

# The reflection-antisymmetric counterpart of the Kármán–Howarth dynamical equation

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## Abstract

We study the isotropic, helical component in homogeneous turbulence using statistical objects which have the correct symmetry and parity properties. Using these objects we derive an analogue of the Kármán–Howarth equation, that arises due to lack of mirror-reflection-symmetry in isotropic flows. The main equation we obtain is consistent with the results of Chkhetiani [JETP 63 (1996) 768] and L’vov et al. [Exact result for the 3rd order correlations of velocity in turbulence with helicity, 1997. <http://xxx.lanl.gov/abs/chao-dyn/9705016>] but is derived using only velocity correlations, with no direct consideration of the vorticity or helicity. This alternative formulation offers an advantage to both experimental and numerical measurements. We also postulate, under the assumption of self-similarity, the existence of a hierarchy of scaling exponents for helical velocity correlation functions of arbitrary order, analogous to the Kolmogorov prediction for the scaling exponents of velocity structure function.

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## 1. Introduction

In their 1938 paper on the statistical properties of homogeneous, isotropic, reflection-symmetric turbulence, von Kármán and Howarth derived the equation for the dynamics of the two-point velocity correlation function [1]. This equation is of fundamental importance since it relates the mean rate of change of energy to the flux of energy across a given correlation length  $r$  in the flow. A form of this equation was used by Kolmogorov in 1941 [2] (K41) to derive one of the few exact results known for isotropic, homogeneous, and reflection-symmetric turbulence, the “4/5th law” which relates the third-order longitudinal structure function to  $\epsilon$ , the mean rate of energy dissipation

$$\langle (\mathbf{u}_L(\mathbf{x} + \mathbf{r}) - \mathbf{u}_L(\mathbf{x}))^3 \rangle = -\frac{4}{5}\epsilon r, \quad (1)$$

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where  $\mathbf{u}_L$  is the component of the velocity along the separation vector  $\mathbf{r}$ . If the flow is not reflection-symmetric however, a new equation may be derived to complement the Kármán–Howarth equation. Three recent works have derived equations for third-order statistics in isotropic helical flows by considering velocity–vorticity correlations [3–5]. In this paper, we show that the Kármán–Howarth equation has a counterpart which arises due to parity-violation in isotropic flows and which can be written solely in terms of two-point velocity correlations. We demonstrate the equivalence of our result with those of [3,4].

We were motivated in this work by a series of investigations which proposed the use of the SO(3) decomposition of tensor quantities, the structure functions, defined by

$$S_{\alpha\beta}(\mathbf{r}) = \langle (u_\alpha(\mathbf{x} + \mathbf{r}) - u_\alpha(\mathbf{x})) (u_\beta(\mathbf{x} + \mathbf{r}) - u_\beta(\mathbf{x})) \rangle \quad (2)$$

in order to study the anisotropic contributions to their scaling. The decomposition of the structure function into rotationally invariant, irreducible subgroups of the SO(3) symmetry group  $S_{\alpha\beta}^{j=0}(\mathbf{r}) + S_{\alpha\beta}^{j=1}(\mathbf{r}) + \dots$  allowed the separation of the isotropic (indexed by  $j = 0$ ) from the anisotropic (indexed by  $j > 0$ ) contributions to the structure function. This procedure has allowed better quantification of the rate of decay of anisotropy of the small scales in turbulence [6–8]. These analyses considered homogeneous, isotropic and reflection-symmetric flows. In the isotropic ( $j = 0$ ) sector, the reflection-symmetric structure function tensor has the form

$$S_{\alpha\beta}(\mathbf{r}) = C_1(r)\delta_{\alpha\beta} + C_2(r)\frac{r_\alpha r_\beta}{r^2}. \quad (3)$$

