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STABILITY OF MOVING MESH SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. Moving mesh methods based on the equidistribution principle (EP) are studied from the viewpoint of stability of the moving mesh system of differential equations. For fine spatial grids, the moving mesh system inherits the stability of the original discretized partial differential equation (PDE). Unfortunately, for some PDEs the moving mesh methods require so many spatial grid points that they no longer appear to be practical. Failures and successes of the moving mesh method applied to three reaction-diffusion problems are explained via an analysis of the stability and accuracy of the moving mesh PDE.

Key words. moving mesh, equidistributing mesh, stability

AMS subject classifications. 65M50, 65M06, 65M12, 65M20

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1. Summary.

Many methods have been proposed for adapting the mesh to achieve good spatial resolution in the solution of PDEs. Some of these methods are *moving mesh methods* in which the meshpoint locations are computed simultaneously with the solution via an augmented system of differential equations. In principle, these methods have the advantage that they are able to move the mesh to follow a rapidly changing solution. However, obtaining a robust implementation of moving mesh methods has proven to be very difficult. This is true especially in problems with more than one spatial dimension.

In this paper we study moving mesh methods from the viewpoint of stability of the moving mesh system of differential equations. Following spatial discretization via the method of lines, a moving mesh PDE which attempts to equidistribute a given mesh function yields a system of differential-algebraic equations (DAEs), where equidistribution is the constraint. In section 2, we give an introduction to moving mesh schemes based on equidistribution. In section 3, the discretized moving mesh system is written as a DAE. Moving mesh methods which have been proposed in the literature are shown to be regularizations of this DAE. In many cases, theory for DAEs

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can be applied directly to yield convergence of the mesh to the equidistributing mesh, as the regularization parameter $\gamma \to \infty$. A technique is presented for investigating the stability of the discretized moving mesh PDE, based on the theory of stability for DAEs. This approach has been used in some simple and very smooth cases to show that the discretized moving mesh PDE inherits the stability properties of the original discretized PDE.

Unfortunately, for some PDEs, moving mesh methods require so many spatial grid points to maintain stability and accuracy that they no longer appear to be practical. In section 4, we present some numerical experiments for three reaction-diffusion problems. For two of the problems, a large number of meshpoints must be used to achieve a reasonable solution with the moving mesh, whereas the results on the fixed mesh are much better. The moving mesh method performs quite well on the third problem. In section 5, these results are explained via an analysis of the stability and accuracy of the moving mesh PDE. Such an analysis can give a good idea of when *not* to use a moving mesh method.

2. Background.

To begin, we consider a one-dimensional time-dependent PDE

$$u_t = f(x, u, t), \qquad x \in \Omega, \quad 0 < t < T, \tag{2.1}$$

with the initial and boundary conditions

$$u(x, t = 0) = u_0(x), \quad b(u, x, t) = 0, \quad x \in \partial\Omega.$$

The subscripts indicate partial derivative operators, Ω is an interval in $\Re$ with boundary $\partial\Omega$, the vector $u(x, t)$ lies in some function space, and f and b are nonlinear spatial differential operators.

In the Lagrangian frame, meshes move continuously with time, and equation (2.1) can be rewritten in the form

$$\dot{u} - u_x \dot{x} = f(x, u, t), \tag{2.2}$$

where $\dot{u} = u_t + u_x \dot{x}$ is a total derivative. Moving mesh methods determine the solutions u and the meshes x simultaneously.

Moving mesh methods are applied for time-dependent PDEs with large gradients. They allow automatic selection of meshes for both spatial x and temporal t according to the behavior of the solution. The idea of moving meshes is easy to understand, while the determination of effective moving mesh strategies has proven to be surprisingly difficult, in part because of problems with instability. Here we will be concerned mainly with finite-difference discretizations of moving mesh methods based on the EP [13]. Another class of moving-grid methods is based on the moving finite element (MFE) method, which was proposed in [28, 29, 18] and analyzed in [36, 25, 37]. Although much interesting research on the MFE method continues, that method now rests on a firmer theoretical foundation than the class of moving mesh methods we will consider. Here, we seek to develop a deeper understanding of moving mesh systems of PDEs based on equidistribution, particularly with respect to the effect of the moving mesh on stability of the spatially discretized system, and to the causes for breakdown of this method.

The EP has been one of the most important concepts in the development of moving mesh methods. Mathematically, the goal of finding mesh *functions* $\{x_i(t)\}_{i=1}^{N-1}$ or moving meshes

$$\Pi : \{a = x_0 < x_1(t) < \cdots < x_{N-1}(t) < x_N = b\}, \tag{2.3}$$

which are equidistributing for all values of t, means that we want

$$\int_{x_{i-1}(t)}^{x_i(t)} M(x,t)dx = \frac{1}{N}\int_a^b M(x,t)dx =: \frac{1}{N}\theta(t), \qquad i = 1, 2, \ldots, N. \tag{2.4}$$

This equidistribution equation can be written equivalently as

$$\int_{x_{i-1}(t)}^{x_i(t)} M(x,t)dx = \int_{x_i(t)}^{x_{i+1}(t)} M(x,t)dx, \quad i = 1, 2, \ldots, N-1, \tag{2.5}$$

where $M(x,t)$ is a positive monitor function. The arclength and curvature monitors are widely chosen to determine the positions of the mesh [38].

In [12], the stability of the equidistribution process has been studied for some moving mesh methods. Differentiating the equidistribution equation (2.5) yields

$$\begin{aligned} M_i\dot{x}_i(t) - M_{i-1}\dot{x}_{i-1}(t) + \int_{x_{i-1}(t)}^{x_i(t)} \frac{\partial}{\partial t}M(x,t)dx \\ = M_{i+1}\dot{x}_{i+1}(t) - M_i\dot{x}_i(t) + \int_{x_i(t)}^{x_{i+1}(t)} \frac{\partial}{\partial t}M(x,t)dx, \end{aligned} \tag{2.6}$$

where $M_i = M(x_i(t), t)$, which leads to instabilities of the moving mesh. In particular, the differentiation of equation (2.4)

$$M(x_i, t)\dot{x}_i + \int_a^{x_i} M_t(x,t)dx = \frac{i}{N}\frac{d\theta}{dt}, \qquad i = 1, \ldots, N, \tag{2.7}$$

is investigated. A linearized perturbation of (2.7) yields

$$M(x_i(t), t)\delta\dot{x}_i(t) + \frac{\partial M}{\partial x}(x_i(t), t)\dot{x}_i(t)\delta x_i(t) + M_t(x_i(t), t)\delta x_i(t) = 0,$$

i.e.,

$$\frac{d}{dt}[M(x_i(t), t)\delta x_i(t)] = 0. \tag{2.8}$$

Integrating from $t = 0$ to t,

$$\delta x_i(t) = \frac{M(x_i(0), 0)}{M(x_i(t), t)}\delta x_i(0). \tag{2.9}$$

If the monitor function is chosen such that $\frac{M(x_i(0),0)}{M(x_i(t),t)} > 1$, the mesh moving is unstable.

