Lie-Poisson Integration for Rigid Body Dynamics*

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Abstract—In this paper, the splitting midpoint rule is presented and proved to be the Lie-Poisson integrators to the rigid body systems. Further discussions are also given. Numerical experiments show that this method has well properties comparing with the Runge Kutta method and ordinary midpoint rule.

Keywords—Free rigid body, Heavy top, Midpoint rule, Lie-Poisson system, Poisson scheme, Splitting system, Composition method.

1. INTRODUCTION

It becomes more and more important to construct the structure-preserving integrators for solving dynamical systems. The Symplectic algorithms of canonical Hamiltonian structure—symplectic structure—are well discussed in many publications. In [1–3], the Poisson schemes for linear Poisson systems have been discussed by several authors. However, for the noncanonical Hamiltonian system—Lie-Poisson system—which exists in rigid body dynamics, celestial mechanics, robotics and biomechanics etc., the theory and algorithm is very rare. Ge-Marsden's [4] generating function methods for the Lie-Poisson system are proved by the author [5] to be unappealing and to only construct a first-order Poisson integrator. The Andersen's constrained algorithm (see [6]) is a good way toward the problems. But it is only practical when the constrained system can be splitting. Maclachlan [7] has brought forward an explicit Lie-Poisson integrator to a kind of splitting Lie-Poisson system. This algorithm must compute the exact solution at each splitting step and evaluate the e^a, which is very time consuming especially when a is large.

The rigid body is a typical Lie-Poisson system. It can also be splitting. Some algorithms have been given. Simo *et al.* [8] proposed an energy and momentum preserving algorithm. Austin *et al.* [2] gave an almost Poisson integration. All these methods cannot be Lie-Poisson integrators. In this paper, the midpoint rule is proved to be a Lie-Poisson integrator to the splitting system, and using the composition methods, Lie-Poisson integrators can be easily constructed. Taking the free rigid body as an example, we present the sufficient and necessary condition for Lie-Poisson integrators among generalized Euler rules. As to heavy top, we proved that the midpoint rule also is a Lie-Poisson integrator. These algorithms can also be changed into momentum-preserving

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(see [9]). Numerical experiments show that it is also faster than other methods and has very good energy preserving property.

2. A LIE-POISSON INTEGRATOR OF FREE RIGID BODY

A Lie-Poisson system has a phase space $M = \mathbf{R}^n(x)$, Lie-Poisson bracket $\{F, G\} = \frac{\partial F}{\partial x_i} J_{ij} \frac{\partial G}{\partial x_j}$, where $J_{ij} = C_{ij}^k x_k$ (with C_{ij}^k the structure constants of a Lie algebra), and Hamiltonian H : $M \longrightarrow R$. The dynamical system is

$$\dot{x} = \{x, H\} = J(x)\nabla H, \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$.

THEOREM 1. The phase flow of equation (1), denoted by g_H^t, preserves the Poisson bracket; i.e.,

$$\{F \circ g_H^t, G \circ g_H^t\} = \{F, G\} \circ g_H^t.$$

Consider now the difference schemes for the Lie-Poisson system (1), restricted mainly to the case of single step schemes; time t is discretized into $t = 0, \pm\tau, \pm2\tau, \ldots, x(k\tau) = x^k$, each 2-lever scheme is characterized by a transition operator relating the old and new state by $\hat{x} = g^\tau x$, $x = x^k$, $\hat{x} = x^{k+1}$, $g^t = g_H^t$ depend on t, H and the mode of discretization. It is natural and mandatory to require g_H^t to be Poisson bracket-preserving, which we call the difference scheme $\longrightarrow$ Poisson scheme.

THEOREM 2. A difference scheme of system (1) is Poisson iff

$$\left(\frac{\partial \hat{x}}{\partial x}\right) J(x) \left(\frac{\partial \hat{x}}{\partial x}\right)' = J(\hat{x}).$$

PROOF.

$$\{F \circ g^t, G \circ g^t\} = \frac{\partial F \circ g^t}{\partial x_i} J_{ij}(x) \frac{\partial G \circ g^t}{\partial x_j}$$

$$= \frac{\partial F}{\partial g^t} \cdot \left(\frac{\partial g^t}{\partial x_i}\right) J_{ij}(x) \left(\frac{\partial g^t}{\partial x_j}\right) \cdot \frac{\partial G}{\partial g^t}$$

$$= (\nabla F \circ g)^T \cdot \left(\frac{\partial g^t}{\partial x}\right) J(x) \left(\frac{\partial g^t}{\partial x}\right)^T \cdot \nabla G \circ g^t,$$

$$\{F, G\} \circ g^t = (\nabla F)' J(\nabla G)(g^t) = (\nabla F) \circ g^t J(g^t)^T (\nabla H) \circ g^t.$$

By Theorem 1, we have

$$\left(\frac{\partial g^t}{\partial x}\right) J(x) \left(\frac{\partial g^t}{\partial x}\right)^T = \left(\frac{\partial \hat{x}}{\partial x}\right) J(x) \left(\frac{\partial \hat{x}}{\partial x}\right)^T = J\left(g^t\right) = J\left(\hat{x}\right).$$

A free rigid body is the simplest and typical Lie-Poisson system. Many other complex Lie-Poisson systems have the same form or property as the rigid body. When complex systems can be split, the splitting system is often a rigid body. Therefore, it is very important to study the algorithm of the rigid body system. The detail information about rigid body has been discussed by the author in another paper [9]. Now, we take the rigid body as an example to construct the Lie-Poisson integrator.

