WENO Schemes for Cylindrical and Spherical Geometry

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Abstract

In this paper, we apply the high order WENO schemes to orthogonal uniform grid in cylindrical and spherical geometry. The 2-D or 3-D equations can be reduced to 1-D equations if the problem has the angular and radial symmetry. Several implementations are considered. It was shown that only high order WENO finite-volume schemes can achieve both the high order accuracy and the conservation. we have also shown that the global flux-splitting may fail to work even for high order WENO finite-difference schemes.

Key words: WENO schemes, cylindrical geometry, spherical geometry, finite volume method.

1 Introduction

WENO schemes become popular in numerical simulations involved discontinuities and sharp fronts. Most of WENO applications in literatures are for Cartesian grid. In this paper, we test WENO schemes for other types of orthogonal curvilinear uniform grid.

We consider a hyperbolic conservation law

$$u_t + \nabla \cdot f(u) = 0, \tag{1}$$

with an initial condition u_0. If we know the solution is radially symmetric, we can rewrite the equations in polar or spherical coordinates, obtaining a system that reduces to a problem in

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a single space variable r. In this paper, we are particularly interested in the Euler equations. In the cylindrical geometry, the radially symmetry can be used to reduce the Euler equations into

$$\begin{aligned}(\rho)_t + \frac{1}{r}(r\rho u)_r &= 0,\\ (\rho u)_t + \frac{1}{r}(r\rho\ u^2)_r + p_r &= 0,\\ E_t + \frac{1}{r}(r(E+p)u)_r &= 0.\end{aligned} \tag{2}$$

This system can be rewritten as

$$\begin{aligned}(\rho)_t + (\rho u)_r &= -(\rho u)/r,\\ (\rho u)_t + (\rho\ u^2 + p)_r &= -(\rho u^2)/r,\\ E_t + (r(E+p)u)_r &= -((E+p)u)/r,\end{aligned} \tag{3}$$

which has exactly the same form as the one-dimensional system of equations in Cartesian grid with the addition of a geometric source term on the right-hand side.

Similarly, for spherical grid, the Euler equations can be reduced to

$$\begin{aligned}(\rho)_t + \frac{1}{r^2}(r^2\rho u)_r &= 0,\\ (\rho u)_t + \frac{1}{r^2}(r^2\rho\ u^2)_r + p_r &= 0,\\ E_t + \frac{1}{r}^2\ (r^2(E+p)u)_r &= 0,\end{aligned} \tag{4}$$

or rewritten with a geometric source term

$$\begin{aligned}(\rho)_t + (\rho u)_r &= -(\rho u)/r^2,\\ (\rho u)_t + (\rho\ u^2 + p)_r &= -(\rho u^2)/r^2,\\ E_t + (r(E+p)u)_r &= -((E+p)u)/r^2.\end{aligned} \tag{5}$$

Even if the real problems of interest must be studied multi-dimensionally, radially symmetric solutions are very valuable in testing and validating numerical codes. A highly accurate solution to the one-dimensional problem can be computed on a fine grid and used to test solutions computed with the multidimensional solver. This is useful not only in checking that the code gives essentially the correct answer in at least some special cases, but also in determining whether the numerical method is isotropic or suffer from grid-orientation effects that lead to the results being better resolved in some directions than in others.

Many numerical solvers solve the reduced problem as a purely one-dimensional problem with a geometric source term (see [7]). This approach has a advantage that the existing numerical solver for the Cartesian grid can be used directly without modification. For the fully discrete methods for conservation law, discretized in both space and time, a Strang splitting [15] strategy is often used to handle the geometric source term separately. For semi-discrete method, where an ODE system is obtained by only space discretization (e.g., high order WENO schemes), the source term can be incorporated into the ODE system directly.

However there is a severe drawback to solve (3) (or (5)) directly. The original multidimensional conservation law is not preserved, i.e., the numerical schemes may not be conservative in multidimensional sense. Remember these equations are reduced from their multidimensional version. The original conservation law in mass and energy is preserved theoretically even in the reduced 1-D equations.

Conservation is a very important requirement in many numerical schemes to produce physical solution, especially for problems involving shock waves. Lax and Wendroff [6] has proved that the numerical solution of a conservative and consistent numerical scheme converges to the weak solution if it converges as mesh is refined. Many numerical examples have shown that a nonconservative method can fail to converge to a weak solution even if it has high order accuracy. Later in our numerical tests, we will show that applying the high order WENO fnite-difference schemes directly to the reduced system with the geometric source term yields wrong shock location and strength.