Homogeneity and incompressibility provide a constraint between the scalar functions  $C_1(r)$  and  $C_2(r)$ . If the condition of reflection-symmetry is dropped, there arises a further tensor contribution to the isotropic sector given by  $\epsilon_{\alpha\beta\gamma}(r_\gamma/r)$ . This contribution is interesting because it is isotropic (rotationally invariant), which implies that it belongs in the  $j = 0$  sector, but is antisymmetric in  $(\alpha, \beta)$  and changes sign under mirror reflection of  $\mathbf{r}$ . Since the second-order structure function is symmetric in its indices and does not change sign under inversion of  $\mathbf{r}$ , it simply cannot be used to observe this antisymmetric contribution. In fact, when the antisymmetric contribution is included in our decomposition, we are effectively using the isotropic irreducible representation of the O(3) symmetry group which includes operations that are not reflection invariant under  $\mathbf{r} \rightarrow -\mathbf{r}$ . Said differently, the elements  $\Lambda$  of the orthogonal group O(3) satisfy  $\det(\Lambda) = \pm 1$ . The elements with determinant +1 form the SO(3) symmetry group of all (even-parity) rotations while those with determinant  $-1$  are (odd-parity) reflections. The present work demonstrates how to access this isotropic, antisymmetric, odd-parity contribution using the tensor object with the appropriate parity and symmetry properties. The dynamics of such an object will provide the antisymmetric counterpart to the Kármán–Howarth dynamical equation.

In Section 2, we present and discuss the second- and third-order velocity correlations and their symmetric and antisymmetric contributions. In Section 3, we derive the antisymmetric, odd-parity counterpart of the Kármán–Howarth equation for the second-order correlation function and show its equivalence to previous results. We also postulate the existence of generalized helical higher-order velocity correlations and their scaling behavior under the assumption of self-similarity. Section 4 provides a summary and discussion.

## 2. The symmetry and parity properties of the two-point velocity correlation functions

### 2.1. The second-order correlation tensor

The two-point correlation tensor function of velocity fluctuations is defined by

$$R_{\alpha\beta}(\mathbf{r}) = \langle u_\alpha(\mathbf{x})u_\beta(\mathbf{x} + \mathbf{r}) \rangle, \quad (4)$$

where  $\mathbf{r}$  is the vector separation between two points, and subscripts  $\alpha, \beta$  are components in a chosen Cartesian coordinate system. In homogeneous, isotropic, and not necessarily reflection-symmetric turbulence, the correlation function may be written as a sum of the dyadics [10,11]

$$R_{\alpha\beta}(\mathbf{r}) = A_1(r)\delta_{\alpha\beta} + A_2(r)\frac{r_\alpha r_\beta}{r^2} + H(r)\epsilon_{\alpha\beta\gamma}\frac{r_\gamma}{r}. \quad (5)$$

Such a tensor may be written as the sum of its symmetric (in  $\alpha, \beta$ ) and antisymmetric components as

$$R_{\alpha\beta}(\mathbf{r}) = \frac{1}{2}(R_{\alpha\beta}(\mathbf{r}) + R_{\beta\alpha}(\mathbf{r})) + \frac{1}{2}(R_{\alpha\beta}(\mathbf{r}) - R_{\beta\alpha}(\mathbf{r})) = R_{\alpha\beta}^S(\mathbf{r}) + R_{\alpha\beta}^A(\mathbf{r}). \quad (6)$$

The symmetric contribution  $R_{\alpha\beta}^S(\mathbf{r})$  consists of the first two terms on the right side of Eq. (5) while the antisymmetric contribution  $R_{\alpha\beta}^A(\mathbf{r})$  is the last term in Eq. (5).

If the flow is statistically homogeneous, then the incompressibility constraint implies

$$\partial_\alpha R_{\alpha\beta}(\mathbf{r}) = \partial_\beta R_{\alpha\beta}(\mathbf{r}) = 0, \quad (7)$$

where  $\partial_\alpha(\cdot)$  denotes the partial derivative with respect to  $r_\alpha$ . The incompressibility condition applies separately to the symmetric and antisymmetric components as  $\partial_\alpha R_{\alpha\beta}^S(\mathbf{r}) = \partial_\beta R_{\alpha\beta}^S(\mathbf{r}) = 0$  and  $\partial_\alpha R_{\alpha\beta}^A(\mathbf{r}) = \partial_\beta R_{\alpha\beta}^A(\mathbf{r}) = 0$  since the symmetric and antisymmetric components are of opposite parity. This is an interesting and useful property of these correlation functions in the isotropic sector and for homogeneous flows—decomposition into symmetric and antisymmetric components automatically separates the even- and odd-parity contributions.

