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A Note for Lie-Poisson Hamilton-Jacobi Equation and Lie-Poisson Integrator

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Abstract—In this paper, a clear Lie-Poisson Hamilton-Jacobi theory is presented. How to construct a Lie-Poisson integrator by generating function methods is also given, which is different from the Ge-Marsden methods [1]. An example on a rigid body has been given to illustrate this point.

Keywords—Lie-Poisson Hamilton-Jacobi equation, Lie-Poisson integrator, Generating function.

1. INTRODUCTION

A Lie-Poisson system is a very common kind of Hamiltonian system which is popular in rigid body, celestial mechanics, robotics and fluid mechanics. Hamilton-Jacobi theory has played an important role in constructing the symplectic integrators (see [2]). How to construct a Lie-Poisson integrator for a Lie-Poisson system using Hamilton-Jacobi theory has been discussed by Ge [3,4], Ge-Marsden [1], Channell and Scovel [5] and the author's paper [6]. When I derived the Lie-Poisson Hamilton-Jacobi theory, I found that a condition should not be ignored by the Lie Poisson Hamilton-Jacobi Equation (LPHJE), which is given by Ge and Marsden [1]. What is more, the condition is essential to construct the momentum-preserving integrators and high order Poisson integrators, which is very important for many practical systems. As to what is shown by us, the momentum-preserving integrators cannot be constructed by the generating methods without the condition. An analysis based on a free rigid body is also given using the generating function method.

2. LIE-POISSON HAMILTON-JACOBI EQUATION

Let G be a compact Lie Group. $t \longrightarrow g(t) \in G$ is a motion on G. Let $\mathbf{g}$ be a Lie algebra of G, $\mathbf{g}^*$ be the dual space of $\mathbf{g}$. Define

$$J_R : T^*G \longrightarrow \mathbf{g}^*$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX is the momentum mapping corresponding to the left translation on G. Then, from the following commutative diagram

$$\begin{array}{ccc} T^*G & \xrightarrow{S} & T^*G \\ J_R \downarrow & & \downarrow J_R \\ \mathbf{g}^* & \overrightarrow{P} & \mathbf{g}^* \end{array}$$

we know that the phase flow on T^*G can induce the phase flow on $\mathbf{g}^*$ and the Poisson transformation on $\mathbf{g}^*$ corresponds to a symplectic transformation on T^*G.

Let $u^t(q, q_0)$(if exists) be the first-kind generating function of symplectic mapping S. Then, we have the following proposition.

PROPOSITION 1. *[1] If $u : G \times G \longrightarrow R$ is invariant under the left action of G, i.e., $u^t(gq, gq_0) = u^t(q, q_0)$, then the symplectic mappings generated by u,*

$$S: \quad (q_0, p_0) \longrightarrow (q, p),$$

where

$$p_0 = -\frac{\partial u}{\partial q_0}(q, q_0), \qquad p = \frac{\partial u}{\partial q}(q, q_0) \tag{1}$$

preserve momentum mapping J_L, which is corresponding to the right translation on G. That is to say

$$J_L \circ S = J_L.$$

DEFINITION. *If Gacts on the configuration space without fixed point, then we say G acts on G freely.*

PROPOSITION 2. *[5] If G acts on G freely, the symplectic mapping S preserves the momentum mapping J_L, then the first-kind generating function of S is left invariant.*

For the left invariant system, such as a generalized rigid body, the Hamiltonian function is left invariant, the phase flow is also left invariant. Furthermore, the momentum mapping J_L is a first integral for this dynamics, i.e., is invariant under the phase flow of $G^t_{H \circ J_R}$. Therefore, if the action is freely (generally speaking, the action is locally freely), the first-kind generating function is left invariant.