In part, the instabilities in the above scheme arise because the equidistribution constraint has been differentiated. Thus the solution can "drift off" the original constraints, a phenomenon which has often been observed in the solution of DAEs when using the differentiated constraint. In this case, the drift can cause the meshpoints even to leave the problem domain. This problem is easily corrected, as proposed in [12], by reintroducing the original constraints, either directly to obtain (following spatial discretization) a DAE or indirectly via a regularization of the DAE. However, the resulting DAE has often been observed to have problems with stability and/or spatial accuracy. Methods based on regularizations of the DAE appear to be more robust but depend on selecting an appropriate value for the regularization parameter which is problem dependent and apparently difficult to determine.

3. Moving mesh DAEs. In this section, we consider the equidistribution (2.5) as a constraint in moving mesh methods for solving time-dependent PDEs. Following spatial discretization and expanding the system by introducing the new variable $y = \dot{x}$ results in a semiexplicit DAE system

$$\dot{x} = y, \tag{3.1a}$$

$$\dot{u} = f(x,u) + h(x,u)y, \tag{3.1b}$$

$$0 = g(x,u), \tag{3.1c}$$

where $x = (x_1, x_2, \ldots, x_{N-1})^T$, $u = (u_1, u_2, \ldots, u_{N-1})^T$, and $g = (g_1, g_2, \ldots, g_{N-1})^T$ where

$$g_i = \int_{x_{i-1}(t)}^{x_i(t)} M(x,u)dx - \int_{x_i(t)}^{x_{i+1}(t)} M(x,u)dx \qquad \text{for} \qquad i = 1, 2, \ldots, N-1. \tag{3.2}$$

$M(x,u)$ is a monitor function which controls the movement of the mesh, and $h(x,u)$ is a diagonal matrix whose ith diagonal element is the difference approximation to u_x at x_i. The DAE system (3.1) is index-2 Hessenberg [9] if $g_x + g_u h(x,u)$ is nonsingular in a neighborhood of the solution.

In moving mesh methods, the PDE and the mesh equation are solved simultaneously as in (3.1). Although the stability of the equidistribution equation has been well studied, the stability of the *system* (3.1) has not to our knowledge been investigated. Moreover, most of the stability results which have appeared [23, 32, 33, 34] apply only to the continuous moving mesh PDE rather than to its discretization.

Our experience with moving mesh schemes based on equidistribution has been that when there is a problem with stability, the solution as well as the mesh is often affected. Any such problems can usually be corrected by allowing enough grid points. However, the number of grid points needed may be much larger than what is desirable for efficiency or necessary for spatial resolution. We have observed that the spatial discretization used in $h(x,u)$ (the discretization of u_x) can dramatically affect stability. For example, noting that the term $u_x\dot{x}$ in a sense convects the solution at a speed which is locally equal to the mesh velocity, it is not surprising that by upwinding this term, rather than using the centered difference as in most of the moving mesh literature, the equidistributing mesh system (3.1) can often be stabilized. High-order upwinding schemes for moving mesh systems are considered in [27] and offer a substantial improvement for convection-dominated problems, but they cannot stabilize the moving mesh for the reaction-dominated problems considered later in this paper.

Thus we believe that a complete understanding of the stability of moving mesh systems of PDEs based on equidistribution can only be obtained by studying the stability of the coupled, discretized system (3.1) and its regularizations. It is important to understand the stability of the equidistributing DAE (3.1) directly, rather than only that of its regularizations. Even if the regularized solutions are shown to converge to the solutions to (3.1) and to possess other good properties, they will not be robust with respect to the choice of the regularization parameter for all problems unless the limiting equation (3.1) is stable.

We will now examine several moving mesh methods which correspond to regularizations of the DAE. Usually, differentiating the constraint equation (3.1c) leads to mild instabilities for the numerical integration. For the stabilization of this type of DAE, some methods are suggested in [5]. One of the popular stabilization techniques is Baumgarte's method [7]. Here the constraint (3.1c) is replaced by

$$\gamma g(x,u) + \frac{d}{dt} g(x,u) = 0, \tag{3.3}$$

where γ is a positive constant to be chosen such that the numerical solution of (3.3) is stable. Application of this technique to stabilize moving mesh equations is suggested by Petzold in [16]. The partial differential form is studied in [21, 22]. If the initial meshes satisfy the equidistribution constraint, then the solution of (3.3) also satisfies the equidistribution constraint. Even if the initial meshes are random, asymptotically we have equidistributing meshes for large enough γ and t.

Some initial results on convergence of the regularizations are immediately available for the discretized system via theory which has recently been developed for DAE systems. For example, consider the regularized DAE arising from Baumgarte's stabilization

$$\dot{x} = y, \tag{3.4a}$$
$$\dot{u} = f(x,u) + h(x,u)y, \tag{3.4b}$$
$$0 = \gamma g(x,u) + \frac{d}{dt} g(x,u). \tag{3.4c}$$

We can apply the results of [6] to show that the manifold $\tilde{M}$ defined by $g(x,u) = 0$ is an asymptotically stable invariant manifold of the ODE (2.2) combined with (2.6) for $\gamma > 0$ and that the flow of (3.4c) on $\tilde{M}$ reduces to the flow of (2.2) and (2.6) restricted to $\tilde{M}$.

Let us now consider another moving mesh method, which can be written following spatial discretization as

$$\dot{x} = y, \tag{3.5a}$$
$$\dot{u} = f(x,u) + h(x,u)y, \tag{3.5b}$$
$$\frac{1}{\gamma} y = A^{-1} g(x,u). \tag{3.5c}$$

This method has been investigated in [1, 2, 4] for

$$A = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}. \tag{3.6}$$

A variant of this method takes A to be

$$A = \begin{pmatrix} -2M_1 & M_2 & & \\ \ddots & \ddots & \ddots & \\ & M_{i-1} & -2M_i & M_{i+1} \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.7}$$

and is closely related to the method of Hyman and Naughton [24].

The moving mesh system (3.5) can be considered to be a regularization of the Hessenberg index-2 DAE (3.1). Applying the theory in [26], we see that the solution to the regularized system (3.5) converges as $\gamma \to \infty$ to the solution of (3.1) if $A^{-1}(g_x + g_u h(x,u))$ is negative definite in a neighborhood of the solution.

It should be noted that the solution to the DAE (3.1) may not be well defined if $g_x + g_u h(x,u)$ becomes singular. At such a point, the DAE is singular and the solution may not exist or be unique. These are places where the moving mesh method can fail. This can happen for example with the arclength monitor at inflection points of u.