The dynamic equation for a free rigid body is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \end{pmatrix}, \tag{2}$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is the angular momentum, $H = (1/2)\langle I^{-1}x, x \rangle$ is the Hamiltonian and energy of the system. I is the symmetry definite positive inertia operator. A general form of H is

$$H = \sum b_i x_i^2 + \sum b_{ij} x_i x_j = \sum a_i x_i^2 + \sum a_{ij}(x_i + x_j)^2. \tag{3}$$

It is very difficult to construct the Poisson schemes for system (2). It has been proved by the author that the generalized Euler schemes and RK methods to system (2) are not Poisson integrators. Using the reduction technique, a generating function method is discussed by several authors [4,10]. However, as the author proved in another paper [5], the Ge-Marsden's generating function methods can only be one-order algorithms and cannot preserve the space angular momentum very well. New generating function methods must be found, but it is difficult. As pointed out by Maclachlan and Scovel [6], the generating function method is very time-consuming. An effective method for splitting systems is to construct the composition methods. As we know, many Hamiltonian systems such as a free rigid body, a heavy top and sine-bracket truncation of 2D Euler equations can be split, and in many cases, the splitting subsystems can be solved analytically or reduced to a symplectic systems. In the following, we will give a method to $SO(3)$ systems.

For the motion of a rigid body fixed at its center of gravity, I^{-1} is a diagonal matrix. Let

$$H = \frac{1}{2}\left(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2\right) = H_1 + H_2 + H_3,$$

where $H_i = (1/2)a_i x_i^2$.

Take one of the subsystems as an example:

$$\dot{x} = J(x) \cdot \frac{\partial H_2}{\partial x} = \begin{pmatrix} -a_2 x_2 x_3 \\ 0 \\ a_2 x_1 x_2 \end{pmatrix}, \tag{4}$$

where

$$J(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

This equation can be reduced to the following symplectic system:

$$\begin{aligned} \dot{x}_1 &= -a_2 x_2 x_3, \\ \dot{x}_3 &= a_2 x_1 x_2, \end{aligned} \tag{5}$$

where x_2 is a constant.

There are a class of symplectic schemes for system (5). However, as we can see later, only a small part of them are Poisson schemes for system (4).

THEOREM 3. *The midpoint rule of (4) is Poisson.*

In order to prove Theorem 3, we give the following lemma first.

LEMMA 1. *For the reduced system (4), the symplectic algorithm of system (5) is Poisson iff the following conditions are satisfied:*

$$\begin{aligned} -x_{11}x_3 + x_{13}x_1 &= -\widehat{x}_3, \\ x_{31}x_3 - x_{33}x_1 &= -\widehat{x}_1, \\ x_{12}\widehat{x}_1 + x_{32}\widehat{x}_3 &= 0, \end{aligned} \tag{6}$$

where $x_i = x_i^n$, $\widehat{x}_i = x_i^{n+1}$, $x_{ij} = \frac{\partial \widehat{x}_i}{\partial x_j}$.

PROOF. By Theorem 2, the schemes are Poisson iff the following equation is satisfied:

$$\left(\frac{\partial \widehat{x}}{\partial x}\right) J(x) \left(\frac{\partial \widehat{x}}{\partial x}\right)^T = J(\widehat{x}).$$

Expand the above equation

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & x_{31} \\ x_{12} & 1 & x_{32} \\ x_{13} & 0 & x_{33} \end{pmatrix} = \begin{pmatrix} 0 & -\widehat{x}_3 & \widehat{x}_2 \\ \widehat{x}_3 & 0 & -\widehat{x}_1 \\ -\widehat{x}_2 & \widehat{x}_1 & 0 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 0 & -x_{11}x_3 - x_{13}x_1 & a_{13} \\ x_{11}x_3 - x_{13}x_1 & 0 & x_{31}x_3 - x_{33}x_1 \\ -a_{13} & x_{33}x_1 - x_{31}x_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \widehat{-x_3} & \widehat{x_2} \\ \widehat{x}_3 & 0 & -\widehat{x}_1 \\ -\widehat{x}_2 & \widehat{x}_1 & 0 \end{pmatrix},$$

where $a_{13} = (x_{12}x_3 - x_{13}x_2)x_{31} + (x_{13}x_1 - x_{11}x_3)x_{32} + (x_{11}x_2 - x_{12}x_1)x_{33}$.

For the system (5), the scheme is symplectic. Thus,

$$-x_{13}x_{31} + x_{11}x_{33} = 1,$$

and a_{13} can be simplified as

$$a_{13} = (x_3x_{31} - x_1x_{33})x_{12} + (x_{13}x_1 - x_{11}x_3)x_{32} + x_2.$$

Comparing the corresponding elements of matrices on both sides of the equation and using the condition $\widehat{x}_2 = x_2$ gives

$$x_{11}x_3 - x_{13}x_1 = \widehat{x}_3,$$
$$x_{31}x_3 - x_{33}x_1 = -\widehat{x}_1,$$
$$x_{12}\widehat{x}_1 + x_{32}\widehat{x}_3 = 0,$$

just the same as equation (6).

Now, we prove Theorem 3 using Lemma 1.