In order to preserve the conservation law, especially to preserve the total mass and total energy, we must adapt the numerical schemes for Cartesian geometry to the cylindrical and spherical geometry. For a second order scheme, this adaptation is easy to implement. However for a higher than second order scheme, this is not trivial.

WENO schemes are based on ENO scheme, which were first introduced by Harten, Osher, Engquist, and Chakravarthy [1] in the form of cell averages. The key idea of ENO schemes is to use the smoothest stencil among several candidates to approximate the fluxes at the cell boundaries to a high order accuracy, and at the same time to avoid spurious oscillations near shocks. The cell-averages (finite-volume) version of ENO schemes involves a procedure of reconstructing point values from cell-averages and could become complicated and costly for multi-dimensional problems. Later, Shu and Osher [12] developed a flux version (finite-difference) of ENO schemes which do not require such a reconstruction procedure.

In one spatial dimension, the finite-volume schemes and the finite-difference schemes are equivalent, both in numerical solution and accuracy and in complexity of coding and CPU timing. However, for multi-spatial dimensions, the finite-volume code becomes much more complicated and costly, while the finite-difference schemes are still very simple to code and fast to compute. It was pointed by Shu [14] that the finite-volume WENO scheme is about four times more expensive than the finite difference WENO scheme in 2-D, and is about nine times more expensive in 3D. However a main restriction on the finite-difference WENO schemes is that the third and higher order of accuracy can only be used on uniform rectangular or smooth curvilinear grid. Even for the smooth curvilinear grid, the finite-difference

WENO schemes have a risk to lose the conservation that they have on a uniform Cartesian grid. We will show it in the next section and in numerical tests.

The reconstruction of the point value from cell-average values is the key for both the finite-difference and the finite-volume WENO schemes. The finite-volume WENO scheme reconstructs the conservative variables at cell-interface, given the cell-average values of them. The finite-difference WENO scheme does not have cell-average values explicitly and hence it takes the point value of the flux at each grid point as a cell-average value of some flux function. Shu [14] have given an excellent review and comparison between these two types of schemes.

The outline of the paper is as follows. In section , we proposed several approached to extend the WENO finite-difference scheme for Cartesian grid to cylindrical and spherical grid. The advantages and disadvantages of each approach are discussed and compared. We also compared the global and local flux-splitting and pointed out the the global splitting is not not robust for curvilinear orthogonal uniform grid. The high order WENO finite-volume schemes are constructed in section 3.1. Finally, the numerical schemes are tested and compared with the Sedov explosion problem.

2 High Order WENO Finite-Difference Schemes

Consider the 1-D scalar hyperbolic conservation law,

$$u_t + f(u)_x = 0. \tag{6}$$

The WENO schemes for a uniform Cartesian grid can be written as

$$\frac{du_j}{dt} + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{h} = 0 \tag{7}$$

where $f_{j+\frac{1}{2}}$ is the numerical flux at cell interface, and h is local grid spacing.

We first consider the finite-difference WENO schemes, since it can be easily extended to multi-dimension. There are several approaches to extend the WENO schemes for Cartesian grid to cylindrical and spherical grid.

2.1 Extension to cylindrical and spherical grid

In the first approach, we can solve the equations with geometric source term, such as (3) and (5). Since the convection term is exactly the same as that in Cartesian grid, the WENO schemes can be plugged in immediately. A semi-discrete ODE system with high order ac-

curate discretization in space can be constructed easily. If we denote the geometric source term to be $g(r,u)$, then the WENO finite-difference scheme yields,

$$\frac{du_j}{dt} + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{h} = g(r_j, u_j), \tag{8}$$

where $g(r_j,u_j) = -(f_j)/r_j^\nu$, and ν is a constant that depends only on the dimension. $\nu = 1$ for 2-D cylindrical grid and $\nu = 2$ for 3-D spherical grid. Note that the order of the discretization is equal to the order of the WENO scheme.