The MMPDEs defined in [21, 22] can all be seen to be regularizations of the type (3.4) or (3.5) of the DAE (3.1). Techniques motivated by temporal and spatial smoothing can also be seen as regularizations to the system (3.1). The temporal smoothing is determined by the time scale $1/\gamma$. The spatial smoothing is determined by the matrix A in (3.5). For example, for the method of Dorfi and Drury (see [14]), the matrix A is of the form

$$
\begin{aligned}
A_{i,i} &= -\left(\tfrac{\mu}{M_i(\Delta x_{i-1})^2} + \tfrac{1+2\mu}{M_i(\Delta x_i)^2} + \tfrac{1+2\mu}{M_{i-1}(\Delta x_{i-1})^2} + \tfrac{\mu}{M_{i-1}(\Delta x_i)^2}\right),\\
A_{i,i-1} &= \tfrac{\mu}{M_i(\Delta x_{i-1})^2} + \tfrac{1+2\mu}{M_{i-1}(\Delta x_{i-1})^2} + \tfrac{\mu}{M_{i-1}(\Delta x_{i-2})^2},\\
A_{i,i-2} &= -\tfrac{\mu}{M_{i-1}(\Delta x_{i-2})^2},\\
A_{i,i+1} &= \tfrac{\mu}{M_i(\Delta x_{i+1})^2} + \tfrac{1+2\mu}{M_i(\Delta x_i)^2} + \tfrac{\mu}{M_{i-1}(\Delta x_i)^2},\\
A_{i,i+2} &= -\tfrac{\mu}{M_i(\Delta x_{i+1})},
\end{aligned}
$$

where $\mu = \kappa(\kappa+1)$ is a spatial smoothing parameter. After introducing the spatial smoothing operator A, the system becomes extremely nonlinear. Different γ and A yield different condition numbers, which result in different convergence speeds when the system is solved by a stiff ODE or DAE solver. We must be careful when we choose γ and A. In our experience, A can be chosen independent of the problem, while γ is often a problem-dependent parameter and determines how fast the mesh tends to the equidistributed mesh. Generally, if strong reactions occur in a very short time interval, a larger γ should be chosen. This often happens in reaction-diffusion problems. However, for hyperbolic equations, a small γ often yields faster convergence than a large one. Note also that some regularizations are less sensitive to different parameters γ. The Dorfi and Drury method is relatively insensitive to the different time scales, compared with the MMPDE(1-6).

We have seen how spatial discretizations of moving mesh PDE systems based on equidistribution can be considered as a DAE or as a regularization of a DAE and how results on the convergence of DAE regularizations can be used to give conditions for convergence of the mesh to the equidistributing mesh (3.1). The next step is to examine the accuracy and stability of the discretized equidistributing PDE itself. We have already explained how the discretized PDE with equidistributing mesh (3.1) can be considered as an index-2 Hessenberg DAE. Fortunately, a theoretical framework has developed over the past few years to understand the stability properties of these types of problems [5, 6, 26].

The stability of a *linear* index-2 Hessenberg DAE system can be investigated by making use of the essential underlying ODE (EUODE) defined in [5]. Roughly speaking, by using the constraints to eliminate some of the degrees of freedom of the system, we can write a smaller ODE which shows how the DAE propagates information from one time to the next. A model for the linearized moving mesh system based on equidistribution can be given by

$$\dot{x} = y, \tag{3.8a}$$

$$\dot{u} = Au + By, \tag{3.8b}$$

$$0 = Cx + Du. \tag{3.8c}$$

To find the EUODE, suppose $[x, u] \in \Re^n$, $y \in \Re^m$. Define $v \in \Re^{n-m}$ by

$$v = -Bx + u. \tag{3.9}$$

Differentiating (3.9), we obtain

$$v' = -Bx' + u' - B'x. \tag{3.10}$$

We can obtain $[x, u]^T$ in terms of v by noting that

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} -B & I \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Solving for $[x, u]^T$,

$$\begin{aligned} x &= -(C + DB)^{-1}Dv, \\ u &= (I - B(C + DB)^{-1}D)v. \end{aligned} \tag{3.11}$$

Multiplying (3.8a,b) by $[-B, I]$ and rewriting in terms of v, we obtain the essential underlying ODE for (3.8),

$$v' = B'(C + DB)^{-1}Dv + A(I - B(C + DB)^{-1}D)v. \tag{3.12}$$

Asymptotic stability of the nonlinear index-2 Hessenberg DAE (3.1) can be studied locally via its linearization [31, 35].

For some simple and very smooth situations, we have been able to show using the EUODE approach that the discretized moving mesh equations (3.1) inherit the stability properties of the original discretized PDE. Huang and Russell [23]) and Smith and Stuart [34] have derived stability bounds for *continuous* forms of moving mesh PDEs based on the EP. A full analysis of the *discretized* moving mesh system, showing in general how coupling between the equidistribution constraint and the original PDE affects the stability of the moving mesh system, has yet to be accomplished.

Based on these limited results, one might hope that in general the stability of the original discretized PDE might be retained by the discretized moving mesh PDE. Unfortunately, these results require that the original system is dynamically stable, and that a sufficiently large number of spatial grid points is used. For many practical problems, in particular some strong reaction problems, the discretized moving mesh system is no longer stable or accurate. For these problems, moving mesh methods can bring about large errors and produce unacceptable solutions except on very fine grids. In the following, we investigate several such problems.

4. Numerical experiments. Many problems have steep wave fronts in their solution. Consider the class of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + g(u). \tag{4.1}$$

We have encountered difficulties in solving these kinds of problems with moving mesh methods. In order to have a clear understanding of the difficulties, we first consider the following two problems for which an analytic solution can be obtained. The moving mesh method we choose is EP with arclength monitor. The first problem is the famous Fisher equation [15]. The general form of the Fisher equation is

$$\frac{\partial u}{\partial t} = \beta \nabla^2 u + \alpha u(1-u).$$

In one dimension, for example, with $u = 1$ for $x < -a$ and $u = 0$ for $x > a > 0$, and some smooth function between, the time asymptotic solution to Fisher's equation is a front with constant thickness $L = 4 * (\frac{\beta}{\alpha})^{\frac{1}{2}}$ propagating to the right with constant speed $(\beta\alpha)^{\frac{1}{2}}$ (see [11]). We consider the following one-dimensional equation with exact solution:

$$\text{(4.2)} \qquad \begin{cases} \frac{\partial u}{\partial t} = u_{xx} + \alpha u(1-u), \\ u = \left(\frac{1}{1+e^{\sqrt{\frac{\alpha}{6}}x - \frac{5}{6}\alpha t}} \right)^2, \quad 0 \le x \le 1, \end{cases}$$

where the large parameter $\alpha = 10^4$. The left boundary and initial condition are derived from the exact solution. The right boundary is set to zero. Since the wave velocity is $c = \sqrt{\frac{25\alpha}{6}} > 0$, the solution reaches the steady state very rapidly (in about 0.005 time). In Figure 4.1, we show the reference solution computed with 200 nodes and central difference on a uniform mesh (UM), compared with a coarser grid (40 nodes) UM solution. All time integration in this section is done via DASPK [9, 10] with RTOL=10^{-6} and ATOL=10^{-6}. We can see that the front moves too fast using the coarser grid. Figure 4.2 shows the results using the moving mesh (Dorfi and Drury [14]) with 40 nodes and spatial smoothing parameter $\kappa = 2$. The results with the moving mesh are worse than with the fixed mesh. We also tried the values of $\alpha = 0.01$ and $\alpha = 0.001$. The results are similar. We also tried computing the solution on the interval $-1 \le x \le 1$. A wave front is formed before propagation at $x = 0$. Even if we use the equidistributed initial mesh, the solution does not improve much. At time $t = 0.003$ there is a strong oscillation in the solution. Results with the MMPDEs(1-6) with large regularization parameter and with the equidistributing DAE are similar.

