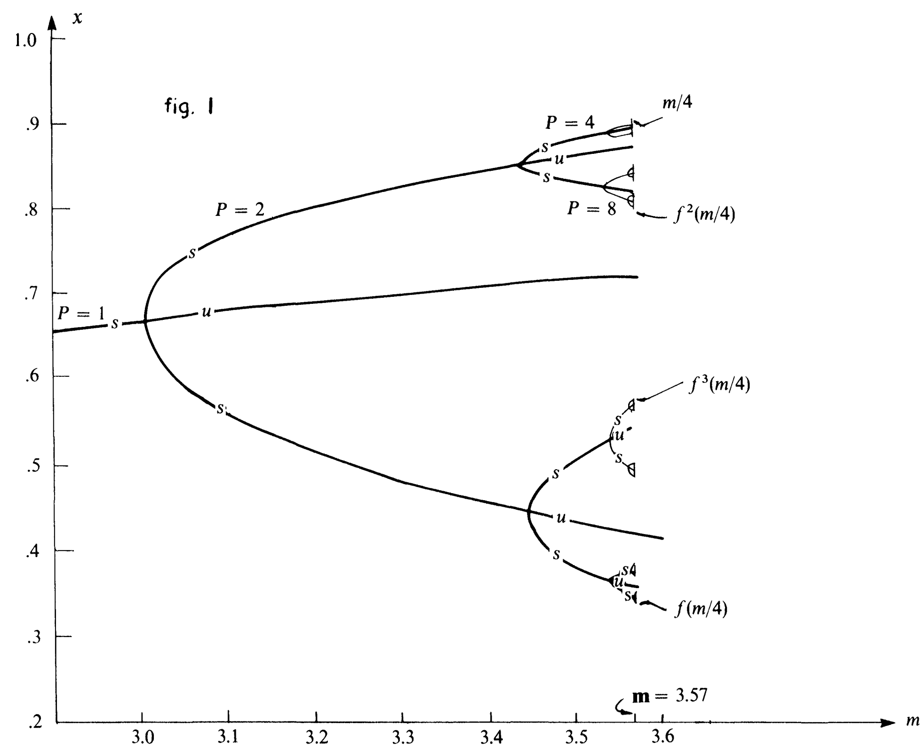

FIG. 1. *The periodic solutions of* (1) *are plotted here for values of* $m \in [2.9, 3.57]$. *Stable branches are labeled S and unstable ones U. The periodic solutions for* $m = \mathbf{m}$ *are contained in the intervals* $[f(m/4), f^3(m/4)]$ *and* $[f^2(m/4), m/4]$.

2.2. $\mathbf{m}_8 \leqq \mathbf{m} \leqq 4$. Certain maxima of iterates of f can be easily found. The function $f^2(x)$ has two maxima. We denote the larger one by $\mathbf{x}_2(m)$, and it lies in the interval $(y_1, 1)$. Thus, $f^2(\mathbf{x}_2(m)) = m/4$. This function is plotted in Fig. 2, which shows that there is a unique value of m for which $\mathbf{x}_2(m) = m/4$. This value is denoted by $\mathbf{m}_2 \doteq 3.24$. In addition, we let x_2^* denote the unique value of $x \in (y_1, 1)$ such that $f^2(x_2^*) = y_1$. (See Fig. 3.)