There is a severe drawback for the approach (8). It does not preserve the original multi-dimensional conservation law, which is

$$\int_0^R u(t,r) r^\nu dr = \int_0^R u(0,r) r^\nu dr, \tag{9}$$

If we discretize (9) with midpoint rule, we obtain

$$\sum_{j=0}^{N} \overline{u}_j^n \frac{1}{\nu+1} (r_{j+\frac{1}{2}}^{\nu+1} - r_{j-\frac{1}{2}}^{\nu+1}) = \sum_{j=0}^{N} \overline{u}_j^0 \frac{1}{\nu+1} (r_{j+\frac{1}{2}}^{\nu+1} - r_{j-\frac{1}{2}}^{\nu+1}) \tag{10}$$

where $\overline{u}$ is the volume average value of u at cell $[r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}]$. In a conservative numerical scheme, we can replace $\overline{u}_j$ with point value u_j. It is clear that the WENO discretization (8) is not conservative in the sense of (10).

The second approach is to do coordinate transformation $x = \frac{1}{\nu+1} r^{\nu+1}$. Then we obtain

$$\frac{\partial (f(u) r^\nu)}{r^\nu \partial r} = \frac{\partial (f(u) r^\nu)}{\partial x}. \tag{11}$$

We can extend the high order WENO scheme to (11) since the grid in x is a smooth curvilinear grid. Denote

$$\tilde{f}(x,u) = f(u) r^\nu = f(u) \left((\nu+1) x \right)^{\frac{\nu}{\nu+1}}.$$

Then as suggested by Shu [14], the WENO scheme for (11) will have form

$$\frac{\partial (f(u) r^\nu)}{r^\nu \partial r} = \frac{\tilde{f}_{j+\frac{1}{2}} - \tilde{f}_{j-\frac{1}{2}}}{r_j^\nu dr}, \tag{12}$$

where $\tilde{f}_{j+\frac{1}{2}}$ is the numerical flux at the interface for uniform grid r. Although (12) does not satisfy conservative form (10), it satisfies an approximate conservation law,

$$\sum_{j=0}^{N} u_j r_j^\nu dr = const,$$

which is also important to have the solution converge to the weak solution. Unfortunately, this approach does not work well near the origin ($r = 0$). It might be because that the flux $f(u)$ is flattened by the scaling factor r^ν, and it has large error near the origin $r = 0$. In numerical experiments, we found the pressure became negative after a short time near the origin.

The third approach is to use the flux generated from the Cartesian grid directly and modify it for use of cylindrical and spherical geometry as

$$\frac{du}{dt} + \frac{r^\nu_{j+\frac{1}{2}} f_{j+\frac{1}{2}} - r^\nu_{j-\frac{1}{2}} f_{j-\frac{1}{2}}}{dV_j} \tag{13}$$

where $dV_j(\nu) = \frac{1}{1+\nu}(r^{1+\nu}_{j+\frac{1}{2}} - r^{1+\nu}_{j-\frac{1}{2}})$ is the local control volume. For the discretization of momentum equation, there is additional source term $\nu p/r$ on the right side of the equation. To be consistent with the conservation law in multi-dimension, this term must be discretized as $p_j \cdot dV_j(\nu)/dV_j(\nu - 1)$.

We will show that the numerical scheme (13) is of only second order if the high order numerical flux $f_{j+\frac{1}{2}}$ is calculated as in (7). We take the spherical geometry as an example. At a fixed $r = r_j$, we have

$$\frac{r^2_{j+\frac{1}{2}} f_{j+\frac{1}{2}} - r^2_{j-\frac{1}{2}} f_{j-\frac{1}{2}}}{r^3_{j+\frac{1}{2}}/3 - r^3_{j-\frac{1}{2}}/3} - \left(\frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{h} + \frac{2f_j}{r_j} \right) = O(h^2)$$

Therefore, (13) is only of second order.

2.2 Flux-splitting for finite-difference schemes

We can apply the 5th order WENO finite-difference scheme to (3) directly to achieve the high order, though it is not a conservative scheme. For achieving numerical stability and for avoiding entropy violation solutions, upwinding and flux-splitting approaches are used in constructing the WENO flux. The flux is written as a sum of the positive and negative fluxes, $f^\pm(u)$,

$$f(u) = f^+(u) + f^-(u),$$

where f^+ has positive eigenvalues and f^- has negative eigenvalues. There are several choices for defining the flux. A simple choice is given by the Lax-Friedrichs splitting, which produces very smooth fluxes,