Let $u^t(q, q_0)$ denote the first-kind generating function of S, then by the left invariance,

$$u^t(q, q_0) = u^t\left(e, q^{-1}q_0\right) = \tilde{u}^t(g), \qquad \text{where } g = q^{-1}q_0.$$

By equation (1), we have

$$p_0 = -\frac{\partial u^t(q, q_0)}{\partial q_0} = -\frac{\partial \tilde{u}^t(q^{-1}q_0)}{\partial q_0} = -\frac{\partial \tilde{u}^t(L_{q^{-1}}q_0)}{\partial q_0} = -L^*_{q^{-1}}\left.\frac{\partial \tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}, \tag{2.1}$$

and

$$p = \frac{\partial u^t(q, q_0)}{\partial q} = \frac{\partial \tilde{u}^t\left(q^{-1}q_0\right)}{\partial q} = \frac{\partial \tilde{u}^t(R_{q_0}V(q))}{\partial q} = V^* R^*_{q_0}\left.\frac{\partial \tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}, \tag{2.2}$$

where $V(q) = q^{-1}$, and $V^* = -L^*_{q^{-1}}R^*_{q^{-1}}$. Therefore, equation (2.2) can be simplified

$$p = -L^*_{q^{-1}}R^*_{q^{-1}}R^*_{q_0}\left.\frac{\partial \tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}. \tag{2.3}$$

Then, the momentum mapping $J_R(q,p)$ is

$$\begin{aligned}\mu_0 = J_R(q_0,p_0) = L^*_{q_0}p_0 = -L^*_{q_0}L^*_{q^{-1}}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0} &= -L^*_{q^{-1}q_0}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}\\ &= -L^*_g\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\mu = J_R(q,p) = L^*_q p &= -L^*_qL^*_{q^{-1}}R^*_{q^{-1}}R^*_{q_0}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}\\ &= -R^*_{q^{-1}q_0}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0} = -R^*_g\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}.\end{aligned} \tag{3.2}$$

By equation (2), we can prove Proposition 2 analytically.

$$\begin{aligned}M_0 = J_L(q_0,p_0) = R^*_{q_0}p_0 &= -R^*_{q_0}L^*_{q^{-1}}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0},\\ M = J_L(q,p) = R^*_q p &= -R^*_qL^*_{q^{-1}}R^*_{q^{-1}}R^*_{q_0}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}\\ &= -L^*_{q^{-1}}R^*_{q_0}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0}\\ &= -R^*_{q_0}L^*_{q^{-1}}\left.\frac{\partial\tilde{u}}{\partial g}\right|_{g=q^{-1}q_0} = M_0.\end{aligned}$$

This means $J_L \circ S = J_L$. Denote $g = q^{-1}q_0$, then, we have the following theorem.

THEOREM 1. *The first-kind generating function u of the symplectic transformation on T^*G define a Poisson transformation*

$$P : \mu_0 \longrightarrow \mu = Ad^*_{g^{-1}}\mu_0$$

on $\mathbf{g}^*$, where μ, μ_0 is defined by equation (3.1), (3.2), respectively.

The first-kind generating function $u^t(q,q_0)$ of the symplectic mapping $S : (q_0,p_0) \longrightarrow (q,p)$ satisfies the following Hamilton-Jacobi equation

$$\frac{\partial u^t(p,q)}{\partial t} + H(p,q) = 0, \tag{4}$$

where $H(p,q)$ is the Hamiltonian on T^*G and the mapping S is defined by equation (1).

Now, we have a Hamiltonian H on $\mathbf{g}^*$. The momentum mapping J_R induces a Hamiltonian on T^*G, which is $H \circ J_R$. By Theorem 1, the Lie-Poisson Hamilton-Jacobi (LPHJ) equation for Lie-Poisson system on $\mathbf{g}^*$ can be derived.

THEOREM 2. *The u, H, J_R is defined as above, then the generating function u induces a generating function $\tilde{u}$ on the $\mathbf{g}^*$, which satisfy*

$$\frac{\partial\tilde{u}}{\partial t}(g) + H\left(-R^*_g\frac{\partial\tilde{u}}{\partial g}(g)\right) = 0, \tag{5}$$

where

$$g = q^{-1}q_0. \tag{6}$$

In the paper of Ge-Marsden [1], condition (6) is ignored. But I think it cannot be ignored, even if we constrain our discussion on $\mathbf{g}^*$. We should use our example to illuminate this point later.

REMARK.

(1) If we can construct the generating function $u(g)$, we then have $u(q_0, q)$. This function can generate a symplectic mapping on T^*G. By the commutative diagram, we have a Poisson mapping on $\mathbf{g}^*$. This is the main point of constructing a Lie-Poisson integrator by generating methods.