PROOF OF THEOREM 3. The midpoint rule for system (4) can be given by

$$\widehat{x}_1 = x_1 - \tau a_2 \frac{\widehat{x}_3 + x_3}{2} x_2,$$

$$\widehat{x}_2 = x_2,$$

$$\widehat{x}_3 = x_3 + \tau a_2 \frac{\widehat{x}_1 + x_1}{2} x_2.$$

The Jacobi matrix is

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 1 & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

where

$$x_{11} = 1 - \frac{\tau}{2}a_2x_2x_{31},$$

$$x_{12} = -\frac{\tau}{2}a_2(\widehat{x}_3 + x_3) - \frac{\tau}{2}a_2x_2x_{32},$$

$$x_{13} = -\frac{\tau}{2}a_2x_{33}x_2 - \frac{\tau}{2}a_2x_2,$$

$$x_{31} = \frac{\tau}{2}a_2x_{11}x_2 + \frac{\tau}{2}a_2x_2,$$

$$x_{32} = \frac{\tau}{2}a_2(\widehat{x}_1 + x_1) + \frac{\tau}{2}a_2x_2,$$

$$x_{33} = 1 + \frac{\tau}{2}a_2x_2x_{13}.$$

Solving the above equations, we have

$$x_{11} = x_{33} = \frac{1 - a^2}{1 + a^2}, \quad x_{13} = -x_{31} = -\frac{2a}{1 + a^2},$$

$$x_{12} = -\frac{\frac{\tau}{2}a_2\widehat{x}_3}{1 + a^2}, \tag{7.1}$$

$$x_{32} = \frac{\frac{\tau}{2}a_2\widehat{x}_1}{1 + a^2},$$

where

$$a = \frac{\tau}{2}a_2 x_2. \tag{7.2}$$

Substituting equation (7.1) into the conditions (6), we can easily see that the conditions are satisfied. Therefore, by Lemma 1, the scheme is Poisson.

LEMMA 2. *[11,12] Consider a dynamical system $\dot{x} = a(x)$. Supposing a admitted a decomposition $a = a_1 + a_2 + \cdots + a_k$, we write $g^s \approx e_a^s$ the phase flow of the dynamical system; then*

$$g_i^s \approx e_{a_i}^s, \quad \text{order } 2 \ \forall i \Longrightarrow g_1^{s/2} \circ \cdots g_k^{s/2} \circ g_k^{s/2} \circ \cdots g_1^{s/2} \approx e_a^s, \quad \text{order } 2.$$

Using Lemma 2 and Theorem 3, we can easily construct the Lie-Poisson schemes. As noted by Maclachlan [7], the less number of the splitting, the better. Using the Casimir $C = |x|^2$, we can see that

$$\widehat{H} = H - \frac{1}{2}a_1 C = \frac{1}{2}(a_2 - a_1)x_2^2 + \frac{1}{2}(a_3 - a_1)x_3^2 = H_1 + H_2$$

has the same dynamic as H. Using $\widehat{H}$ as the new Hamiltonian, the step number which the scheme needs can be reduced.

For the symplectic system (5), the generalized Euler scheme

$$\widehat{x} = x + \tau J\nabla H\left(B\widehat{x} + (1 - B)x\right)$$

is symplectic iff

$$B = \frac{1}{2}(I + C), \qquad JC + C'J = 0. \tag{8}$$

It is natural to ask such questions as: among the symplectic schemes of system (5), which is the Poisson scheme for system (4)? In the following derivation, we give a condition to the generalized Euler schemes.

Let

$$C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix};$$

then by the symplectic condition (8), we have $c_4 = -c_1$. So,

$$B = \frac{1}{2}\begin{pmatrix} 1 + c_1 & c_2 \\ c_3 & 1 - c_1 \end{pmatrix}$$

and

$$B\widehat{x} + (1 - B)z = \frac{1}{2}\begin{pmatrix} (1 + c_1)\widehat{x}_1 + (1 - c_1)x_1 + c_2(\widehat{x}_3 - x_3) \\ c_3(\widehat{x}_1 - x_1) + (1 - c_1)\widehat{x}_3 + (1 - c_1)x_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} z_1 \\ z_3 \end{pmatrix}. \tag{9}$$

The Euler difference scheme is

$$\widehat{x}_1 = x_1 - az_3,$$
$$\widehat{x}_3 = x_3 - az_1, \tag{10.1}$$

where a is defined by equation (7.2), and z_1, z_2 is defined by equation (9).

After computing, we have the Jacobi matrix of the algorithm and $\widehat{x}_3$

$$x_{11} = \frac{(1 + ac_3)(1 - ac_2) - a^2(1 - c_1)^2}{(1 + ac_3)(1 - ac_2) + a^2(1 - c_1^2)},$$

$$x_{13} = \frac{-2a(1 - ac_2)}{(1 + ac_3)(1 - ac_2) + a^2(1 - c_1^2)},$$

$$\widehat{x}_3 = \frac{\left((1 + ac_3)(1 - ac_2) - a^2(1 - c_1)^2\right)x_3 + 2a(1 + ac_3)x_1}{(1 + ac_3)(1 - ac_2) + a^2(1 - c_1^2)}.$$

Solving the first equation of (6)

$$x_{11}x_3 - x_{12}x_1 = \widehat{x}_3,$$

we get

$$c_1 = 0, \qquad c_2 = -c_3 \tag{10.2}$$

for arbitrary real number x_1, x_3.