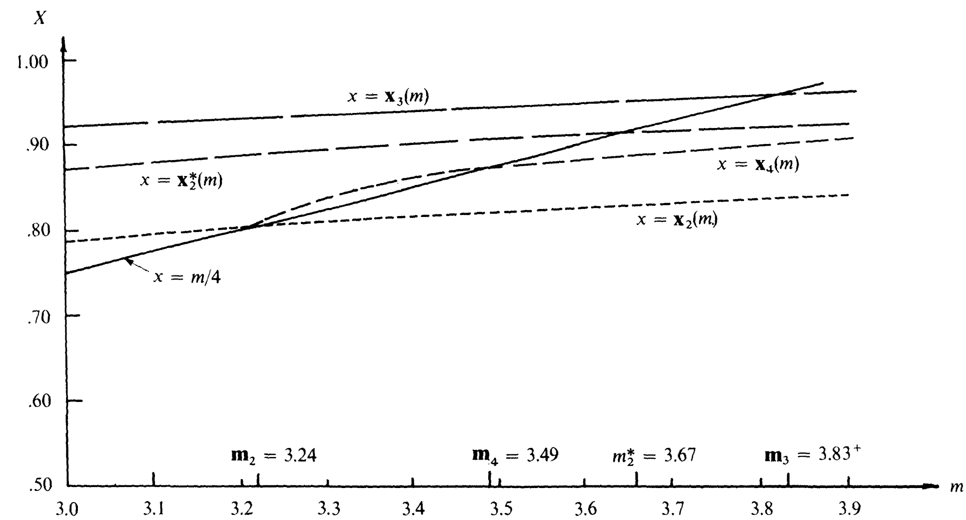

FIG. 2. *This contains some technical information used in the text.*

For $m > \mathbf{m}_2$, a polygonal path P joining y_1 to $f(\mathbf{x}_2)$ can be constructed as shown in Fig. 3. By reflecting the horizontal segments of P into f^2 and f^3, respectively, we find monotone sequences $\{\mathbf{x}_{2k}\}$ and $\{\mathbf{x}_{2k+1}\}$ approaching x_2^* from the left and right respectively. Moreover, $f^{2k}(\mathbf{x}_{2k}) = m/4$ and $f^{2k+1}(\mathbf{x}_{2k+1}) = m/4$ for $k = 1, 2, \cdots$. The function $\mathbf{x}_3(m)$ is also plotted in Fig. 2, and we let $\mathbf{m}_3$ denote the value of m for which $\mathbf{x}_3(m) = m/4$.

It follows from the intermediate value theorem that if $x_l(m) \leqq m/4$, then f^l has a fixed point (i.e., the graph of $f^l(x)$ crosses the line $i(x) = x$) at some value in the interval $x_l(m) \leqq x < 1$. In addition, if $\mathbf{x}$ is a fixed point of f^p and has period q, then q divides p. If this were not the case, we could write $p = kq + r$ for some integers k and r, $1 \leqq r \leqq q - 1$. Then

$$\mathbf{x} = f^p(\mathbf{x}) = f^r[f^{kq}(\mathbf{x})] = f^r(\mathbf{x}),$$

and so $\mathbf{x}$ has period $r < q$. This contradiction shows that $r = 0$ and that p is a multiple of q.

We will now use the fact established in [2] that the existence of a solution of period three implies the existence of solutions having all periods.

Let m_2^* denote the value of m such that $x_2^*(m) = m/4$. It follows that for $m \geqq m_2^*$, the numbers $\mathbf{x}_{2k}(m)$, $k = 1, 2, \cdots$, satisfy $\mathbf{x}_{2k} < m/4$. Since $\mathbf{x}_6 < x_2^*$, there is a value $\mathbf{m}_6$ such that $\mathbf{x}_6 < m/4$ for $m > \mathbf{m}_6$. Thus, for $m > m_6$, $f^{2 \cdot 3}$ has a fixed

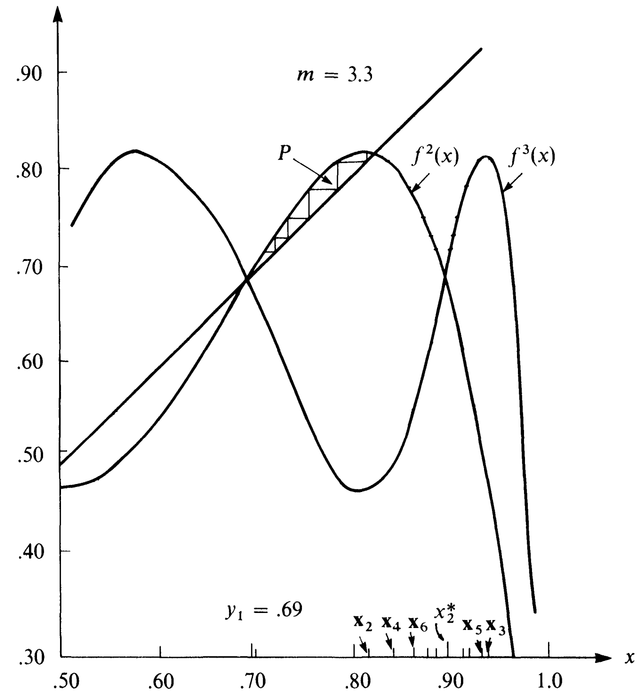

FIG. 3. *For $m > \mathbf{m}_2 = 3.24$, the polygonal path P can be drawn connecting $(\mathbf{x}_2, m/4)$ with (y_1, y_1). The horizontal segments of P are reflected in f^2 and f^3 giving sequences $\{\mathbf{x}_{2k}\}$ and $\{\mathbf{x}_{2k+1}\}$, respectively, which converge monotonically to x_2^*. These points define maxima for the corresponding iterates of f. In particular, for one of these points $\mathbf{x}_n$, the function f^n satisfies $f^n(\mathbf{x}_n) = m/4$.*