$$f^\pm(u) = \frac{1}{2}\left(f(u) \pm \alpha u\right),$$

where α is taken as $\max_u |f'(u)|$ over the relevant of u. If the range is locally defined, it is called the local Lax-Friedrichs (LLF) splitting; if the range is global, it is called the global Lax-Friedrichs splitting (LF). For lower order schemes the quality of the solution is usually very sensitive to the choice of the splitting, and the Lax-Friedrichs flux is very diffusive. But

it is claimed in [3] that this sensitivity is much less important for a higher-order method. It is recommended in [4] that the global LF be used for fifth order WENO scheme. In numerous examples of Cartesian grid, we have not observed the fifth order WENO scheme failed with global LF. We have pointed out in [2] that LLF is required to achieve the conservation law strictly in adaptive mesh method and parallel computation via domain decomposition. Even in those cases, we found that a global LF can give reasonable results. Later we will find that this may no longer true for spherical and cylindrical grid.

We remark that the LF may fail in other cases where the global α is far more larger than the local α near the shock and other discontinuities.

We have investigated the possibility of developing conservative high order finite-difference WENO schemes of form (13). As indicated in Shu [14], the high order WENO finite-difference schemes can be constructed only for smooth non-uniform mesh.Our mesh in $x = \frac{1}{\nu} r^\nu$ satisfies this condition. However, the high order WENO scheme is constructed for $\frac{\partial (f(u) r^\nu)}{\partial r}$ only. The numerical example showed that the (12) is no better than other discretization.

3 High Order WENO Finite-Volume Scheme

3.1 Finite-Volume discretization

WENO (or ENO) scheme can be of higher than second order. The high order WENO finite-volume scheme of form (13) can be developed following the way of Liu et al. [8]. For finite-volume scheme, we have cell-average values at each cell, defined by

$$\overline{u}_j(\cdot, t) = \frac{1}{dV} \int_{r_{j-\frac{1}{2}}}^{r_{j+\frac{1}{2}}} u(r, t) r^\nu dr.$$

Integrate the hyperbolic system

$$u_t + \frac{\partial r^\nu f(u)}{r^\nu \partial r} = 0$$

over each cell, we obtain

$$\overline{u}_t = -\frac{1}{dV} (r^\nu_{j+\frac{1}{2}} f(u(r_{j+\frac{1}{2}}, t)) - r^\nu_{j-\frac{1}{2}} f(u(r_{j-\frac{1}{2}}, t))).$$

To achieve the high order accuracy, we need to evaluate $u(r_{j+\frac{1}{2}}, t)$ at interface $r = r_{j+\frac{1}{2}}$ to high order. First, from the given cell average $\overline{u} = \{\overline{u}_j\}$, we can reconstruct a piecewise high order interpolation polynomial $R(r)$. Next the $u(r_{j+\frac{1}{2}}, t)$ can be evaluated to high order by $R(x)$ locally. The flux at the interface then is evaluated by a numerical flux (approximate or exact Riemann solvers). For example, one could use the simple Lax-Friedrichs flux, which is

given by

$$f(u(r_{j+\frac{1}{2}}),t) = \frac{1}{2}(f(u^+,t) + f(u^-,t) - \alpha(u^+ - u^-)),$$

where α is taken as the upper bound for the eigenvalues of the Jacobian at $r_{j+\frac{1}{2}}$, and u^+ and u^- are the values of $u(r_{j+\frac{1}{2}})$ from the right and left side of interface.

3.2 Reconstruction procedure

Following the reconstruction procedure of Liu et al. [8], given cell averages $\{\overline{u}_j\}$, we can immediately evaluate the point values of the solution's primitive function $W(r)$ at interfaces $W(r_{j+\frac{1}{2}})$, where the primitive function is defined as

$$W(r) = \int_{r^{j'-\frac{1}{2}}}^{r} u(\xi,t)\xi^\nu d\xi. \tag{14}$$

Note that there is a difference between Cartesian and other orthogonal grids to define the primitive function. However, if we set $x = \frac{1}{\nu} r^\nu$ then

$$W(x) = \int_{x^{j'-\frac{1}{2}}}^{x} u(\xi,t)d\xi, \tag{15}$$

which is more like the primitive function defined by Liu et al. [8]. Differentiating (15) with respect to x yields

$$u(x,t) = \frac{d}{dx}W(x). \tag{16}$$

It is also obvious from (15) that

$$W(x_{j+\frac{1}{2}}) = \sum_{i=j'}^{j} \overline{u}_i dV_i, \tag{17}$$

where

$$dV_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{1}{\nu+1}(r^\nu_{i+\frac{1}{2}} - r^\nu_{i-\frac{1}{2}})$$

is the local control volume. Unlike the conservation law in 1-D uniform grid, the local control volume is no longer constant here. It depends on the location of the cell.