The symmetric part  $R_{\alpha\beta}^S(\mathbf{r})$  with tensor basis as follows:

$$R_{\alpha\beta}^S(\mathbf{r}) = A_1(r)\delta_{\alpha\beta} + A_2(r)\frac{r_\alpha r_\beta}{r^2} \quad (8)$$

has been analyzed extensively (see, for example [9]) under the assumption of homogeneous, isotropic and mirror-symmetric turbulence. These three conditions imply the translational, rotational and reflectional invariance, respectively, of a given statistical quantity used to describe the flow. Note that the structure function (Eq. (2)) is twice the symmetrized correlation function  $R_{\alpha\beta}^S$  plus twice the mean-square velocity fluctuation. The latter addition makes the structure function Galilean invariant and hence a suitable candidate for the study of universal statistics of the small scales.

The form of the antisymmetric tensor in the  $j = 0$  sector of the  $O(3)$  representation is

$$R_{\alpha\beta}^A(\mathbf{r}) = \langle u_\alpha(\mathbf{x})u_\beta(\mathbf{x} + \mathbf{r}) \rangle - \langle u_\beta(\mathbf{x})u_\alpha(\mathbf{x} + \mathbf{r}) \rangle = H(r)\epsilon_{\alpha\beta\gamma}\frac{r_\gamma}{r}. \quad (9)$$

Let us apply the incompressibility constraint to the antisymmetric tensor form:

$$\partial_\alpha \left( H(r)\epsilon_{\alpha\beta\gamma}\frac{r_\gamma}{r} \right) = \epsilon_{\alpha\beta\gamma}\frac{r_\gamma}{r}\partial_\alpha H(r) + \epsilon_{\alpha\beta\gamma}\frac{H(r)}{r} \left( \delta_{\alpha\gamma} - \frac{r_\alpha r_\gamma}{r^2} \right) = \epsilon_{\alpha\beta\gamma}\frac{r_\alpha r_\gamma}{r^2}\frac{\partial H(r)}{\partial r} \equiv 0. \quad (10)$$

In going from the first to the second equality of Eq. (10), we have used the fact that contracting an antisymmetric tensor with a symmetric one gives identically zero. We conclude that incompressibility does not provide any constraint on the scalar coefficient  $H(r)$  of the antisymmetric tensor contribution.

We can give an argument that the antisymmetrized correlation function is Galilean invariant by definition. Suppose we are in a frame moving with velocity  $\mathbf{U}$ , then

$$R_{\alpha\beta}^A(\mathbf{r}) = \langle (u_\alpha(\mathbf{x}) + \mathbf{U})(u_\beta(\mathbf{x} + \mathbf{r}) + \mathbf{U}) \rangle - \langle (u_\beta(\mathbf{x}) + \mathbf{U})(u_\alpha(\mathbf{x} + \mathbf{r}) + \mathbf{U}) \rangle. \quad (11)$$

It is seen that, because of homogeneity and the minus sign used to antisymmetrize, any dependence on  $\mathbf{U}$  drops out. Therefore, we can hope that, as in the case of the structure functions, the object  $R_{\alpha\beta}^A(\mathbf{r})$  will display the (universal) properties of the small scales.

## 2.2. The third-order correlation tensor

Our goal is to derive the dynamical equation for the second-order antisymmetric correlation  $R_{\alpha\beta}^A(\mathbf{r})$  as a counterpart to the Kármán–Howarth dynamical equation for the second-order symmetric correlation  $R_{\alpha\beta}^S(\mathbf{r})$  (denoted in their paper of 1938 [1] by  $R_{ik}(\xi)$ ). Since such an expression will involve the two-point third-order correlation function, we will first review its properties.

$$S_{\alpha\gamma,\beta}(\mathbf{r}) = \langle u_\alpha(\mathbf{x})u_\gamma(\mathbf{x})u_\beta(\mathbf{x} + \mathbf{r}) \rangle \quad (12)$$

has the following properties in homogeneous turbulence. It is clearly symmetric in indices  $\alpha, \gamma$ , with mixed symmetry in other combinations  $\alpha, \beta$  and  $\gamma, \beta$  and in general of mixed parity. By “mixed” we mean that the symmetry and parity properties are indeterminate. In the isotropic tensor representation then, there are four terms [3,9]

$$S_{\alpha\gamma,\beta}(\mathbf{r}) = S_1(r)\delta_{\alpha\gamma}\frac{r_\beta}{r} + S_2(r)\left(\delta_{\alpha\beta}\frac{r_\gamma}{r} + \delta_{\beta\gamma}\frac{r_\alpha}{r}\right) + S_3(r)\frac{r_\alpha r_\gamma r_\beta}{r^3} + S(r)\left(\epsilon_{\alpha\beta\gamma}\frac{r_\nu r_\nu}{r^2} + \epsilon_{\gamma\beta\nu}\frac{r_\nu r_\alpha}{r^2}\right). \quad (13)$$