point. Since this point cannot have period two or period three, this implies that f^2 has a fixed point of period three, and so (1) has solutions of all even periods for $m \geqq m_6$.

As m increases from m_6 to $\mathbf{m}_3$, solutions having successively lower odd periods are added to the solution set until a value $m_3 < \mathbf{m}_3$ is reached when solutions of period three arise, and hence solutions of all periods are present.

Finally, we focus attention on the interval $m_8 < m < m_2^*$. First, consider f^4. Since $\mathbf{x}_4(m) < x_2^*(m)$, there is a value $\mathbf{m}_4$ such that $\mathbf{x}_4(m) = m/4$ for $m = \mathbf{m}_4$. The construction of a polygonal path P_2 can be carried out also for f^4 when $m > \mathbf{m}_4$. In particular, there is a point $x_4^* \in (y_2, x_2^*)$ such that $f^4(x_4^*) = y_2$ for $m \geqq \mathbf{m}_4$, and there are monotone sequences $\{\tilde{\mathbf{x}}_{4k}\}$ and $\{\tilde{\mathbf{x}}_{4k+1}\}$ converging to x_4^* from the left and right, respectively, and having properties similar to those of $\{\mathbf{x}_{2k}\}$ and $\{\mathbf{x}_{2k+1}\}$. We define m_4^* to be the value of m for which $x_4^* = m/4$; ($m_4^* \doteq 3.6$). For $m \geqq m_4^*$, there is a solution of period $4k$ for each prime number k. In particular, there is a solution of period $4 \cdot 3$. Therefore, f^4 has a three-period point, and so (1) has solutions of all periods of the form $4k$ provided $m \geqq m_4^*$.

This construction can be continued with the results being sequences $\{\mathbf{m}_{2^l}\}$ and $\{m_{2^l}^*\}$ which satisfy

$$\mathbf{m}_2 < \mathbf{m}_4 < \mathbf{m}_8 < \cdots < m_8^* < m_4^* < m_2^*.$$

We define $\mathbf{m} = \lim\,(l \to \infty) m_{2^l}^*$. Our calculated values show that $\mathbf{m} \doteq 3.57$.

By the arguments given above, for $m \geqq m^*_{2^l}$, solutions having all periods of the form $2^l k$ exist for all $k = 1, 2, 3, \cdots$. This shows that the solution set is chaotic for $m > \mathbf{m}$ and that $\mathbf{m}$ plays the role of a threshold of chaos.

3. Numerical description of the solution set of (1). In an effort to gain a clearer description of the structure of the chaotic solution set, we calculated frequency distributions of iterates of f. These demonstrate several interesting phenomena.

The calculations were performed by iterating 500 times each of 100 points in (0, 1), and then counting the number of iterates which entered each of 1000 cells. Some sample calculations are described in Figs. 4–7. In these, $\varphi(x)$ measures the number of iterates after the first 50 which entered the cell which contains x. For each m in the interval $0 \leqq m \leqq \mathbf{m}$, there were only finitely many peaks, they were quite sharp, and they were located over the stable orbit corresponding to the particular choice of m. In the terminology of ergodic theory, these reflect an invariant measure whose density function is an appropriate linear combination of delta functions with support on the particular periodic points.