Substituting equation (10.2) into equation (10.1) and computing the Jacobi matrix again, we have

$$x_{31} = \frac{2a(1 - ac_2)}{a^2 + (1 - ac_2)^2},$$

$$x_{33} = \frac{(1 - ac_2)^2 - a^2)}{a^2 + (1 - ac_2)^2},$$

$$\widehat{x}_1 = \frac{\left((1 - ac_2)^2 - a^2\right)x_1 - 2a(1 - ac_2)x_3}{a^2 + (1 - ac_2)^2}.$$

We can easily see that equation (6),

$$x_{31}x_3 - x_{33}x_1 = -\widehat{x}_1,$$

is satisfied. Similarly, we can prove that another equation of (6) is satisfied as well.

Equation (10.2) means that $C = cJ$, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. THE LIE-POISSON INTEGRATORS FOR HEAVY TOP

In Section 1, we discussed the Lie-Poisson integrators for a free rigid body, which is a Lie-Poisson system on the dual space of semisimple Lie algebra. We now consider another kind of Lie-Poisson system—heavy top—which is constructed on the semidirect product of Lie algebra and linear space. The symmetric group for heavy top is three-dimensional Euclidean space $E(3)$, to which some important systems arising in hydrodynamics are also connected. On the phase space of $e^*(3)$, there are 6 coordinates $\{x_1, x_2, x_3, p_1, p_2, p_3\}$ and the Lie-Poisson brackets

$$\{x_i, x_j\} = \varepsilon_{ijk}x_k, \quad \{x_i, p_i\} = \varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = 0, \tag{11}$$

where

$$\varepsilon_{ijk} = \begin{cases} \text{the signum of the permutation } (i, j, k), & \text{if } i, j, k \text{ are all different,} \\ 0, & \text{if there is a pair of coinciding indiced } i, j, k. \end{cases}$$

The bracket (11) possesses two independent Casimir functions $f_1 = \sum p_i^2$, $f_2 = \sum p_i x_i$.

Let $H(x,p)$ be a Hamiltonian. Let us introduce the notation $u_i = \frac{\partial H}{\partial p_i}$, $\omega_i = \frac{\partial H}{\partial x_i}$. The Lie-Poisson equation will assume of "*Kirchhoff's equation*"

$$\dot{p} = [p, \omega], \qquad \dot{x} = [x, \omega] + [p, u], \tag{12}$$

where the square brackets denote the vector product. Equation (12) coincides (for quadratic Hamiltonians $H(x,p)$) with Kirchhoff's equations for the motion of a rigid body in a fluid which is perfect, incompressible, and at rest at infinity. The energy $H(x,p)$, quadratic in x,p and positive definite, can be given in the form

$$2H = \sum a_i x_i^2 + \sum b_{ij}(p_i x_j + x_i p_j) + \sum c_{ij} p_i p_j. \tag{13.1}$$

For the heavy top (the details are discussed and properties of the heavy top can be seen in [12]), the Hamiltonian has the reduced form

$$H(x,p) = \frac{x_1^2}{2I_1} + \frac{x_2^2}{2I_2} + \frac{x_3^2}{2I_3} + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3, \tag{13.2}$$

where I_i is defined as Section 2, and γ_i are the coordinates of the center of mass.

The structure matrix for this Lie-Poisson system is

$$\begin{pmatrix} J(x) & J(p) \\ J(p) & 0 \end{pmatrix},$$

where $J(x)$ is defined as in Section 2.

It is more difficult to construct Lie-Poisson integrators for heavy top than for free rigid body, for the generating function methods are not valid in this case. As done to free rigid body, the Lie-Poisson system for heavy top can also be split. So, using the composition methods and Lemma 2, we can easily construct the Lie-Poisson integrator for heavy top.

Splitting the Hamiltonian as $H = \sum H_i$, where

$$H_i = \frac{x_i^2}{2I_i}, \qquad H_{i+3} = \gamma_i p_i, \qquad \text{for } i = 1, 2, 3,$$

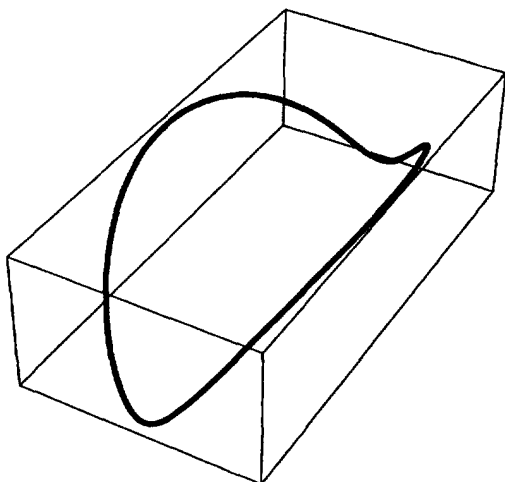we take H_1, H_4 as examples to construct the Lie-Poisson algorithm for subsystems.