(2) The above theory of a generating function on T^*G can be reformulated by the *exponential* mapping in terms of algebra variables, which has been done by Channell and Scovel [2]. We now give some results. For $g \in G$, choose $\xi \in \mathbf{g}$ so that $g = \exp(\xi)$. Then, the LPHJ equation can be transformed into

$$\frac{\partial u}{\partial t} + H(-du \cdot \psi(ad_\xi)) = 0, \tag{7}$$

where

$$M_0 = -du \cdot \chi(ad_\xi), \qquad M = -du \cdot \psi(ad_\xi). \tag{8}$$

The function χ and ψ can be expanded as Taylor series, which give

$$\begin{aligned} \chi(ad_\xi) &= I_d + \frac{1}{2} ad_\xi + \frac{1}{12} ad_\xi^2 + \cdots, \\ \psi(ad_\xi) &= \chi(ad_\xi) - ad_\xi, \end{aligned} \tag{9}$$

and the condition (6) $g = q^{-1}q_0$ is transformed into $\xi|_{t=0} = I_d$.

3. THE GE-MARSDEN INTEGRATORS AND THEIR DRAWBACKS

According to the Ge-Marsden algorithm, condition (6) is ignored and only gives a generating function which can generate an identity mapping on $\mathbf{g}^*$. We use Channell and Scovel's [5] representation.

The generating function is given by

$$u = u_0 + \sum_{n=1}^{\infty} \frac{(\delta t)^n}{n!} u_n, \tag{10}$$

where $u_0 = (\xi, \xi)/2$ can generate the identity on the regular quadratic Lie algebras.

After substituting (10) into the LPHJ equation, we find

$$u_1 = -H(V), \qquad u_2 = \frac{\partial H}{\partial V} \cdot du_1 \cdot \psi(ad_\xi), \ldots. \tag{11}$$

In the following, we will use $so(3)^*$ as an example to illustrate the algorithm and its shortcoming.

For $so(3)^*$, $u_0 = \xi^2/2$, and $V = \xi$. Thus, the first-order and second-order integrator can be given by

$$S_1 = u_0 + \tau u_1 = \frac{\xi^2}{2} - \tau H(\xi) = \frac{\xi^2}{2} - \frac{\tau}{2} \xi \cdot I^{-1} \xi,$$

and

$$\begin{aligned} S_2 = S_1 + \frac{\tau^2}{2} u_2 &= \frac{\xi^2}{2} - \tau H(\xi) + \frac{\tau^2}{2} \frac{\partial H}{\partial V} \cdot du_1 \cdot \psi(ad_\xi) \\ &= \frac{\xi^2}{2} - \frac{\tau}{2} \xi \cdot I^{-1} \xi - \frac{\tau^2}{2} I^{-1} \xi \cdot (I^{-1} \xi \cdot \psi(\xi)). \end{aligned}$$

By equations (8) and (9), we have

$$M - M_0 = -du \cdot ad_\xi. \tag{12}$$

Now, we will prove that the generating equation S_1 surely generates the first-order integrator of the Euler equation. But the S_2 does not generate the second-order integrator, and in the terms of this algorithm we cannot construct the momentum preserving integrators.

Since

$$dS_1 = d\left(\frac{\xi^2}{2} - \frac{\tau}{2}\xi \cdot I^{-1}\xi\right) = \xi - \tau I^{-1}\xi,$$

and $M_0 = -dS_1 \cdot \chi(ad_\xi) = (-\xi + \tau I^{-1}\xi) \cdot \chi(\xi)$, we have that $\xi = -M_0 + O(\tau)$. By equation (12), and using the fact that $\xi \cdot ad_\xi = 0$, we have

$$\begin{aligned} M - M_0 &= (\xi - \tau I^{-1}\xi) \cdot ad_\xi = -\tau I^{-1}\xi \cdot ad_\xi \\ &= \tau\left[\xi, I^{-1}\xi\right] = \tau\left[-M_0 + O(\tau), I^{-1}(-M_0 + O(\tau))\right] \\ &= \tau\left[M_0, I^{-1}M_0\right] + O(\tau^2), \end{aligned}$$