The calculations described in Figs. 4–7 were carried out for several values of m in the chaotic regime. First, the sensitivity of density functions to changes in m is illustrated for values of m near the critical point m_3. For $m = 3.825$, the solutions migrate under iterates of f in a chaotic way, and the calculation of φ suggests that there is a regular (with respect to Lebesgue measure) density function which describes the dynamics of the solutions. For $m = 3.83$, the solutions are converging to the solution of period three, and the solution set appears to

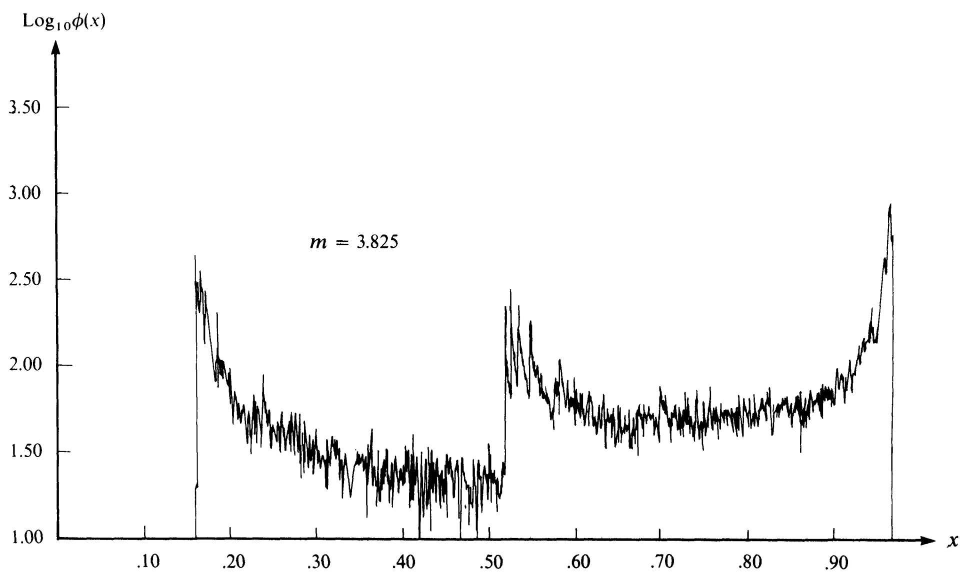

FIG. 4. *The numerical calculation described in the text gives this distribution for* $m = 3.825$. $\varphi(x)$ *measures the number of* 500 *iterates of each of* 100 *points in* (0, 1) *which entered the cell containing* x. *Here* $\log_{10}\varphi(x)$ *is plotted. Points migrate according to some regular invariant measure whose density function has its graph reflected here.*

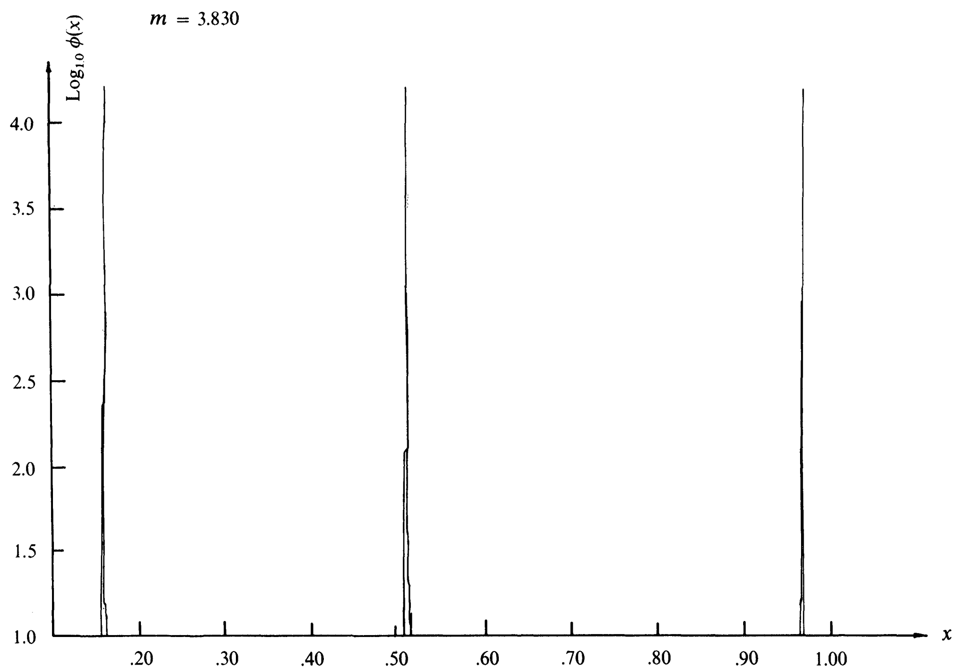

FIG. 5. *The sensitivity of density functions to changes in m is strikingly demonstrated here. m has been increased* 0.005 *from Fig.* 4, *and the results are radically different. Figure* 5 *shows that almost all points x approach the stable three-period solution.*

be quite regular. As m increases from $m = 3.83$, another heirarchy of bifurcations occurs with stable six-period, twelve-period, etc., points arising. This is illustrated in Fig. 6 where the density function is calculated for $m = 3.845$. It indicates that a solution of period six has bifurcated from the solution of period three. Of course, it is stable. Finally, the calculation at $m = 4$ gives a distribution which reflects the known density function which is proportional to $[x(1-x)]^{-1/2}$. This calculation also gives some idea of the variance of our simulation of solutions to (1).