Figure 1. The orbit curve of the Lie-Poisson for free rigid body.
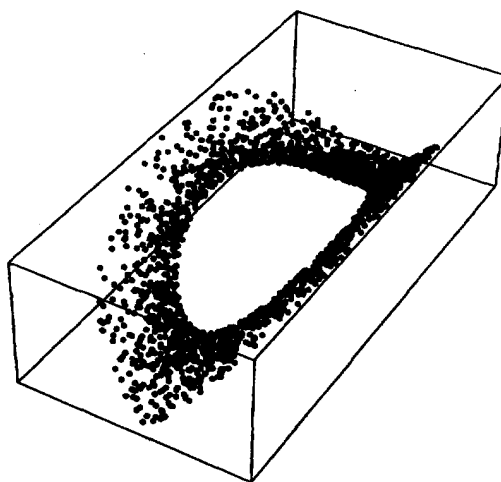
Figure 2. The orbit curve of the RK4 for free rigid body.
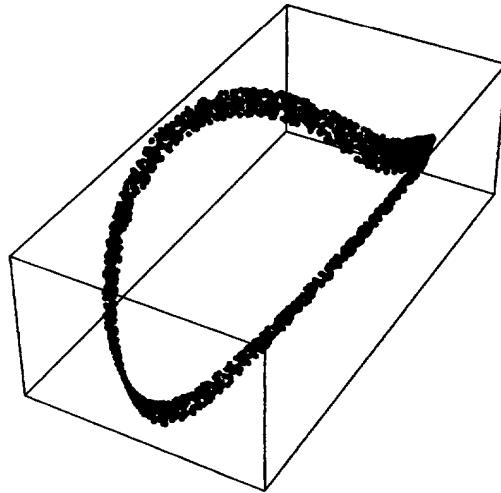
Figure 3. The orbit curve of the MD for free rigid body.
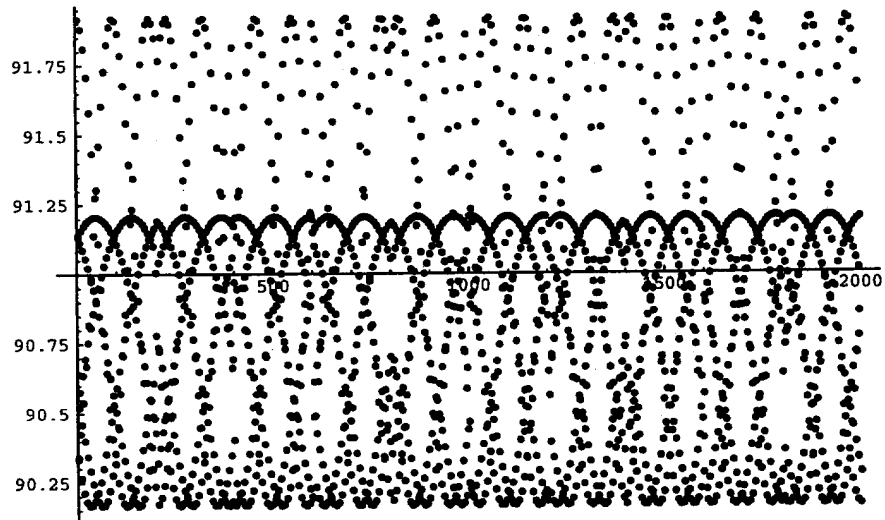

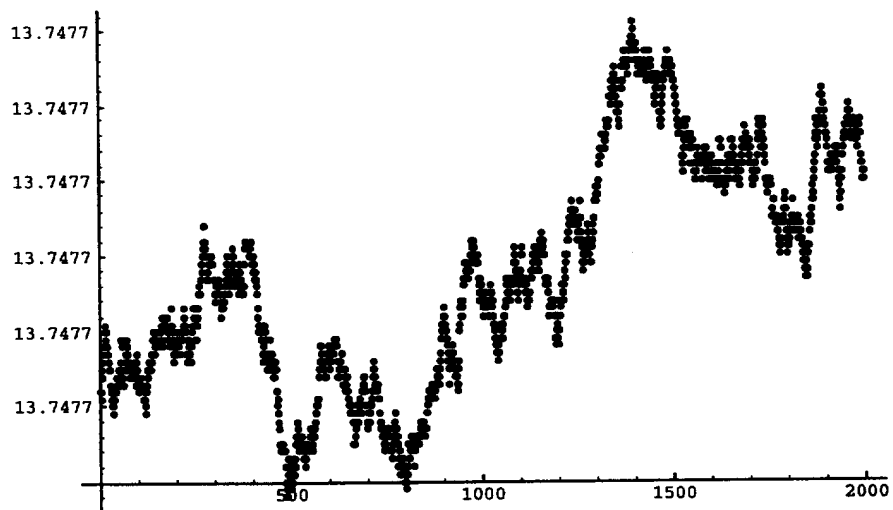Figure 4. The energy curve of the LP method.