In anticipation of separating the terms of opposite symmetry as was done in the case of the second-order correlation function, we write

$$S_{\alpha\gamma,\beta}(\mathbf{r}) = \frac{1}{2}(S_{\alpha\gamma,\beta}(\mathbf{r}) + S_{\beta\gamma,\alpha}(\mathbf{r})) + \frac{1}{2}(S_{\alpha\gamma,\beta}(\mathbf{r}) - S_{\beta\gamma,\alpha}(\mathbf{r})) = S_{\alpha\gamma,\beta}^S(\mathbf{r}) + S_{\alpha\gamma,\beta}^A(\mathbf{r}), \quad (14)$$

where  $S_{\alpha\gamma,\beta}^A$  is antisymmetric in  $\alpha, \beta$  and has tensor contributions as follows:

$$\begin{aligned} S_{\alpha\gamma,\beta}^A(\mathbf{r}) &= \frac{\langle u_\alpha(\mathbf{x})u_\gamma(\mathbf{x})u_\beta(\mathbf{x} + \mathbf{r}) \rangle - \langle u_\beta(\mathbf{x})u_\gamma(\mathbf{x})u_\alpha(\mathbf{x} + \mathbf{r}) \rangle}{2} \\ &= \frac{S_1(r) - S_2(r)}{2}\left(\delta_{\alpha\gamma}\frac{r_\beta}{r} - \delta_{\beta\gamma}\frac{r_\alpha}{r}\right) + \frac{S(r)}{2}\left(2\epsilon_{\alpha\beta\gamma}\frac{r_\nu r_\nu}{r^2} + \epsilon_{\gamma\beta\nu}\frac{r_\nu r_\alpha}{r^2} - \epsilon_{\gamma\alpha\nu}\frac{r_\nu r_\beta}{r^2}\right). \end{aligned} \quad (15)$$

These are the terms which were excluded in the Kármán–Howarth equation for reflection-symmetric flows.

## 3. The antisymmetric component of the Kármán–Howarth equation

We now derive in a simple manner the dynamical equation for  $R_{\alpha\beta}^A(\mathbf{r})$ . As in Hinze’s [9] equation 1.48, starting from the Navier–Stokes equation for homogeneous turbulence we can write the equation for  $R_{\alpha\beta}$

$$\frac{\partial}{\partial t}R_{\alpha\beta} - \partial_\gamma S_{\alpha\gamma,\beta} + \partial_\gamma S_{\alpha,\gamma\beta} = -\frac{1}{\rho}(-\partial_\alpha K_{p,\beta} + \partial_\beta K_{\alpha,p}) + 2\nu\partial_{\gamma\gamma}R_{\alpha\beta}, \quad (16)$$

where  $K_{\alpha,p} = \langle u_\alpha(\mathbf{x})p(\mathbf{x} + \mathbf{r}) \rangle$  and  $p$  is the pressure. We write a similar equation for  $R_{\beta\alpha}$  which we subtract from Eq. (16) and divide throughout by 2.