4. Discussion. Equation (1) arises in various models of physical phenomena and population biology. While it may be difficult to make a good case for its being a realistic representation of any phenomenon, it serves as a prototype of more realistic models. It is therefore of interest to investigate this model in detail. The geometric arguments and computer simulations described here give a reasonably complete picture of the dynamics of solutions governed by (1).

It was shown here that as m increases from zero, a nontrivial one-period solution and an accompanying succession of harmonics of it arise. This continues until the critical value $\mathbf{m} \doteq 3.57$. The value $\mathbf{m}$ acts as a threshold of chaos: for $m > \mathbf{m}$, the solution set of (1) contains an infinity of distinct periodic solutions.

As m increases from $\mathbf{m}$, the chaotic solution set successively acquires more periodic solutions until a value m_6 is reached at which solutions of all even periods are present. Finally, as m increases from m_6 to m_3, solutions having successively

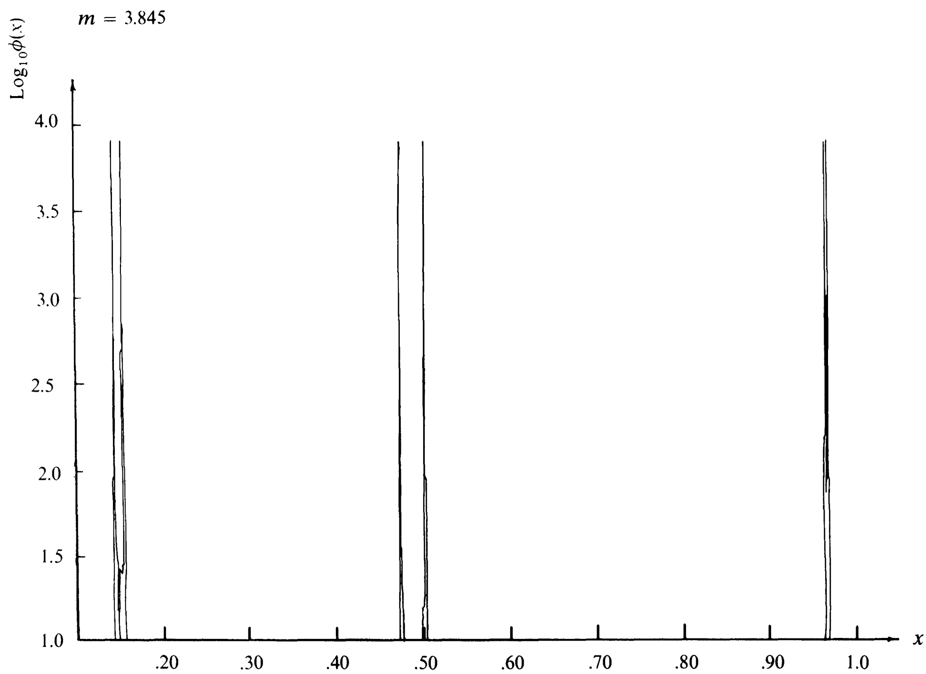

FIG. 6. *This shows that the three-period solution branches into a stable harmonic having period six. In Figs.* 5 *and* 6, *the vertical scale begins at* 1.0. *The deleted parts of the graph describe transient states which evolve into the stable three- and six-period orbits.*

smaller odd periods arise until m_3 is reached at which solutions of period three, and so solutions of all periods, are present. In this sense, solutions of period three are the last to arise. In particular, the analysis used here suggests that

$$m_2 < m_{2^2} < m_{2^3} < \cdots < m_{2^3 \cdot 5} < m_{2^3 \cdot 3} < \cdots < m_{2^2 \cdot 5}$$
$$< m_{2^2 \cdot 3} < \cdots < m_{2 \cdot 5} < m_{2 \cdot 3} < \cdots < m_5 < m_3.$$

The stability properties of these various solutions are quite sensitive to changes in m. As m increases through the chaotic regime, some new solutions arising are stable for short m intervals, and some are unstable. Once a new stable solution arises, a tree of bifurcating harmonics of it arises as m increases. For example, the solution of period three is stable for m near m_3, and all points near it are attracted to this orbit under iterations of f. The numerical calculations indicate that this solution is globally stable, but in the weak sense that the interval (0, 1) less a set of measure zero is attracted to the three-period orbit. The structure at m_5, m_7, etc., is similar. Thus, the term chaotic is not entirely appropriate for describing the solution set for all $m > \mathbf{m}$.