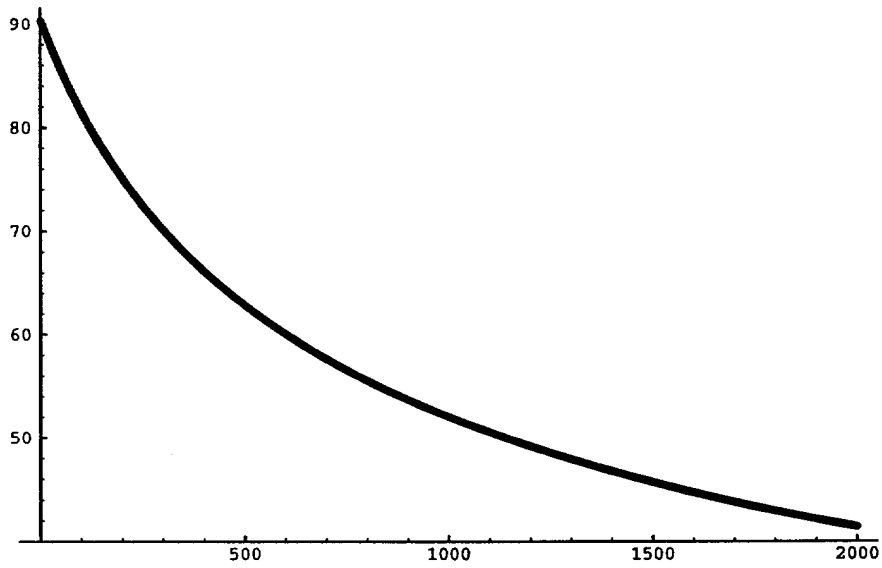Figure 5. The Casimir function curve of the LP method.
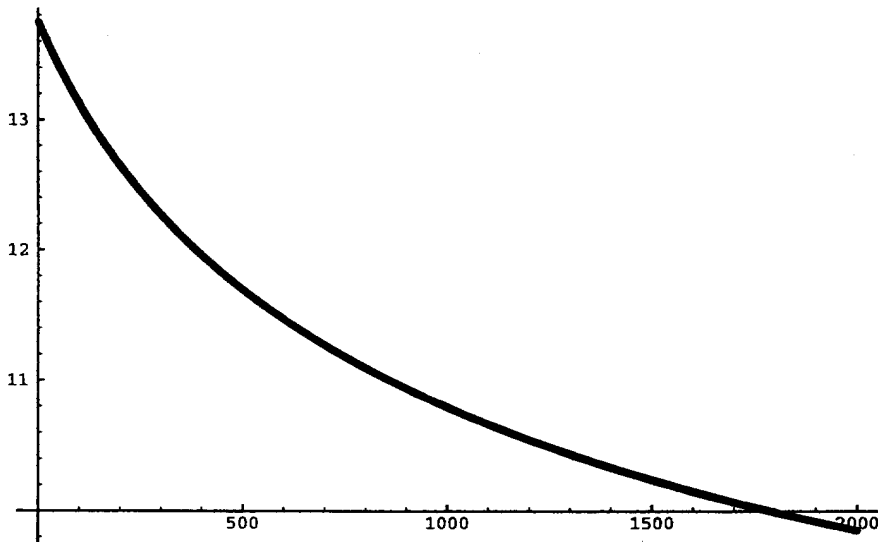
Figure 6. The energy curve of RK4 method.

Figure 7. The Casimir function curve of the RK4 method.

The equation for H_1 is

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} J(x) & J(p) \\ J(p) & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_1}{\partial x} \\ 0 \end{pmatrix}.$$

Expand this equation:

$$
\begin{aligned}
x_1 &= 0, \\
x_2 &= \frac{x_3 x_1}{I_1}, \\
x_3 &= -\frac{x_2 x_1}{I_1}, \\
p_1 &= 0, \\
p_2 &= \frac{x_1 p_3}{I_1}, \\
p_3 &= -\frac{x_1 p_2}{I_1}.
\end{aligned}
\tag{14}
$$

THEOREM 4. *The midpoint rule for subsystem (14) is Poisson.*
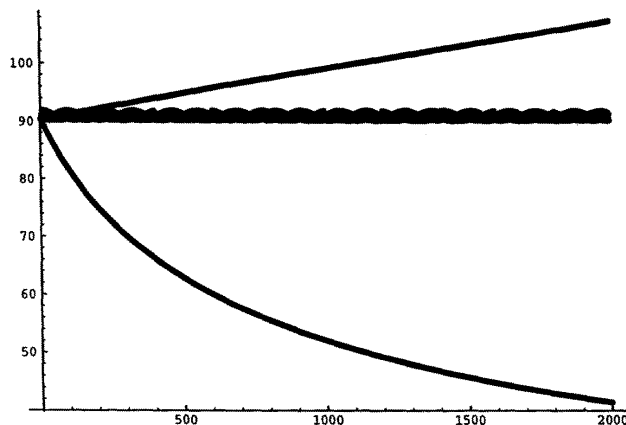
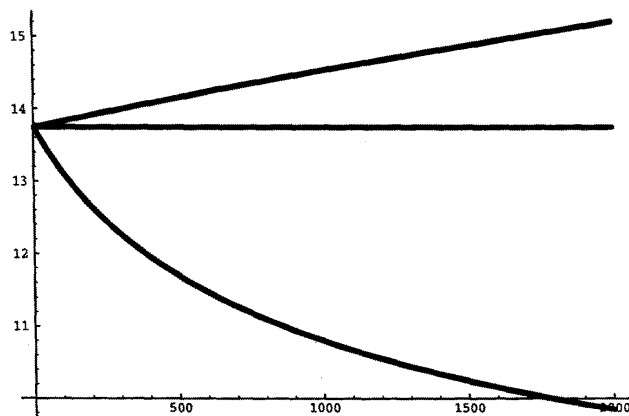Figure 8. The energy curve comparison of LP-MD-RK4 methods.

Figure 9. The Casimir curve comparison of LP-MD-RK4 methods.

Proof. By Theorem 2, to prove the midpoint rule for system (14) is Poisson iff to prove the Jacobi matrix of transformation $(x, p) \longrightarrow (\widehat{x}, \widehat{p})$ satisfy

$$\begin{pmatrix} \frac{\partial \widehat{x}}{\partial x} & \frac{\partial \widehat{x}}{\partial p} \\ \frac{\partial \widehat{p}}{\partial x} & \frac{\partial \widehat{p}}{\partial p} \end{pmatrix} \begin{pmatrix} J(x) & J(p) \\ J(p) & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \widehat{x}}{\partial x} & \frac{\partial \widehat{p}}{\partial x} \\ \frac{\partial \widehat{x}}{\partial p} & \frac{\partial \widehat{p}}{\partial p} \end{pmatrix} = \begin{pmatrix} J(\widehat{x}) & J(\widehat{p}) \\ J(\widehat{p}) & 0 \end{pmatrix}. \tag{15}$$

