LA-UR-03-1741

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Submitted to: Proceedings of Workshop on Mesh Quality and Dynamic Meshing, Sandia National Laboratories, Livermore, CA, Jan 2003.

UNTANGLING OF 2D TRIANGULAR AND QUADRILATERAL MESHES IN ALE SIMULATIONS

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Abstract: A procedure is presented to untangle unstructured 2D meshes containing inverted elements by node repositioning. The inverted elements may result from node movement in flow simulations and in large deformation problems such as metal forming. Meshes with inverted elements may also be created due to the limitations of mesh generation algorithms particularly for non-simplicial mesh generation. The untangling procedure uses a combination of direct node placement based on geometric computation of the feasible set and node repositioning driven by numerical optimization of an element area based objective function. It is shown that a combination of the feasible set based method and the optimization method achieves the best results in untangling the mesh. Preliminary results are also presented for untangling of 3D unstructured meshes by the same approach.

Key Words: Mesh untangling, inverted elements, triangles, quadrilaterals

Introduction

Arbitrary Lagrangian-Eulerian or ALE methods are a popular class of methods for simulating ow problems and large deformation problems [1, 2]. ALE methods consists of a Lagrangian step in which the mesh nodes move according to the ow of the material, a rezone step in which the mesh is modi ed to improve its quality and the remapping step in which the solution is transferred from the old mesh to the new, improved mesh. In [3, 4, 5], methods were described to improve quality of the mesh while keeping it close to the original mesh. However, in order to improve the meshes by the methods described in [3, 4, 5], all elements of the starting mesh must be valid or non-inverted. Therefore, if the Lagrangian step of an ALE simulation causes the mesh to become tangled (i.e., it has some elements that become inverted), the mesh must be untangled before the mesh improvement procedures are applied to it.

The need for untangling meshes also exists when a mesh generation procedure is unable to create all valid elements in a mesh. This situation may be encountered

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in the generation of all hexahedral meshes and or general polyhedral meshes where there is no guaranteed method of directly generating a valid mesh [6]. It may also be encountered in advancing front based mesh generation of tetrahedral and hexahedral [7, 8, 9] where it is possible to generate small cavities that cannot be filled with all positive volume elements. In these cases, it is very useful to have a tool that can untangle the mesh after the initial generation assuming that the right mesh connectivity has been generated.

Several researchers have recently begun focusing on the problem of untangling unstructured meshes by node repositioning [10, 11, 12, 13]. Freitag and Plassman [11, 14] untangle meshes by optimization of a local function based on maximizing the minimum element area at each mesh vertex. Knupp [12] performs a global optimization of the difference between the absolute and signed values of element volumes in order to optimize the mesh. Kovalev et.al. [13] visit each vertex connected to at least one invalid element and reposition the vertex directly to a point in its feasible set (or kernel) to make all connected elements valid. They define the feasible set of a vertex to be the set of all locations of the vertex for which all elements connected to the vertex will be valid. The new location of the vertex in the feasible set is found by a clever use of the simplex method to find three corners of the feasible set and taking their mean. Most of these procedures are capable of successfully untangling many tangled mesh configurations. However, there are conditions under which they may fail to untangle the mesh or they may not be able to produce elements with sufficient positive volume so as to provide a good starting point for mesh improvement procedures.

In this paper, a method is presented for untangling unstructured 2D meshes containing triangles and quadrilaterals. The method uses a multi-step approach based on relocating vertices to points in its feasible set and on minimizing a function to untangle the mesh elements. The approach adopted here focuses only on ensuring that all elements of the mesh meet at least bare minimum validity criteria as defined by the simulation procedures that will use them. However, the methods can easily be combined with separate procedures for improving the quality of the untangled mesh like the ones described in [4, 5].

1 Definitions

In this study, an element is considered valid if each corner of the element is considered valid. An element corner is valid if the Jacobian determinant of its mapping to a right corner is positive.

The ***feasible set*** *for a vertex can be defined as the set of all positions of the vertex for which each connected element is valid at the corners affected by the position of the vertex.*

The feasible set for a vertex connected to a single triangle is a half-space as shown in the 2D example in Figure 1a. For a vertex connected to a general polygonal element, the feasible set of a vertex is defined by the intersection of three half-spaces as shown in Figure 1b for a pentagon. Figure 1c shows the feasible set for a vertex interior to a patch of elements. From the definition of the feasible set, it is clear that the feasible set is a convex polygon in 2D while it is a convex polyhedron in 3D. Note that the feasible set for a vertex can be empty depending on the geometry and configuration of the elements connected to the vertex. This implies that given the positions of the boundary vertices of the patch, there is no location for the vertex which will simultaneously make all the elements valid.

2 Untangling by Finding the Feasible Set

The definition of a feasible set given above leads to a natural method of untangling meshes in which the feasible set of a vertex is determined and the vertex is positioned inside it. This is referred to here as the *feasible set method*. In this approach each vertex of the mesh is visited and its connected elements examined to see if any of them are invalid. If an invalid element exists among the elements connected to the vertex, the feasible set of the vertex is computed and the vertex is placed inside the feasible set. The method loops over the mesh until all elements are valid or no invalid element can be fixed.