To reconstruct the solution, we interpolation $W(x)$ on each stencil $S_j = (x_{j-j_1+\frac{1}{2}}, ..., x_{j+j_2+\frac{1}{2}})$ to obtain a polynomial $p_j(x)$, i.e.,

$$p_j(x_{l+\frac{1}{2}}) = W(x_{l+\frac{1}{2}}), \quad l = j - j_1, ..., j + j_2.$$

Obviously the corresponding polynomial $p_j'(x)$ approximate the solution $u(x,t)$ to high order. The difference between our reconstructions and those in Liu et al. [8] is that the grid in x is non-uniform in cylindrical and spherical grid.

The ENO schemes choose the least oscillation one among all possible stencils. The WENO scheme instead is a convex combination of all possible stencils. Unlike in the uniform mesh, the optimal weights for the WENO scheme now depend on the local grid spacing in x. For example, the optimal weights for the third-order WENO scheme on stencil $[x_{i-1}, x_i, x_{i+1}]$ at $x_{i+\frac{1}{2}}$ is

$$C = \begin{cases} \frac{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{3}{2}}} - \frac{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}{x_{i+\frac{3}{2}} - x_{i-\frac{3}{2}}}, & \text{for stencil } [x_{i-1}, x_i] \\ \frac{(x_{i+\frac{1}{2}} - x_{i-\frac{3}{2}})(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})}{(x_{i+\frac{3}{2}} - x_{i-\frac{1}{2}})(x_{i+\frac{3}{2}} - x_{i\frac{3}{2}})}, & \text{for stencil } [x_i, x_{i+1}] \end{cases} \tag{18}$$

It can be verified that the weights are $\frac{1}{3}$ and $\frac{2}{3}$ for uniform grid. In case of $x = r^\nu$, we have

$$C_0 = \begin{cases} \frac{r_i + h}{6r_i - 3h} & \text{if } \nu = 2 \\ \frac{(3r_i^2 + \frac{1}{4}h^2)(3r_i^2 + 6r_i h + \frac{13}{4}h^2)}{6(3r_i^2 - 3r_i h + \frac{7}{4}h^2)(3r_i^2 + \frac{9}{4}h^2)} & \text{if } \nu = 3 \end{cases}$$

$$C_1 = \begin{cases} \frac{2r_i - h}{6r_i + 3h} & \text{if } \nu = 2, \\ \frac{(6r_i^2 - \frac{3}{2}r_i h + \frac{7}{2}h^2)(3r_i^2 + \frac{1}{4}h^2)}{(6r_i^2 + \frac{3}{2}r_i h + \frac{7}{2}h^2)(9r_i^2 + \frac{27}{4}h^2)}, & \text{if } \nu = 3 \end{cases}$$

The actual weight of the stencil in the convex combination can be evaluated by smoothness indicator in each stencil. Here we use the measure taken from Jiang and Shu 1997 [4], which amounts to a measure on the L^2-norms of the derivatives. Again, when we evaluate the smoothness measure, the non-uniformity of the grid should be considered. For the third-order WENO scheme, it yields the measure as for the uniform grid. However for the fifth-order, the results are different from those of the uniform grid.

We remark that for system of equations, the WENO reconstruction should be performed in local characteristic fields.

4 Numerical Results

In this section, we show some numerical results for Sedov problems. We focus ourselves to the results of WENO scheme. The reference solutions were calculated with second order HLLC

Riemann solve with very fine grid.