Let us denote the Jacobi matrix $\left(\frac{\partial y}{\partial z} \right) = \widehat{y}_z$. After expanding equation (15), we have

$$\begin{aligned} \widehat{x}_x J(x) \widehat{x}'_x &= J(\widehat{x}), \\ \widehat{x}_x J(x) \widehat{p}'_x + \widehat{x}_x J(p) \widehat{p}'_p &= J(\widehat{p}), \\ \widehat{p}_x J(x) \widehat{p}'_x + \widehat{p}_p J(p) \widehat{p}'_x + \widehat{p}_x J(p) \widehat{p}'_p &= 0. \end{aligned} \tag{16}$$

Using the result of Section 2, it is easy to verify the first equation of (16). Note that

$$\widehat{p}_x = \begin{pmatrix} 0 & p_{21} & p_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widehat{p}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & p_{23} \\ 0 & & \end{pmatrix}$$
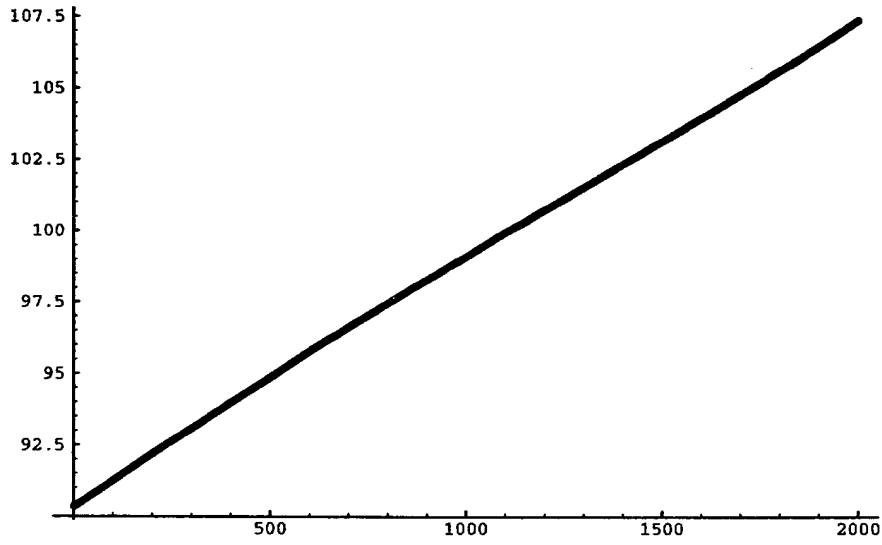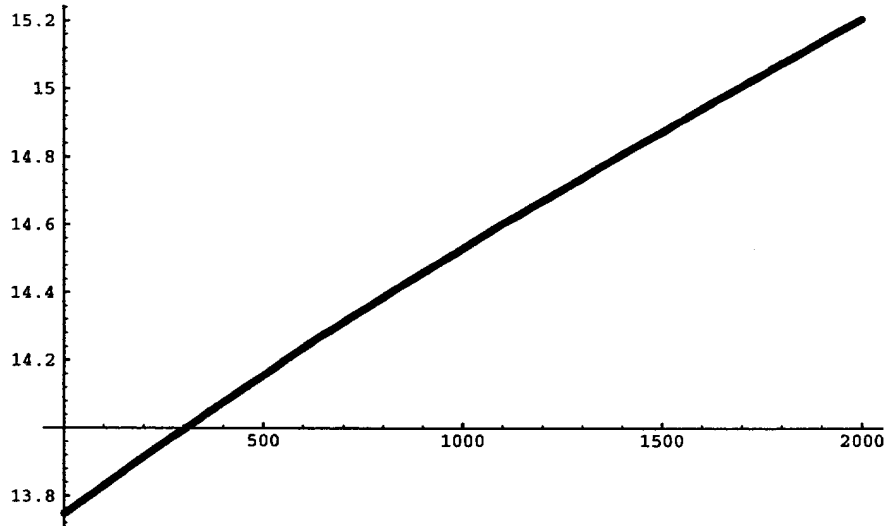
Figure 10. The energy curve of the MD method.

Figure 11. The Casimir function curve of the MD method.

where $p_{21} = \frac{\widehat{\partial p_2}}{\partial x_1}$, $p_{31} = \frac{\widehat{\partial p_3}}{\partial x}$, $p_{ij} = \frac{\widehat{\partial p_i}}{\partial p_j}$; $i, j = 2, 3$. After computation, we have

$$p_{22} = x_{22}, \quad p_{23} = x_{23}, \qquad p_{32} = x_{32},$$

$$p_{33} = x_{33}, \quad p_{21} = \frac{\tau \widehat{p_3}/I_1}{1 + a^2}, \quad p_{31} = \frac{-\tau \widehat{p_2}/I_1}{1 + a^2}, \tag{17}$$

where a is defined by equation (7).

Substitute (17) into equation (16). The equation is satisfied. Therefore, we can say that the midpoint rule for system (14) is Poisson.

It is easier to construct the Lie-Poisson integrator for the Hamiltonian H_4, for the equation is turned into a constant equation.