The second problem is a heat conduction problem which has been considered by Coyle, Flaherty, and Ludwig [12] and Ren [32],

$$\text{(4.3)} \qquad \begin{cases} \frac{\partial u}{\partial t} = \mu u_{xx} + (r_2 + 2r_1^2 \mu u)(1-u^2) \\ u = \tanh(r_1 x + r_2 t), \quad -3 \le x \le 3, \end{cases}$$

with small diffusion term $\mu = 10^{-3}$ and $r_1 = r_2 = 5.0$. The wave velocity is $c = -\frac{r_2}{r_1} < 0$. The solution is a steep wave front propagating to the left side, which will reach the left boundary at time $t = 3$. The boundary and initial conditions are derived by the given exact solution. We tested it with 20 nodes and central difference in the spatial discretization. The moving mesh method we chose is MMPDE(6). We chose different time scales $\tau = \frac{1}{\gamma}$ to test this problem. The time scale τ determines how fast the mesh tends to the equidistributing mesh. The smaller the τ, the faster the mesh tends to the equidistributing mesh. The reference solution (solid line in the plots) is computed using the exact solution. We see that the UM method is surprising accurate. However, for the moving mesh method, the wave front moves too fast.

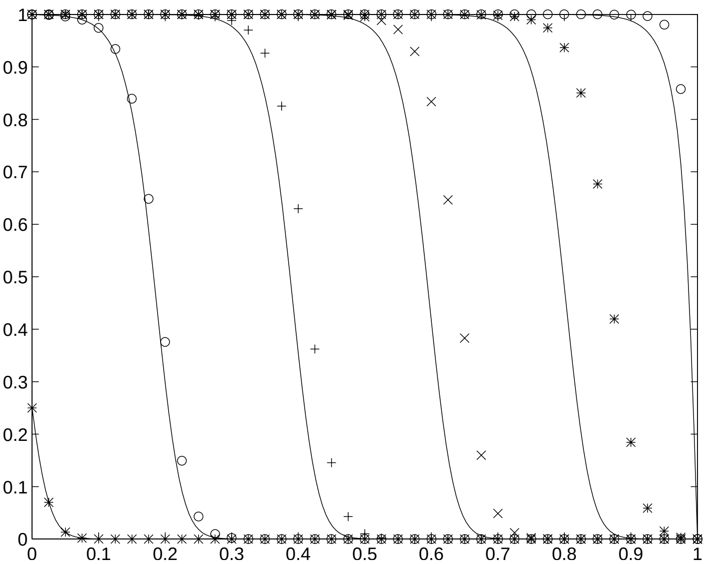

FIG. 4.1. *Results of uniform grid (41 nodes) for Fisher equation; reference solution: solid line; marked solution at* $t = 0.0(*), 0.001(o), 0.002(+), 0.003(x), 0.004(*), 0.005(o)$.

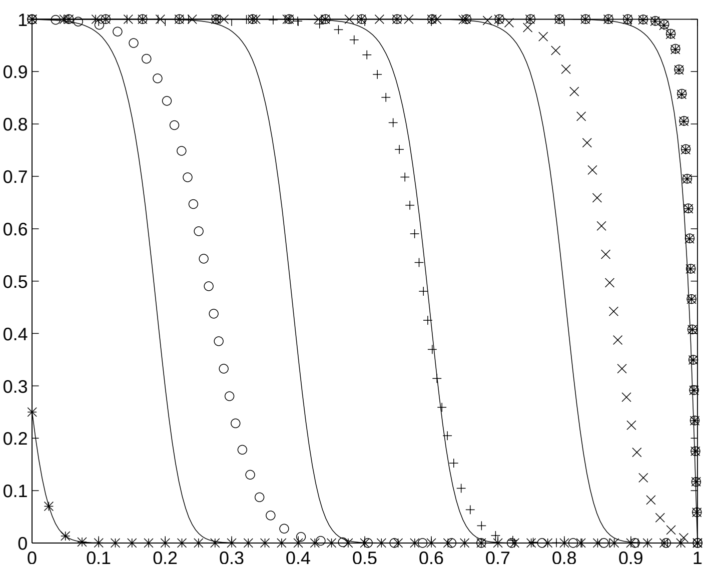

FIG. 4.2. *Results of moving mesh (Dorfi and Drury) method (41 nodes) for Fisher equation; time scale* $\tau = 10^{-6}$.

Moreover, there is a strong oscillation in the solution after time $t = 1.0$, which causes much difficulty for the Newton iteration of the DAE solver. We only plot the solution at time $t = 1$. The results are shown in Figure 4.3. Because in this problem the diffusion coefficient μ is independent of the exact solution, this allows us to choose different values of μ for the same solution. For example, we can choose $\mu = 1$ in (4.3). The results for the moving mesh are better than the fixed mesh now as shown in Figure 4.4. It should be pointed out that the wave front for this problem is not

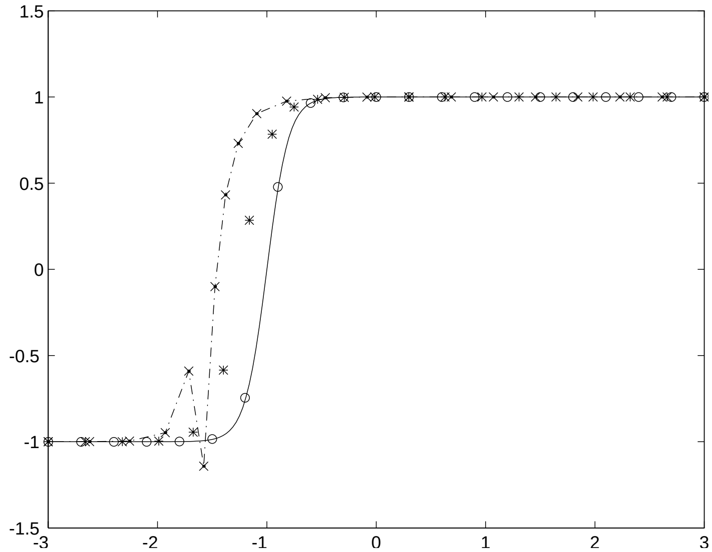

FIG. 4.3. *Results of moving mesh (21 nodes) for heat conduction problem ($r_1 = r_2 = 5.0$ and $\mu = 10^{-3}$) at $t = 1.0$; exact solution: solid line; marked solution: o(UM), $*(\tau = 1.0)$, $-x-$ $(\tau = 0.001)$.*