4.1 Sedov Explosion Problem

The Sedov explosion problem (Sedov 1959) is a purely hydrodynamical example. The problem involves the self-similar evolution of a cylindrical or spherical blast wave from a delta-function initial pressure perturbation in an other wise homogeneous medium. To initialize the code, we deposit a quantity of dimensionless energy $\epsilon = 1$ into a small region of radius δr at the center of the grid. The dimensionless pressure inside this volume, P_0, is given by

$$P_0' = \frac{3(\gamma - 1)\epsilon}{(\nu + 2)\pi\delta r^{\nu+1}},$$

where $\nu = 1$ for cylindrical geometry and $\nu = 2$ for spherical geometry. In running this problem, we choose δr to be as large as the width of the ghost cells in order to minimize the effects of the reflection boundary conditions. The density is set to $\rho = 1$ throughout the grid and the pressure is set to a small value $P_0 = 10^{-5}$ except in the explosion region. The fluid is initially at rest. In the self-similar blast wave that develops for $t > 0$. the density, pressure, and radial velocity are all functions of $\xi = r/R(t)$. where

$$R(t) = \beta_\nu(\gamma)\left(\frac{\epsilon t^2}{\rho_0}\right)^{1/(\nu+3)}$$

Here β_ν is a dimensionless constant depending only on ν and γ; for $\gamma = 1.4$, $\beta_2 \approx \beta_3 \approx 1$ to within a few percent. Just behind the shock front at $\xi = 1$ we have

$$\rho = \rho_1 = \frac{\gamma + 1}{\gamma - 1}\rho_0 \tag{19}$$

$$P = P_1 = \frac{2}{\gamma + 1}\rho_0 w^2 \tag{20}$$

$$v = v_1 = \frac{2}{\gamma + 1}w \tag{21}$$

where $w = dR/dt$ is the speed of the shock wave. The analytical solution of the Sedov problem can be derived implicitly. Near the center of the grid (origin), the solutions are

$$\rho(\xi)/\rho_1 \propto \xi^{(\nu+1)/(\gamma-1)}$$

$$P(\xi)/P_1 = \text{constant} \tag{22}$$

$$v(\xi)/v_1 \propto \xi \tag{23}$$

Although there is an analytical solution for the reduced 1-D problem. It is defined implicitly and not readily available. Therefore, we calculated the reference solutions with very fine grid and conservative second order method.

4.2 Results for two-dimensional Sedov problem

The 2D Sedov problem with cylindrical geometry can be reduced to 1-D problem with a source term. The WENO scheme of (12) failed at $t = 0.002424$ due to the negative pressure. At $t = 0.002$, the solution seems OK but has large difference from the reference solutions, especially near the origin.

The fifth order WENO finite-difference scheme with LF flux-splitting failed to produce steep shock front (see Fig.4.1-a) for both conservative 13 and nonconservative version 8. We initially thought that it was due to the steep initial conditions for pressure. So we calculated the solution with our conservative second order method to time $t = 0.04$ and then we advance the solution from $t = 0.04$ using the WENO with LF flux-splitting. However, we obtained almost the same results.

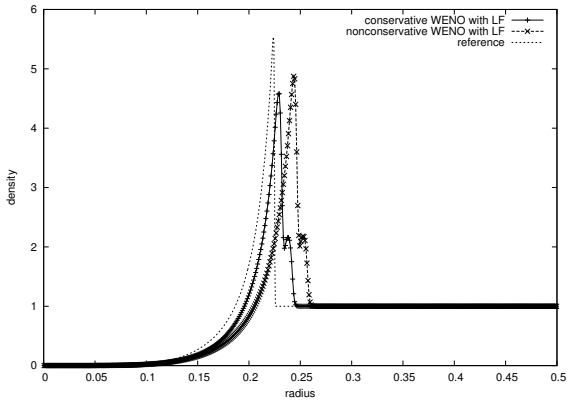

Fig. 4.1-a. WENO finite-difference with LF flux-splitting at $t = 0.05$

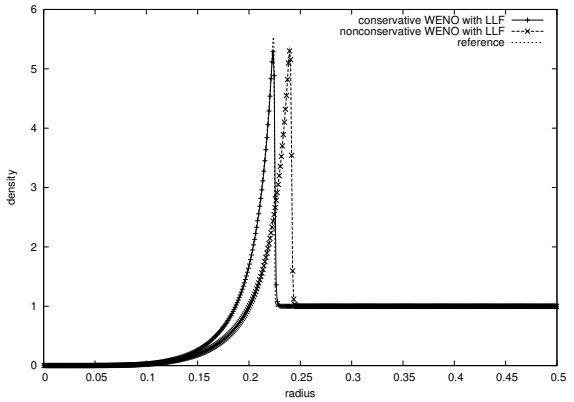

Fig. 4.1-b. WENO finite-difference with LLF flux-splitting at $t = 0.05$

The WENO finite-difference scheme with LLF flux-splitting gives steep shock front (see 4.1-b). However, Without the conservation, the shock front has wrong location. It is interesting to see that the results for the modified conservative WENO scheme (13) match very well with the reference solution although it has only second order accuracy.