4. FURTHER DISCUSSION ABOUT THE RIGID BODY

In Sections 2 and 3, we have given the Lie-Poisson integrators for the special cases of rigid body. That is to say, we used the special Hamiltonian to construct the Lie-Poisson integrator. In
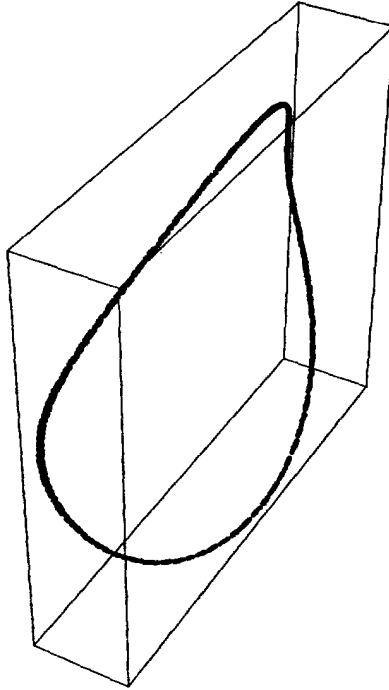
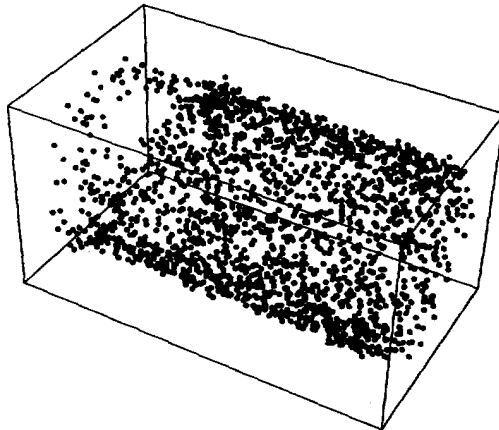Figure 12. The orbit curve of Lie-Poisson method for heavy top.

Figure 13. The orbit curve of Runge-Kutta 4 method for heavy top.

this section, we will give the Lie-Poisson integrator for the general form of Hamiltonian on rigid body.

Using free rigid body as an example, the general form of Hamiltonian has been given by equation (3). So we only construct a Lie-Poisson integrator for the subsystem when the Hamiltonian is $H_{ij} = (1/2)a_{ij}(x_i + x_j)^2$.

Consider the system

$$\dot{x} = J(x)\frac{\partial H_{12}}{\partial x}. \tag{18}$$

It is easy to see that $x_i + x_j$ is a Casimir function of equation (18). Expanding the equation gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -a_{12}x_3(x_1 + x_2) \\ a_{12}x_3(x_1 + x_2) \\ a_{12}(x_1^2 - x_2^2) \end{pmatrix}. \tag{19}$$

Since $x_1 + x_2$ is constant, we denote $c = x_1 + x_2$. Equation (19) is turned into

$$\dot{x}_1 = -ca_{12}x_3,$$
$$\dot{x}_2 = ca_{12}x_3, \qquad (20)$$
$$\dot{x}_3 = ca_{12}(c - 2x_2).$$

The last two equations of equation (20) can form the canonical equations on symplectic structure. The midpoint rule for (20) is not a Lie-Poisson scheme. However, we can numerically solve equation (20) explicitly, which, of course, is a Lie-Poisson integrator. Things are similar to the Lie-Poisson system for heavy top.

As pointed out in the introduction, the above Lie-Poisson integrator can also be angular momentum-preserving. See the author's paper [9].

5. NUMERICAL EXPERIMENT AND CONCLUSION

Using the above algorithms, we have computed several examples. In Example 1, we consider the motion of a free rigid body. In our numerical test, we first take the initial values $x_1 = 0.5$, $x_2 = 0.8$, $x_3 = 1.0$, the inertia operators to be $I_1 = 1.$, $I_2 = 2.$, $I_3 = 3$. (In the following examples, we also take this inertia operator.) The step length is 0.5 for Lie-Poisson and Midpoint methods and 0.1 for Runge-Kutta methods, the step number is 100000. Figures 1–3 give the orbit tracing of Lie-Poisson (LP) method, 4-order Runge-Kutta (RK4) method and Midpoint rule (MD). We find that the LP method is well orbit-preserving and the RK4 method is a poor orbit-preserving algorithm, though its step length is smaller and accuracy order is higher. In Figures 4–11, we have given their energy (i.e., the Hamiltonian of the system) function and Casimir function ($f = \sqrt{x_1^2 + x_2^2 + x_3^2}$) curves and their comparison. From the energy and Casimir preserving, we also can see that LP integrator is better than RK4 method and MD method. (In MD methods, we chose the relative iterative error to be 10^{-10}.) The computations also show that the LP algorithm is the fastest among the three methods and MD method is the slowest. Their CPU time to computing the same number of steps is, respectively, 3.729, 7.648, 31.933. All our computings are done on SGI workstation using the double precision. For the midpoint rule, because of solving nonlinear equations, the energy and the Casimir function is not well preserving so that the orbit have some spectral dissipation. If we chose the relative iterative error to be 10^{-17} (the digit number needed for double precision), MD method is also well energy-preserving and Casimir-preserving; but the method becomes slower.

In Example 2, we consider the motion of heavy top. We assume that the center of mass is in the z axes; i.e., $\gamma_1 = \gamma_2 = 0$. We chose $\gamma_3 = 1.0$, the initial value of x_i as $(2, 3, 4)$, p_i as $(0.3, 0.4, 0.5)$; the step length is 0.1 for LP and 0.01 for RK4, the step number is 100000. We only gave (in Figures 12 and 13) the phase trajectories of LP and RK4.

In conclusion, we can say that the Lie-Poisson Integrator is very good for long time tracing and very well orbit-preserved, which is very important for Celestial mechanics.

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