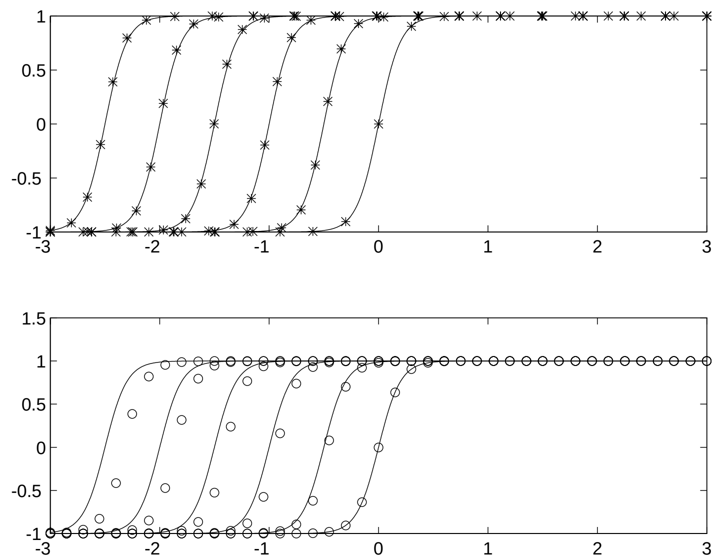

FIG. 4.4. *Results of moving mesh (top one with 21 nodes, $\tau = 0.001$) and UM (bottom one with 41 nodes) for heat conduction problem ($r_1 = r_2 = 5.0$, and $\mu = 1$); exact solution: solid line. Plot at times $t = 0.0, 0.5, 1.0, 1.5, 2.0, 2.5$.*

very steep. We can choose a big r_1 to steepen the wave front. We also have changed μ to other values with the new r_1 and r_2 and find that if

$$2r_2 + 2r_1^2\mu - 6r_1^2\mu < 0, \tag{4.4}$$

no matter how steep the wave front, the moving mesh can get good results (see Figure 4.5). Note that (4.4) is the derivative of the reaction term $g(u)$ when $u = -1$.

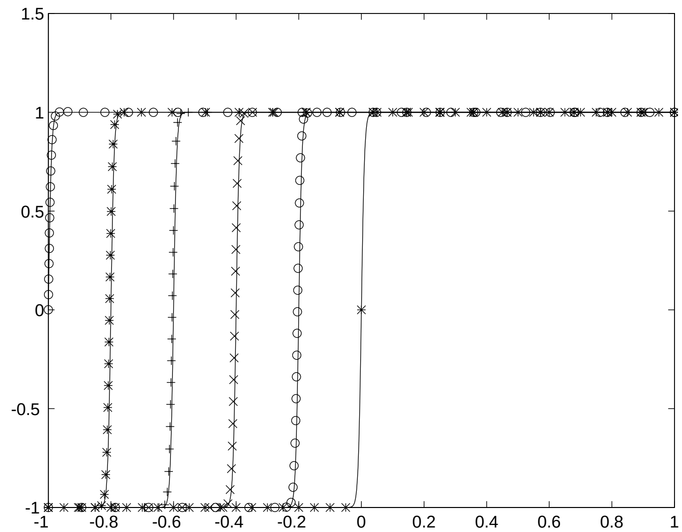

FIG. 4.5. *Results of moving mesh for heat conduction problem* ($r_1 = 100$, $r_2 = 10$ *and* $\mu = 10^{-3}$, $\tau = 0.001$) *with* 40 *nodes; reference: solid line; marked solution plotted at times* $t = 0.0, 2.0, 4.0, 6.0, 8.0, 10$.

We also have experienced other problems in related numerical experiments as follows.

1. The smaller the time scale τ, the bigger the errors, i.e., the front moves faster.

2. There is a constant time shift δt for each time interval, i.e., $u_{i+1} \approx u(x(\tilde{t}_{i+1} + \delta t), \tilde{t}_{i+1} + \delta t)$, where $\tilde{t}_{i+1}$ is the expected time with true value at t_i. δt changes very little between two adjacent intervals. The shift causes the computed solution to be further and further from the true solution.

3. When using a more stable scheme for the discretized PDEs, such as first-order upwinding, the results show some improvement. However, the above problems remain.

4. If the initial mesh is chosen to be the equidistributing mesh, the results are improved a bit, but the wave front still goes faster than expected after some time.

5. We experimented with higher-order spatial discretizations; however, the results did not improve a lot.

It should be noted that the moving mesh method does not fail for all reaction problems. For some problems, the moving mesh method can work very well. Take the following scalar combustion model (see [30] and [17] for details) as an example. The equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + D(1 + a - u)\exp(-d/u), \quad 0 < x < 1, \quad 0 < t, \tag{4.5}$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(1, t) = 1, \quad 0 < t,$$

$$u(x, 0) = 1, \quad 0 \le x \le 1,$$

where $D = Re^d/(ad)$ and R, d, a are constants. In numerical testing, the parameters are often chosen to be $a = 1$, $d = 20$, $R = 5$. We used the central difference for spatial

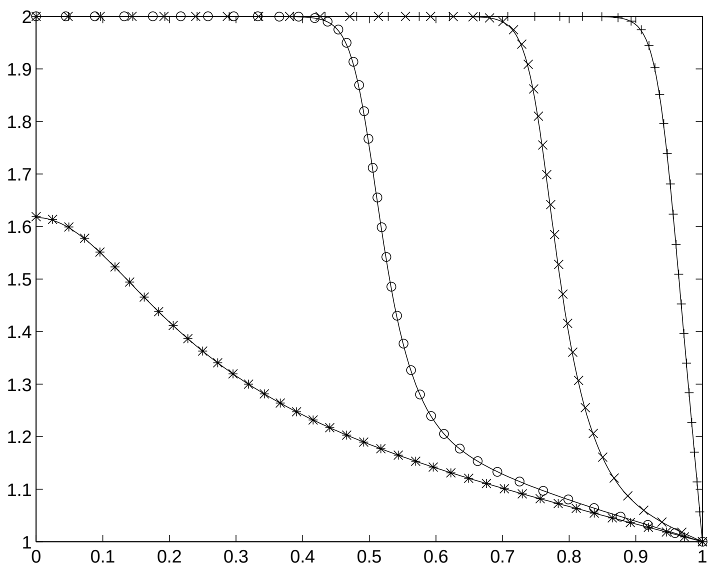

FIG. 4.6. *Results of moving mesh ($\tau = 0.001, 41$ nodes) for scalar combustion problem. Reference: solid line; marked solution plotted at times $t = 0.26, 0.27, 0.28, 0.29$.*

discretization and the Dorfi and Drury [14] method to control the mesh moving. The result is good as shown in Figure 4.6. The MMPDEs (1-6) [22] achieve similarly good results for this problem.