which is an approximation solution to the Euler equation

$$\dot{M} = \left[M, I^{-1}M\right]. \tag{13}$$

For generating function S_2, we first compute the function $\chi(\xi)$. Let

$$\chi(\xi) = 1 + a_1\xi + a_2\xi^2,$$

where a_1, a_2 can be solved analytically as follows (see the Appendix):

$$a_1 = \frac{1 - \cos(|\xi|)}{\sin^2(|\xi|) + (1 - \cos(|\xi|))^2},$$

$$a_2 = \frac{\dfrac{(\cos|\xi| - 1)^2}{|\xi|^2} + \dfrac{\sin|\xi| - |\xi|}{\xi} + (\sin|\xi| - |\xi|)|\xi|}{\sin^2|\xi| + (1 - \cos|\xi|)^2}.$$

Thus,

$$u_2 = -I^{-1}\xi \cdot (I^{-1}\xi \cdot \psi(\xi)) = -\langle I^{-1}\xi, I^{-1}\xi\rangle - a_2 I^{-1}\xi\left(I^{-1}\xi \cdot \xi^2\right),$$

and

$$dS_2 = \xi - \tau I^{-1}\xi - \tau^2\left(I^{-1}\right)^2\xi - \frac{\tau^2}{2}d\left(a_2 I^{-1}\xi \cdot \left(I^{-1}\xi \cdot \xi^2\right)\right).$$

By

$$M_0 = -dS_2 \cdot \chi(\xi) = -\xi - \tau\xi \cdot \chi(\xi) + O(\tau^2),$$

we have

$$\xi = -M_0 + \tau\xi \cdot \chi(\xi) + O(\tau^2).$$

Therefore, by equation (12),

$$\begin{aligned} M - M_0 &= -dS_2 \cdot ad_\xi = -\left(\xi - \tau I^{-1}\xi - \tau^2\left(I^{-1}\xi\right)^2 - \frac{\tau^2}{2}d\left(a_2 I^{-1}\xi \cdot \left(I^{-1}\xi \cdot \xi^2\right)\right)\right) \cdot \hat{\xi} \\ &= \tau\left[M_0, I^{-1}M_0\right] + a_1\tau^2\left(\left[M_0, I^{-1}M_0\right], I^{-1}M_0\right] + \left[M_0, I^{-1}\left[M_0, I^{-1}M_0\right]\right]\right) \\ &\quad + a_2\left\{I^{-1}M_0\left(I^{-1}M_0 \cdot M_0^2\right) + I^{-1}\left(I^{-1}M_0 \cdot M_0^2\right) \cdot M_0\right\} \\ &\quad - \frac{\tau^2}{2}d\left(a_2 \cdot I^{-1}\xi\left(I^{-1}\xi \cdot \xi^2\right)\right) \cdot \hat{\xi} + O\left(\tau^3\right). \end{aligned} \tag{14}$$

According to the Euler equation (13), the second-order approximation solution should be

$$M - M_0 = \tau\left[M_0, I^{-1}M_0\right] + \frac{\tau^2}{2}\left\{\left[\left[M_0, I^{-1}M_0\right], I^{-1}M_0\right] + \left[M_0, I^{-1}\left[I^{-1}M_0, M_0\right]\right]\right\} + O(\tau^3). \tag{15}$$

If $\xi \longrightarrow 0$, as $t \longrightarrow 0$, the equation (14) approximates (15). But in this case, ξ approximates to M_o as $\tau \longrightarrow 0$, after complex computation, we find that the equation (14) does not approximate to the equation (15). Thus, the generating function S_2 cannot generate the second-order integrator to the Euler equation.

As we have proved, the generating function S_1 surely generates a first-order integrator to the Euler equation. But the momentum mapping preserving integrators should satisfy $J_L(q, M) = J_L(q_0, M_0)$, which in terms of $T^*SO(3)$ is $qM = q_0M_0$, i.e., $M = q^{-1}q_0M_0$. If we want to construct the momentum mapping preserving integrators, we should estimate the $q \in G$ in the meantime. If we have a formula $M = gM_0$, it is natural for us to let $g = q^{-1}q_0$, and thus, $q = q_0g^{-1}$. If the algorithm on $so(3)^*$ is designed well, the above algorithm about $q \in SO(3)$ is correct. In the next section, we should use our LPHJ equation theory to construct such integrators. But in the algorithm of Ge and Marsden, the algorithm on G cannot be given, for the condition (6) is ignored. Now, we would give a detailed explanation.