The feasible set of a vertex can be found by computing the intersection of half-spaces representing the feasible region of the vertex with respect to each element. In 2D, this is accomplished by the intersection of pairs of lines demarcating the feasibility half-planes [15].

In Figure 2, an example of mesh untangling by the intersection based feasible set method is illustrated step by step. In the figure, the invalid quadrilaterals are shown shaded and nodes connected to at least one invalid quadrilateral are shown in black.

While the intersection based computation of the feasible set works well in 2D, finding the feasible set polyhedron by intersection of half-spaces is impractically complex in 3D. Therefore, an alternate method for positioning the vertex inside the feasible set is implemented based on the simplex method as described in [13]. In [13] the idea is proposed that the boundary lines of the half-planes forming the feasible set in 2D can be interpreted as inequality constraints on the minimization of an arbitrary function. If the function is a linear function then its minimum must occur at the intersection of two of these inequality constraints or in other words the corner of the feasible set polygon. Therefore, the corners of the feasible set polygon can be found by minimizing different linear functions along with the appropriate inequality constraints using the simplex method [16, 17].

This idea is further simplified by recognizing that for untangling, it is sufficient to find any one position for the vertex inside its feasible set to make all connected elements valid. This implies that it is unnecessary to find all corners of the feasible set polygon. Therefore, for untangling a patch of elements in 2D, it can be inferred that it is sufficient to find three distinct corners of the feasible set polygon and reposition the vertex to the center of the triangle formed by these corners. Therefore, the procedure minimizes simple linear functions such

Figure 1: Illustration of Feasible Sets (a) Feasible set for vertex connected to single element (b) Feasible set for vertex connected to a pentagon (c) Polygonal feasible set for vertex connected to patch of triangular and quadrilateral elements.

Figure 2: Untangling of mesh by feasible set method (a) Initial mesh (b),(c) Intermediate meshes (d) Final mesh. The shaded quadrilaterals indicate invalid elements and vertices represented in black are connected to at least one invalid quadrilateral.

as $f(x, y) = x$, $f(x, y) = -x$ and $f(x, y) = y$ to nd three distinct corners of the feasible set polygon. The vertex is then repositioned to the center of the triangle formed by these corners.
The feasible set approach to untangling is very useful because it is a direct way of xing inverted elements a ecting only those nodes connected to invalid elements. However, the shortcoming of this approach is that some elements cannot be xed because the feasible set associated with each of their vertices is empty.

3 Optimization Approach to Untangling

An alternative approach for untangling meshes is by minimizing an appropriate objective function so as to make all the invalid elements valid [11, 12]. Knupp [12] proposed optimization of a global objective function based on the diference between signed volumes of elements and their corresponding absolute values. If the area of the i'th element in a mesh is α_i, then the function to be minimized is

$$f(x) = \sum_{i}^{n} (\mid \alpha_i \mid - \alpha_i) \tag{3.1}$$

Minimization of this function can only bring elements to a zero area (volume) state which is still considered unusable in numerical simulations. Therefore, Knupp suggested that a user controlled parameter β be added to the function modifying it to be

$$f(x) = \sum_{i}^{n} (\mid \alpha_i - \beta \mid - (\alpha_i - \beta)) \tag{3.2}$$

The role of β is to force the function to reach a minimum when the elements have a small positive volume instead of zero volume.
In this work, the objective function proposed by Knupp in [12] has been modi ed so that it is quadratic and smooth as shown in Eq. 3.3 below.

$$f(x) = \sum_{i}^{n} (\mid \alpha_i - \beta \mid - (\alpha_i - \beta))^2 \tag{3.3}$$

This smooth function can then be minimized using a numerical optimization method such as the conjugate gradient method [16, 17]. It has been found that, in pratice, minimization of the quadratic form of the objective function untangles the mesh more reliably than the linear form.
The advantage of the optimization approach to untangling described above is that is guaranteed to make all elements at least non-negative. However, the method can have a non-local e ect on the mesh since it may have to mean several nodes in a local neighborhood in order to x an invalid element.

Figure 3: Invalid mesh of Figure 2 optimized using optimization with quadratic objective function (a) Without β (barely valid elements shown shaded) (b) With β = 0.106 = 10% of bounding box diagonal normalized by number of elements in x or y directions

Figure 3 shows example of the mesh shown in Figure 2, untangled by the optimization procedure. Figure 3a shows the mesh optimized using the quadratic objective function without the use of β or $\beta = 0$. The procedure untangles the mesh but because β is zero, some of the valid elements are barely valid. The barely valid elements and the interior vertices connected to these elements are shaded in the figure. Figure 3b shows the same optimization but with a finite β, which is calculated as 10% of the problem size (diagonal of the bounding box) normalized by the number of elements in either the x or y directions and has the value of 0.106. As seen from the figure, the mesh is better in this case since all the elements are positively valid.