5. Stability and accuracy analysis for reaction-diffusion problems. In this section, we examine the stability and accuracy of moving mesh methods applied to reaction-diffusion equations and explain the results of the previous section. We explain the reasons why the moving mesh methods fail for some problems and point out the kind of problems which may not be practical to solve by moving mesh methods.

5.1. Discretization error. Suppose the coupled system of mesh equation and physical equation is

$$\begin{cases} \dot{u} = \mu u_{xx} + g(u) + u_x \dot{x} = f_1(x, u), \\ \dot{x} = f_2(x, u). \end{cases} \tag{5.1}$$

The truncation error introduced by the physical equation discretization is very significant for our problems. For moving finite-difference (MFD) methods, central space differencing is often chosen to discretize the terms u_{xx} and u_x because of its conservative and simple form. Suppose we have three consecutive nodes: $x - h_2, x$, and $x + h_1$. We know from the Taylor expansion that the central difference discretization satisfies

$$\begin{aligned} u'(x) &= \tfrac{u(x+h_1) - u(x-h_2)}{h_1+h_2} - \tfrac{1}{2}(h_1 - h_2)u''(x) - \tfrac{1}{6}\tfrac{h_1^3 u'''(\eta_1) + h_2^3 u'''(\eta_2)}{h_1+h_2}, \\ u''(x) &= u_{x\widehat{x}} - \tfrac{1}{3}(h_1 - h_2)u'''(x) - \tfrac{1}{12}\tfrac{h_1^3 u^{(4)}(\xi_1) + h_1^3 u^{(4)}(\xi_2)}{h_1+h_2}, \end{aligned} \tag{5.2}$$

where u' here denotes the derivative of u with respect to x, and $u_{\widehat{xx}}$ denotes the central difference approximation to u_{xx}. Inserting (5.2) into (5.1),

$$
\begin{aligned}
\frac{du}{dt} &= \mu u_{\widehat{xx}} - \mu\left[\frac{1}{3}(h_1-h_2)u''(x) + \frac{1}{12}\frac{h_1^3u^{(4)}(\xi_1)+h_2^3u^{(4)}(\xi_2)}{h_1+h_2}\right] \\
&\quad + g(u) + u_{\widehat{x}}\cdot\dot{x} - \left[\frac{1}{2}(h_1-h_2)u''(x) + \frac{1}{6}\frac{h_1^3u^{(3)}(\eta_1)+h_2^3u^{(3)}(\eta_2)}{h_1+h_2}\right]\dot{x} \\
&= \mu u_{\widehat{xx}} + g(u) + u_{\widehat{x}}\dot{x} + \tilde{f}_1(x,u),
\end{aligned}
\tag{5.3}
$$

where $u_{\widehat{x}}$ denotes the central difference approximation to u_x and the truncation error term is

$$
\begin{aligned}
\tilde{f}_1(x,u) = &-\mu\left(\frac{1}{3}(h_1-h_2)u'''(x) + \frac{1}{12}\frac{h_1^3u^{(4)}(\xi_1)+h_2^3u^{(4)}(\xi_2)}{h_1+h_2}\right) \\
&-\frac{1}{2}(h_1-h_2)u''(x)\dot{x} - \frac{1}{6}\frac{h_1^3u^{(3)}(\eta_1)+h_2^3u^{(3)}(\eta_2)}{h_1+h_2}\cdot\dot{x} \\
= &-(h_1-h_2)(\tfrac{1}{2}\dot{x}u'' + \tfrac{1}{3}u'''\mu) - \frac{1}{6}\frac{h_1^3}{h_1+h_2}(\dot{x}u^{(3)}(\eta_1) + \tfrac{1}{2}u^{(4)}(\xi_1)\mu) \\
&-\frac{1}{6}\frac{h_2^3}{h_1+h_2}(\dot{x}u^{(3)}(\eta_2) + \tfrac{1}{2}u^{(4)}(\xi_2)\mu).
\end{aligned}
\tag{5.4}
$$

Because the spatial discretization error in the moving mesh equation does not affect the accuracy of the solution variable, we can assume that there is no spatial truncation error for the moving mesh equations and consider only the spatial discretization error introduced in the original augmented equation.

There are two important factors which determine the size of the errors. One is the perturbation term $\tilde{f}_1(x,u)$. The other is the stability of the equation, which we will discuss in the next subsection. From the error term equation (5.4), we find that if the diffusion term μu_{xx} is much smaller than the convection term $u_x\dot{x}$, the truncation error will be dominated by the discretization error of the convection term. This term is introduced by the moving mesh and does not exist when using the UM. From (4.2) and (4.3), we note that when implementing the moving mesh, the mesh nodes at the wave front will move at the speed of the wave propagation velocity c. It is easily seen that the truncation error introduced by discretization of the convection term in these two cases is much larger than that of the diffusion term. For Fisher's equation, the node velocity $\dot{x}$ is very large. For the heat conduction problem, the diffusion parameter μ is very small. This explains in part why the results of the moving mesh methods are so much worse than that of the UM for both of these problems. The second-order derivative in the heat conduction problem has little to do with the solution in the parameter testing. If we drop it, the problem becomes a system of uncoupled ODEs, where no spatial truncation error is introduced.

One of the objectives of moving mesh methods is to gain accuracy. Hence, for our two problems, we should control the truncation error introduced in the convection term $u_x\dot{x}$. This can be done by redistributing mesh nodes according to the rule that where the leading terms of the truncation error are large, the spatial grid size should be small, whereas where the leading terms of the truncation error are small, the grid size can be larger. However, for some equidistributing moving mesh methods this objective is only partially realized. For example, the leading error term of the above problems is $u''(x)$ where the grid size varies in space, and $u'''(x)$ where the grid size is equally spaced. We cannot expect in general that the term $(h_1 - h_2)u''(x)$ is small. We can illustrate this from another point of view. It is well known that the EP is equivalent to the transformation from the uniform computational mesh to the physical mesh; i.e., there exists a mapping $\xi \to x$. From this, we get

$$h_j = x_{j+1} - x_j = x_\xi \Delta\xi + O(\Delta\xi^2), \quad h_{j+1} - h_j = x_{\xi\xi}\Delta\xi^2 + O(\Delta\xi^4),$$