Using another form of equations (7) and (8), we have

$$M_0 = -du \cdot \chi(ad_\xi), \qquad M = \exp(ad_\xi)M_0. \tag{16}$$

As we have proved, $\xi = -M_0 + \tau I^{-1}\xi \cdot \chi(\xi)$ for the first-order integrator. If we let $q = q_0g^{-1} = q_0\exp(-\xi) = q_0\exp(M_0 - \tau I^{-1}\xi \cdot \chi(\xi))$, the q is not what we want, i.e., the approximation of the motion equation $\dot{q} = qI^{-1}\hat{M}$. Indeed, from the Ge-Marsden algorithm, we cannot even give the form of q, let alone construct the momentum preserving algorithm.

4. THE EXACTLY MOMENTUM PRESERVING LIE-POISSON INTEGRATOR

In this section, we will use the LPHJ equation and condition (6) to construct the momentum preserving Lie-Poisson integrator. For convenience, we take $so(3)^*$ as an example.

The Hamiltonian for $so(3)^*$ is $H(M) = \frac{1}{2}M \cdot I^{-1}M$. By equation (7), we have

$$M = -du \cdot \psi(ad_\xi) = -du \cdot \left(1 - \frac{1}{2}ad_\xi + \frac{1}{12}ad_\xi^2 + O(ad_\xi^3)\right) = -du + \frac{1}{2}du \cdot ad_\xi + O\left(ad_\xi^2\right).$$

After substituting H into equation (7) and using the expansion of ψ, we have

$$\begin{aligned}
&\frac{\partial u}{\partial t} + H\left(-du + \frac{1}{2}du \cdot ad_\xi - O\left(ad_\xi^2\right)\right) \\
&= \frac{\partial u}{\partial t} + \frac{1}{2}\left(-du + \frac{1}{2}du \cdot ad_\xi + O\left(ad^2\xi\right)\right) \cdot I^{-1}\left(-du + \frac{1}{2}du \cdot ad_\xi + O\left(ad^2\xi\right)\right) \\
&= \frac{\partial u}{\partial t} + \frac{1}{2}I^{-1} \cdot du \cdot du - \frac{1}{2}I^{-1} \cdot du \cdot du(ad_\xi) + O\left(ad_\xi^2\right) \\
&= \frac{\partial u}{\partial t} + \frac{1}{2}I^{-1} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \xi} - \frac{1}{2}I^{-1} \cdot \frac{\partial u}{\partial \xi} \cdot \left(\frac{\partial u}{\partial \xi}ad_\xi\right) + O\left(\tau^2\right) \\
&= \frac{\partial u}{\partial t} + \frac{1}{2}I^{-1} \cdot \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \xi} + O\left(\tau^2\right) \\
&= 0,
\end{aligned} \tag{17}$$

for $du \cdot du \cdot ad_\xi = du \cdot [du, \xi] = 0$ and $\xi = O(\tau)$.

We choose $u = (I\xi \cdot \xi)/(2\tau)$, then equation (17) is satisfied. Therefore, the algorithm can be given by equation (16)

$$M_0 = -I\xi \cdot \chi(\xi), \tag{18.1}$$

$$M = \exp(\xi)M_0. \tag{18.2}$$

Solve equation (18.1) for ξ and then substitute ξ into (18.2), we can have the algorithm. For $q \in SO(3)$, we let $q = q_0 exp(-\xi)$, which is a first-order approximation to the motion equation. In this algorithm, how to solve ξ from equation (18.1) is very important. Using the linearlized technique and the expansion of χ, we give an iteration formula as follows:

$$\left(1 + \tau \left[c1\hat{\xi} - (c3\xi \cdot \xi + c4)(I^{-1}\hat{M}_0 \times \xi) + c2(I^{-\hat{1}}M_0)\right]\right) \delta\xi = Re = \xi_{k+1} - \xi_k,$$

where

$$\begin{aligned} c1 &= \frac{2 - |\xi| \sin |\xi| - 2\cos|\xi|}{|\xi|^4}, & c2 &= \frac{\cos|\xi| - 1}{|\xi|^2}, \\ c3 &= \frac{-2|\xi| - |\xi|\cos|\xi| + 3\sin|\xi|}{|\xi|^5}, & c4 &= \frac{2|\xi| - \sin|\xi|}{|\xi|^3}. \end{aligned}$$

The above algorithm can be applied to a generalized rigid body, where the Hamiltonian of the system is quadratic. But using the above derivation to construct the high order integrator as having done to the symplectic mapping would be very difficult. A composition method [6] is recommended in this case.