4.3 Results for three-dimensional Sedov problem

The 3-D case is very similar to the 2-D case, except that now the finite-difference discretization and finite-volume discretization of the source term are no longer the same. Since for the conservative form, the source term is added only to the momentum equation, whether the finite-difference or finite-volume discretization of the source term does not make any difference in conservation of the mass and energy. In fact, if r is large enough, the difference between these two discretization is $O(h^2)$. Near the origin ($r = 0$), the difference between these two is large as (O(1/h)). We observed that the velocity and pressure is totally wrong near the origin if the finite-difference discretization of the source term is used.

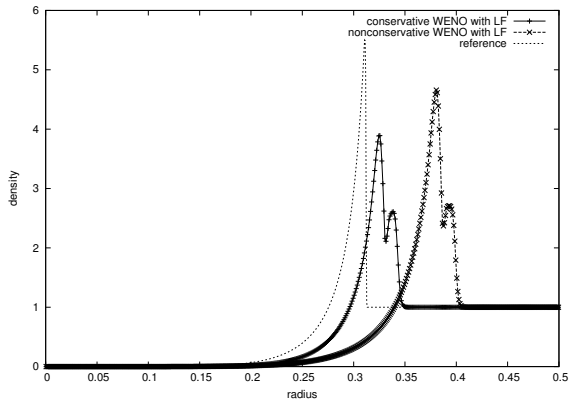

Fig. 4.3-a. WENO finite-difference with LF flux-splitting at $t = 0.05$

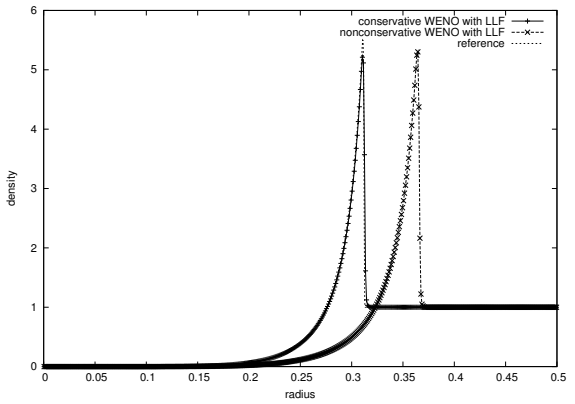

Fig. 4.3-b. WENO finite-difference with LLF flux-splitting at $t = 0.05$

We found that the local maximum sound speed near the shock front is about 1.5, while it is about 310 near the origin. For the global LF flux-splitting, the flux $f^{\pm}(u)$ deviated the local flux $f(u)$ so much that it cannot reflect the local characteristics any more. That is why the global LF failed for this example. For other examples we have tested so far, the location of the global maximum sound speed is always close to the shock front. However, for this example, the density is extremely low (about $O(10^{-5})$ for 3-D case) near the origin and pressure is relatively large (0.188), the local sound speed calculated by $c^2 = \gamma p/\rho$ near the origin is much larger than that near the shock front.

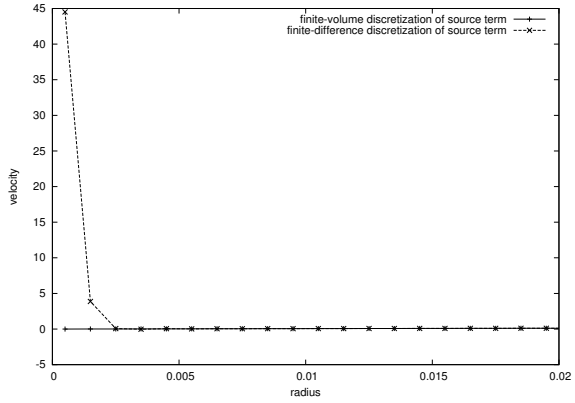

Fig. 4.3-a. Results of the different discretizations for the source term near the origin.