so that

$$\begin{aligned} \tfrac{u_{i+1}-u_{i-1}}{x_{i+1}-x_{i-1}} &= u_x + \tfrac{1}{2}\Delta\xi^2(x_{\xi\xi}u_{xx} + \tfrac{1}{3}x_\xi^2 u_{xxx}) + \text{HOT} \\ &= u_x + \tfrac{1}{2}h_j^2\left(\tfrac{x_{\xi\xi}}{x_\xi^2}u_{xx} + \tfrac{1}{3}u_{xxx}\right) + \text{HOT}, \end{aligned}$$

where HOT denotes the higher-order terms. The difference scheme is of second order if and only if

$$\frac{x_{\xi\xi}}{x_\xi^2} = O(1). \tag{5.5}$$

Equation (5.5) is also called the quasi-EP [25]. We can see that the first term is proportional to the second derivative of the solution and hence represents a numerical diffusion, which depends on the variation of the grid point spacing. This numerical diffusion may even be negative and hence destabilizing. Attention must be therefore be paid to the variation of the spacing. Large changes in spacing from point to point cannot be allowed, or else significant truncation error will be introduced. If the spacing doubles from one point to the next, i.e.,$\frac{h_{i+1}}{h_i} = 2$, we have approximately $x_{\xi\xi} = 2x_\xi - x_\xi = x_\xi$ (we assume here that $\Delta\xi = 1$), so that the ratio of (5.5) is inversely proportional to the spacing x_ξ. Thus, for small spacing, such a rate of change of spacing would clearly be much too large. The important point is that the spacing not be allowed to change too rapidly in high gradient regions such as steep wave fronts or shocks. Because the local stretching factor $r_j = h_{j+1}/h_j$ is

$$r_j = \frac{h_{j+1}}{h_j} = \frac{x_\xi\Delta\xi + \frac{1}{2}\Delta\xi^2 x_{\xi\xi}}{x_\xi\Delta\xi - \frac{1}{2}\Delta\xi^2 x_{\xi\xi}} + \text{HOT}$$

or

$$r_j = 1 + h_j x_{\xi\xi}/x_\xi^2 + \text{HOT},$$

condition (5.5) implies that the stretching is quasi-uniform; i.e., the mesh should be smooth enough that it will not change greatly between two adjacent intervals. This can be achieved, for example, by the Dorfi and Drury method [14]. However, although this method achieves better results, the results for the first two problems are still unsatisfactory and much inferior to the equally spaced mesh. We must examine another important property, stability of the physical equation, to explain the poor results of the moving mesh methods for these problems.

5.2. Stability. We have seen that the truncation error is sometimes much larger when the moving mesh is introduced. If the resulting ODE is dynamically stable, we can get reasonable results by controlling this error term. In this section, we will see that stability is very significant to our problems.

From the form of the truncation error, we know that the error introduced by discretization of the convection term is much greater than that of the diffusion term.

Since the diffusion term is much smaller than the reaction and convection terms, for simplicity we drop this term and the truncation error introduced by it and consider only the truncation error introduced by the convection terms in (5.1). The reduced system is given by

$$\begin{cases} \dot{u} = g(u) + u_{\widehat{x}} \cdot \dot{x}, \\ \dot{x} = f_2(x, u), \end{cases} \tag{5.6}$$

and the perturbed system is

$$\begin{cases} \dot{u} = g(u) + u_{\widehat{x}} \cdot \dot{x} + \tilde{f}_1(x, u), \\ \dot{x} = f_2(x, u). \end{cases} \tag{5.7}$$

The following theorem will be used to give the difference caused by the perturbation term $\tilde{f}_1(x, u)$. The theorem is from [38].

THEOREM 5.1 (see [36, 37]). *Denote by y and z the solutions of the initial value problem*

$$\begin{aligned} \dot{y} &= f(t, y), & y(t_0) &= y_0, \\ \dot{z} &= f(t, z) + g(t, z), & z(t_0) &= y_0, \end{aligned}$$

respectively, and suppose that $\partial f / \partial y$ exists and is continuous. Then the solutions are related by

$$z(t) - y(t) = \int_{t_0}^{t} \frac{\partial y}{\partial y_0}(t, s, z(s)) \cdot g(s, z(s)) ds,$$

where the derivative with respect to initial values

$$\frac{\partial y(t, t_0, y_0)}{\partial y_0} = \Phi(t)$$

satisfies

$$\begin{aligned} &\Phi'(t) = \tfrac{\partial f}{\partial y}(t, y(t, t_0, y_0)) \cdot \Phi(t), \\ &\Phi(t_0) = I. \qquad \square \end{aligned}$$

From this theorem, we know that the global error is determined by the Jacobian $\frac{\partial f}{\partial y}$ and the perturbation term $g(t, z)$. Applying this theorem to our problem (5.7), the perturbation term is the truncation error $\tilde{f}_1(x, u)$. It will be very large if we use the UM at the initial time step. That is why the initial equidistributed mesh works better than the initial UM. When the mesh has been equidistributed, the truncation error term $\tilde{f}_1(x, u)$ will be a function independent of time t for the wave propagation problems. So we can think of it as a constant vector δ approximately. The Jacobian will be

$$C = \begin{pmatrix} g_u + C_1 & C_2 \\ (f_2)_u & (f_2)_x \end{pmatrix},$$

where (C_1, C_2) are the Jacobian of $u_{\widehat{x}} \dot{x}$ with respect to (u, x), and $(f_2)_u$ and $(f_2)_x$ are the Jacobian of f_2 with respect to (u, x). The global truncation error is not simply the sum of local truncation errors. To see this, we must realize that at each step, the numerical solution must use as its initial value the approximate ordinate computed at the preceding step. Thus, using the above theorem and ignoring the time truncation error, we have

$$\begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} = e^{C(t-t_0)} \begin{pmatrix} \delta_{01} \\ \delta_{02} \end{pmatrix} + \int_{t_0}^{t} e^{(t-\tau)C} d\tau \begin{pmatrix} \delta \\ 0 \end{pmatrix}. \tag{5.8}$$

The initial value

$$(\Delta u, \Delta x)_0 = (\delta_{01}, \delta_{02})$$

is due to the truncation error before reaching the state of equidistributed mesh. From (5.7) and (5.8), we know that once g_u is positive and very large, the error will grow very rapidly. Because the meshpoints are globally equidistributed by the moving mesh, the error in one node in the turning point or wave front can cause a similar error for its neighbor and then the truncation error will propagate quickly to all nodes. The truncation error, which is large at the turning points (for our two problems, this happens at the bottom corner of the wave front) will cause the nodes in that area to advance faster than expected.