APPENDIX
THE FORMULA OF $\chi(x)$ IN $so(3)$

According to the definition, we have

$$\begin{aligned} iex(\xi) &= \sum_{n=0}^{\infty} \frac{\xi^n}{(n+1)!}, \\ \chi(\xi) iex(-\xi) &= Id_{\xi}. \end{aligned}$$

For $\widehat{\xi} \in so(3)$, by $(\widehat{\xi})/(\|\xi\|)^3 = -(\widehat{\xi})/(\|\xi\|)$, we have

$$\begin{aligned} iex\left(-\widehat{\xi}\right) &= \sum_{n=0}^{\infty} \frac{\left(-\widehat{\xi}\right)^n}{(n+1)!} = \sum_{k=0}^{\infty} \frac{\left(\widehat{\xi}\right)^{2k}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{\left(-\widehat{\xi}\right)^{2k+1}}{(2k+2)!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\|\xi\|^{2k}}{(2k+1)!} \left(\frac{\widehat{\xi}}{\|\xi\|}\right)^2 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\|\xi\|^{2k+1}}{(2k+2)!} \cdot \frac{\widehat{\xi}}{\|\xi\|} \\ &= 1 + \frac{|\xi| - \sin|\xi|}{|\xi|^3}\widehat{\xi}^2 + \frac{\cos|\xi| - 1}{|\xi|^2}\widehat{\xi} \\ &= 1 + c_1\widehat{\xi} + c_2\widehat{\xi}^2, \end{aligned}$$

where $c_1 = (\cos|\xi| - 1)/(|\xi|^2), c_2 = (|\xi| - \sin|\xi|)/(|\xi|^3)$.

Let $\chi(\widehat{\xi}) = 1 + a_1\widehat{\xi} + a_2\widehat{\xi}^2$, then

$$\begin{aligned} \chi\left(\widehat{\xi}\right) iex\left(-\widehat{\xi}\right) &= \left(1 + a_1\widehat{\xi} + a_2\widehat{\xi}^2\right)\left(1 + c_1\widehat{\xi} + c_2\widehat{\xi}^2\right) \\ &= 1 + (a_1 + c_1)\widehat{\xi} + (a_1c_1 + c_2 + a_2)\widehat{\xi}^2 + (a_1c_2 + a_2c_1)\widehat{\xi}^3 + a_2c_2\widehat{\xi}^4 \\ &= 1 + \left(a_1 + c_1 - (a_1c_2 + a_2c_1)|\xi|^2\right)\widehat{\xi} + \left(c_2 + a_2 + a_1c_1 - a_2c_2|\xi|^2\right)\widehat{\xi}^2 \\ &= I_d, \end{aligned}$$

thus,

$$\begin{cases} a_1 + c_1 - (a_1c_2 + a_2c_1)|\xi|^2 = 0, \\ a_1c_1 + c_2 + a_2 - a_2c_2|\xi|^2 = 0. \end{cases}$$

Solving the above equations, we have

$$a_1 = \frac{-c_1}{(1-c_2|\xi|^2)^2 + c_1^2|\xi|^2} = \frac{1-\cos|\xi|}{(\sin|\xi|)^2 + (1-\cos|\xi|)^2},$$

$$a_2 = \frac{-c_2 + c_2|\xi|^2 + c_1^2}{(1-c_2|\xi|^2)^2 + c_1^2|\xi|^2} = \frac{\frac{(\,cos|\xi|-1)^2}{|\xi|^2} + \left(\frac{\sin|\xi|-|\xi|}{|\xi|}\right) + (\sin|\xi|-|\xi|)|\xi|}{(\sin|\xi|)^2 + (1-\cos|\xi|)^2}.$$

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