The analysis carried out here shows that a quite simple deterministic model can have a solution set having a random structure. This, of course, is just another illustration of a well-known phenomenon which has been studied by many

$m = 4.000$

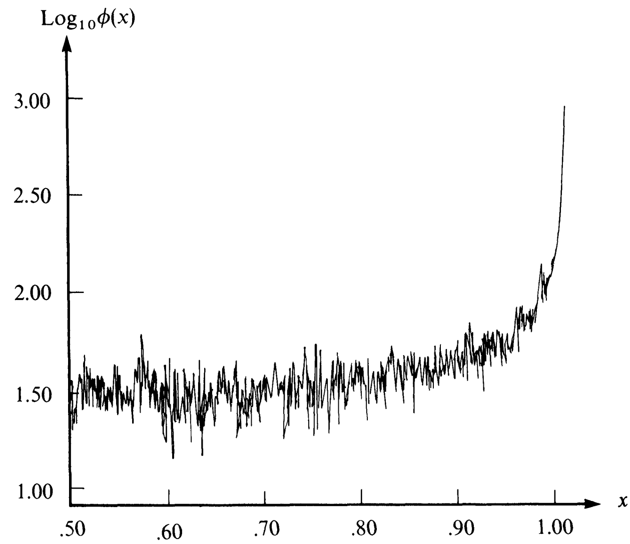

FIG. 7. *These data are symmetric about* $x = \frac{1}{2}$, *and they reflect the known density function for* $m = 4$ *which is proportional to* $[x(1-x)]^{-1/2}$.

investigators. For practical purposes, the solution set should probably be considered as chaotic for $\mathbf{m}_8 < m < m_7$, since for most m in this interval, there are either stable solutions with high period which are difficult to distinguish from chaos, or no stable periodic solutions. For $m_7 < m < 4$, the solution set is intermittently dominated by stable low (odd) period solutions and by chaotic behavior.

It is clearly illustrated here that a standard perturbation approach based on the implicit function theorem does not give a satisfactory description of the solution set. This approach becomes unmanageable even at $m = m_8$, and it cannot proceed past $\mathbf{m}$. In particular, such methods will not give an indication of the interesting dynamics in the chaotic regime. The problem considered here is sufficiently simple to allow extensive use of global methods to derive the threshold of chaos and some properties of the solution set in the chaotic regime.

The methods and results derived here carry over to equations more general than (1); for example, $f(x)$ can be replaced by $mg(x)$, where g is a concave function satisfying $g(0) = g(1) = 0$. Of course, the critical values of m will change with different choices of g.

Equation (1) was encountered by Lorenz [3] in modeling atmospheric turbulence. Explicit calculations of the solution set for $3 \leqq m \leqq m_8$ can be found there. Equation (1) and similar equations have been used in many studies of population dynamics [1], [4], [8]. A heuristic argument describing the successive branchings and the corresponding exchanges of stability which occur in the interval $m_8 < m < \mathbf{m}$ is given in [4]. Poincaré, Fatou, Julia and many other mathematicians have studied iterations of this and other functions from other points of view. After this manuscript was submitted, Dr. I. N. Baker brought to our attention the work of Myrberg [9] who, with a detailed analysis special to iterations of real quadratic polynomials, found the intervals denoted here by $[\mathbf{m}_{2^l}, \mathbf{m}_{2^{l+1}})$ and the corresponding stable orbits and showed that these intervals accumulate at $\mathbf{m}$. The methods used in [9] are based on those of Fatou and Julia, and they are not easily generalized.

Equation (1) has been studied in the chaotic regime from yet another point of view in [5]. Finally, Ulam [6] derived a regular invariant measure for (1) when $m = 4$ (this is reflected in Fig. 7) and the case $m > 4$ has been studied in [7], [10].

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