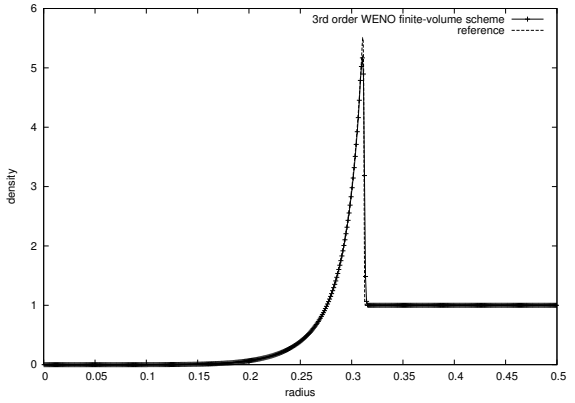

Fig. 4.3-b. Results of the third-order WENO scheme for Sedov problem.

We also test the conservative WENO finite-difference scheme (12). If we start from $t = 0$, the integration stops after a few time steps due to the negative pressure. We started the simulation from $t = 0.01$ (scheme Eq.(13) is used before $t = 0.01$), the results are in good agreement with the reference solutions except at the origin. Fig. 4.3-a shows that the high order scheme (12) has almost the same results as the second order scheme (13). Fig. 4.3-b shows that the scheme (12) has large error near the origin. From equation (22), the pressure should be constant near the origin. However, the result from scheme (12) has an incorrect large jump near the origin.

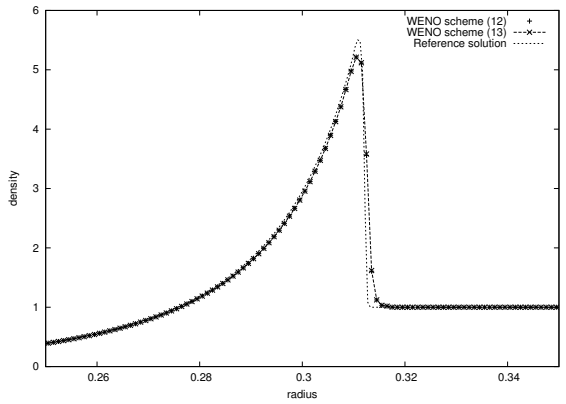

Fig. 4.3-a. WENO finite-difference with LLF flux-splitting at $t = 0.05$.

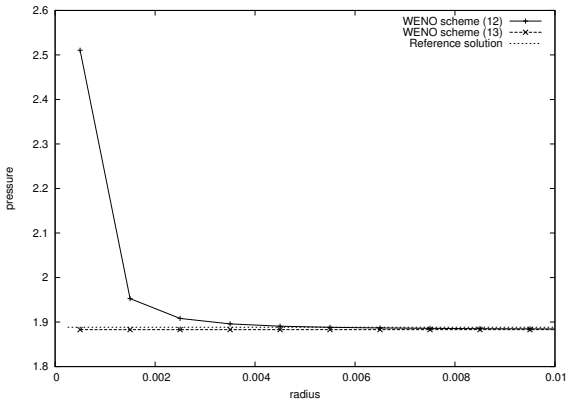

Fig. 4.3-b. WENO finite-difference with LLF flux-splitting at $t = 0.05$ near the origin.

5 Conclusion

We tested WENO scheme for radially symmetric cylindrical and spherical grid. It is shown that conservativity of the numerical scheme is crucial to have accurate shock location and strength. Traditional implementation to solve the cylindrical and spherical problem with the geometric source terms does not preserve the mass and energy, although it can achieve higher than second order accuracy. Treating the cylindrical or spherical grid as smooth nonuniform grid as WENO finite-difference scheme applied to general curvilinear grid has large error near the origin so that it failed for some initial conditions. There is also a risk with this approach that the discontinuity will be spread out because the weight in WENO scheme will be smoothed out by the geometric factor (r for 2-D and r^2 for 3-D).

The modified conservative WENO finite-difference scheme produces good results. However, it is only of second order accuracy in theory. This leads to a question that the complexity and effort of the WENO may not be worthy for such low order of accuracy. Actually, a second-order TVD scheme may work much efficient in this case.

We should mention that the global flux-splitting may not work even for high order method. The spurious solution is generated if the global maximum characteristics speed is far more larger than the local speed near the shock or contact. For robustness, we suggest the local flux-splitting method be used.

The WENO finite-volume scheme can achieve both the conservation law and the high order accuracy. However, it might be too much costly for multi-dimensional problem. Since we have an orthogonal uniform grid, a low order method with fine grid may work more efficient than the high order WENO finite-volume scheme.

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