In using the optimization procedure for untangling meshes, the choice of β must be made carefully. Without β (i.e., $\beta = 0.0$), the optimization procedure can only make all the elements valid in the sense that the volume of every element is positive or at least zero. On the other hand, using an indiscrimanetely large β can be detrimental since the objective function minimum can become non-zero. This implies that for the given boundary configuration it is not possible for all elements to achieve a volume equal to or greater than β. Therefore, the approach adopted here is to set β to the minimum acceptable area or volume of the elements with respect to the mesh optimization procedures that will subsequently improve the mesh.

4 Mesh Untangling by 3 Stage Procedure

In the previous sections, it was seen that the feasible set approach had a local effect on the mesh but could not always fix the mesh. On the other hand, the optimization approach, usually fixed the mesh by making all elements non-negative but could affect a larger number of nodes and resulted in barely valid elements. Therefore, the procedures have been combined here into a 3-step procedure for maximizing the possibility of untangling the mesh with minimal impact on the mesh.

The 3-step procedure for untangling the mesh first performs untangling by the feasible set method so as to fix as many elements as possible with minimal impact to the valid part of the mesh. The second step of the procedure performs a minimization of the quadratic objective function described earlier in order to fix any remaining invalid elements. The optimization procedure first performs a local or vertex-by-vertex optimization loop over the boundary vertices in order to try and fix as many elements as possible by their movement. The movement of the boundary vertices is constrained to the original discrete boundary by a local parameterization technique [5].

The third step of the procedure performs another round of untangling on barely valid elements by the feasible set approach to try increase their volume. This step requires redefinition of the feasible set boundaries to account for the desired element volumes. Once this is done, the mesh is usually suitable for use as input to a mesh quality improvement procedure such as the one described in [4, 5].

5 Results

The first example shown in Figure 4 illustrates the untangling of a mesh arising from a Rayleigh-Taylor simulation. The mesh is made invalid during the Lagrangian step of an ALE simulation of the problem and must be fixed before the simulation can proceed. Figure 4a shows a part of the overall domain, Figures 4b shows the tangled portion of the mesh (with squares marking nodes connected to invalid elements). Figures 4c shows the untangled mesh corresponding to the tangled meshes shown in Figures 4a,b. The stages of untangling of this mesh are illustrated further in Figure 5. Figure 5a shows a zoom-in of the tangled portion of the original mesh. Figure 5b shows the mesh after the first untangling step by the feasible set method which is unable to fix all the elements. Figure 5c shows the mesh after untangling by optimization during which all elements were made at least barely valid. Finally, Figure 5d shows the mesh after the second round of untangling by the feasible set method during which all elements were brought to a positive volume state.

Figure 6 presents the example of untangling a mesh of the state of Texas. An originally valid mesh shown in Figure 6a was tangled by a random perturbation of a subset of the interior vertices to result in the mesh shown in Figure 6b. The maximum perturbation was 20% of the domain size. This mesh was then successfully untangled using the 3-step procedure to give the mesh shown in

Figure 4: Untangling of Lagrangian mesh from Rayleigh-Taylor Simulation (a) Part of original tangled mesh (b) Zoom-in of tangled mesh (nodes shown by a square are connected to at least one invalid element) (c) Zoom-in of untangled mesh

Figure 6c. For this example the feasible set method alone left behind 17 patches with negative area triangles and the optimization method alone left behind 6 zero area triangles. The value of β was zero for the optimization step. The result of mesh improvement on the untangled mesh is presented in Figure 6d to illustrate that good quality meshes can be obtained when the untangling procedure is combined with mesh improvement procedures.

Conclusions

A multi-step method for successful untangling of unstructured 2D meshes has been presented in this paper. The method uses a combination of the feasible set method and optimization method to achieve the greatest degree of success in untangling the mesh while keeping the mesh close to the original mesh as required for remapping in ALE simulations. The methods have shown a high degree of success in untangling complex 2D meshes. The formulation of the procedures allows easy extension to 3D problems and preliminary results are promising.

Figure 5: Stages of untangling of mesh from Rayleigh-Taylor simulation (a) Zoom-in of original mesh (nodes connected to invalid elements shown as squares) (b) Mesh after application of feasible set approach, some elements remain with less than a minimum volume (c) Mesh after optimization, some elements remain with zero volume (d) Mesh after second application of the feasible set approach, all elements have volume greater than required minimum.

Figure 6: Untangling of triangular mesh of Texas outline (a) Original valid mesh (b) Tangled Mesh (random movement of interior vertices) (c) Untangled mesh with poor quality elements (d) Mesh improved based on optimization of element condition numbers.

6 Acknowledgments

The work of the authors was performed at Los Alamos National Laboratory operated by the University of California for the US Department of Energy under contract W-7405-ENG-36. Los Alamos National Laboratory strongly supports academic freedom and a researcher's right to publish; as an institution, however, the Laboratory does not endorse the viewpoint of a publication or guarantee its technical correctness.
The work of P. Vachal was also supported in part by the Czech Grant Agency grant 201/00/0586

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