We now analyze our problems. For the Fisher equation, $\frac{\partial g}{\partial u} = \alpha(1-2u)$; when α is very large, it is strongly unstable at the bottom corner around $u = 0$.[1] For the heat conduction problem,

$$\frac{\partial g}{\partial u} = -2r_2 u + 2r_1^2 \mu - 6r_1^2 \mu u^2.$$

When μ is very small, $g_u \simeq 2r_2$ at the bottom corner around $u = -1$, which can cause strong instability in that area. So the truncation error will soon become large in that region and propagate to other regions. In contrast, for the scalar combustion model (4.5) the derivative of the reaction term is

$$\frac{\partial g}{\partial u} = -D\exp(-20/u) + D(2-u)(20/u^2)\exp(-20/u) = D\exp(-20/u)\left[\frac{40}{u^2} - \frac{20}{u} - 1\right].$$

It is unstable if $u \leq 1.8$. However, we can see that the truncation error is very small in that area and the instability is not so severe where there is greater truncation error. At the bottom of the wave front (around $u = 1.0$), g_u is very small; thus, the truncation error will not grow as much for this problem. The diffusion term also plays an important role in stabilizing the problem.

The importance of stability can also be seen by our second example. If we change the parameter μ in (4.3) (because it is not related to the true solution of the equation), we find that as soon as μ is large enough so that g_u is negative or very small, i.e., the equation is stable or mildly unstable, the moving mesh will work very well no matter how steep the wave front is. This has been seen in Figure 4.4.

We have considered two strategies for improving the performance of moving mesh methods based on stability and accuracy. Because stability is more important, we would like to improve the stability first. One strategy to improve the stability of the discretized system is to use a more stable scheme. This can be done by adopting an

[1] Fisher originally developed this equation as a deterministic continuous model of a stochastic model of spread of genetic traits in a population. The unstable stationary solution was an important part of the model; recessive genes are unstable compared with dominate genes in a population of individuals.

upwinding scheme for the $u_x\dot{x}$ term. From equation (5.8), we know that if $g_u + C_1$ is negative or very small, the error can be controlled. When using the first-order upwinding scheme (without loss of generality, we assume $\dot{x} > 0$), C_1 will be of the following form:

$$\begin{pmatrix} \frac{-\dot{x}_2}{x_3-x_2} & \frac{\dot{x}_2}{x_3-x_2} & & & & \\ & \ddots & \ddots & & & \\ & & \frac{-\dot{x}_i}{x_{i+1}-x_i} & \frac{\dot{x}_i}{x_{i+1}-x_i} & & \\ & & & \ddots & \ddots & \\ & & & & & \frac{-\dot{x}_{n-1}}{x_n-x_{n-1}} \end{pmatrix} + \mathrm{diag}(u_{\widehat{x}})(f_2)_u.$$

If the region of instability is very small, the stability will be improved. However, this is at a cost of accuracy in that area. High-order upwinding schemes are considered in [27]. These offer a substantial improvement for the moving mesh method applied to convection-dominated problems but cannot stabilize the moving mesh method for problems like the Fisher equation so that it is competitive with a fixed grid method. Another approach is to use more nodes in the unstable area so that the spacing is small enough for stability of the stable difference schemes to dominant the instability due to the reaction terms. When the spacing is very small, the diffusion term contributes more to the stability (because the eigenvalues for it are proportional to $\frac{1}{(\Delta x)^2}$). We can use the monitor to distribute the nodes to the area where we want. One kind of monitor for our problem to distribute more nodes to the unstable region is to incorporate the reaction term into the monitor function, i.e., to choose the monitor as

$$M(x,u) = \left(1 + \frac{1}{2}(g_u(u) + |g_u(u)|)\right)\sqrt{1+u_x^2}. \tag{5.9}$$

This has a down side that the top corner is not resolved very well.

The next step is to reduce the truncation error. There are also two ways. One is to use higher-order schemes. However, the high-order accuracy schemes often decrease the stability of the systems. This is in contradiction to our objective of the first step, so we do not recommend it. Another is to place many nodes at the location where the spatial truncation error is large. This can reduce the truncation error sharply. This can be achieved by choosing a monitor function which includes some measurement of truncation error. Numerical experiments show that if we choose the curvature monitor instead of the arclength monitor, we can get a better result. We would like to use truncation error expressions in our monitor to distribute the nodes and get better accuracy. However, it is well known that numerical evaluation of higher-order derivatives can be subject to considerable computational noise. Therefore, it is usually not practical to use formal truncation error expressions in the monitor function in dynamic mesh moving. Hence lower-order derivative expressions are often adopted such as the arclength or curvature monitors. Some problems may arise even with solution curvature, i.e., with second derivatives, in rough transit. Numerical experiments [8] also show that implementations of the moving mesh methods with the curvature monitor are much less efficient than those with the arclength monitor. We have tested the Fisher equation using the strategies we have given. To reduce the truncation error due to using a uniform initial mesh, we use an equidistributed initial mesh. We use the interval $-1 \le x \le 1$ as our computing domain. We first use the

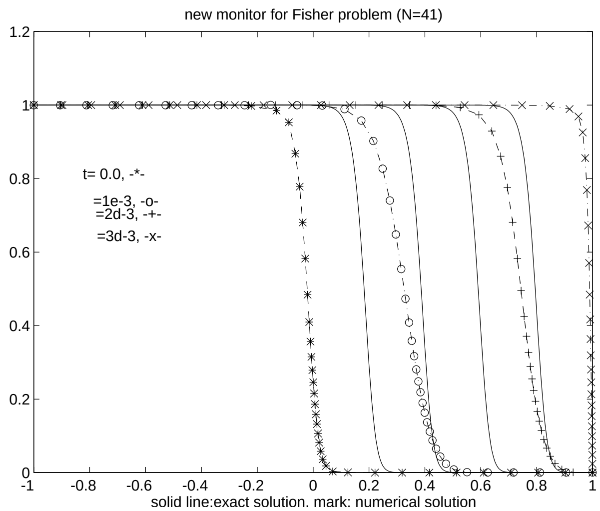

FIG. 5.1. *The results of moving mesh with monitor function* (5.9) *for the Fisher equation. The moving mesh still does not work even with the improved monitor.*

first-order upwinding scheme with the arclength monitor. The result is very bad. So we change to our new monitor (5.9). The result is still not very good as shown in Figure 5.1. Even for a small parameter, $\alpha = 100$, the front moves too rapidly.

Although all the strategies can be used for our problems, the result does not improve a lot if we do not use many nodes. However, the moving mesh methods are no longer advantageous for so many nodes. Therefore, for problems with strongly unstable regions, if the diffusion is much smaller than the reaction term, the fixed mesh method is recommended. Because there is no convection term appearing in the equation and the truncation error is relatively smaller and fast convergence can be obtained, the fixed mesh in this case can gain a better result than the moving mesh even if the wave front is very steep. When we use the moving mesh to solve a physical problem, we must be careful to observe whether the moving mesh term could introduce large truncation errors (compared with the original truncation error) and if the resulting system is strongly unstable. If either is true, the moving mesh methods cannot be recommended.

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