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{R_{\alpha\beta} - R_{\beta\alpha}}{2}\right) - \partial_\gamma\left(\frac{S_{\alpha\gamma,\beta} - S_{\beta\gamma,\alpha}}{2}\right) + \partial_\gamma\left(\frac{S_{\alpha,\gamma\beta} - S_{\beta,\gamma\alpha}}{2}\right) \\ = \frac{1}{2}\left(-\frac{1}{\rho}(-\partial_\alpha K_{p,\beta} + \partial_\beta K_{\alpha,p}) + \frac{1}{\rho}(-\partial_\beta K_{p,\alpha} + \partial_\alpha K_{\beta,p})\right) + 2\nu\partial_{\gamma\gamma}\left(\frac{R_{\alpha\beta} - R_{\beta\alpha}}{2}\right). \end{aligned} \quad (17)$$

The pressure terms may be shown to vanish identically using homogeneity and incompressibility and assuming regularity as  $r \rightarrow 0$ , as in the reflection-symmetric, isotropic case [12,13]. The homogeneity condition  $S_{\alpha\gamma,\beta}(\mathbf{r}) = S_{\beta,\gamma\alpha}(-\mathbf{r})$  adds a further constraint, giving

$$\frac{\partial}{\partial t}R_{\alpha\beta}^A - 2\partial_\gamma S_{\alpha\gamma,\beta}^A = 2\nu\partial_{\gamma\gamma}R_{\alpha\beta}^A. \quad (18)$$

This equation is the antisymmetric counterpart to the Kármán–Howarth equation for the second-order correlation functions. All the quantities in this equation are relatively easily measured in experiments and numerical simulations since no velocity derivatives are involved in the correlation functions, only the velocities themselves. Substituting in Eq. (18) the tensor forms for the antisymmetric correlation functions (Eqs. (9) and (15)) we arrive at the dynamical relation relating the scalars  $H(r)$  and  $S(r)$

$$\frac{\partial}{\partial t} H(r) - \left( 2 \frac{\partial S(r)}{\partial r} + 6 \frac{S(r)}{r} \right) = 2\nu \left( \frac{\partial^2 H(r)}{\partial r^2} + \frac{2}{r} \frac{\partial H}{\partial r} - \frac{2}{r^2} H(r) \right). \quad (19)$$

This equation was derived by Chkhetiani [3] using the dynamics of velocity–vorticity correlations. In the present derivation, we have arrived at the conclusion without the need to directly consider vorticity or helicity. We only used the O(3) tensor representation for the correlation function in homogeneous, isotropic flows in which symmetry and parity properties are trivially separated.

### 3.1. Derivation of KH-helical scaling law

We apply the curl operator to the second-order antisymmetrized correlation function equation (9), and obtain the leading-order behavior of  $H(r) = \mathcal{H}r/3$  (see Eq. (A.3) and associated details in Appendix A) where the mean helicity  $\mathcal{H} = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle / 2$ . We now substitute this leading-order dependence of  $H(r)$  back into the KH law,

$$\frac{\partial}{\partial t} \left( \frac{\mathcal{H}r}{3} + \dots \right) - \left( 2 \frac{\partial}{\partial r} + \frac{6}{r} \right) S(r) = 2\nu \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \right) H(r). \quad (20)$$

Here, if we make the same assumption as in [3], that the main contribution to the time-derivative comes from the linear term with the next order terms not changing in the inertial range, and neglect the right-hand side in the limit as  $\nu \rightarrow 0$ ,

$$S(r) = \frac{1}{30} hr^2, \quad (21)$$

where  $h$  is the mean helicity dissipation rate. This agrees with the scaling law derived in [3]. (There is a difference of a factor of 1/2 in the definition of mean helicity between [3] and the present work.) The assumption made in deriving this law is that we have fully developed, freely *decaying* turbulence. These are the same assumptions made by Kolmogorov in deriving the 4/5ths law and the energy spectrum. It is with this assumption that the following holds [14]

$$\frac{\partial}{\partial t} \mathcal{H} = \nu \langle (\partial_k v_i)(\partial_k \omega_j) \rangle = h. \quad (22)$$

If a driving force is introduced, additional terms arise in the helicity balance equation (22) (for example  $\langle f \cdot \boldsymbol{\omega} \rangle$ ) which may not directly allow us to derive Eq. (21). It is, however, not unreasonable to expect that Eq. (21) will hold for the steady-state, forced high-Reynolds number case. An argument similar to that which Frisch [13] used to prove the 4/5ths law for the forced case, would have to be used. This aspect will not be covered in the present work.

In Appendix A, we show that an alternative form of Eq. (21) may be derived in the form of the following pair of equations

$$\mathbf{u}_L \cdot (\mathbf{u}_T \times \mathbf{u}'_T) = \frac{1}{15} hr^2, \quad \mathbf{u}_T \cdot ((\mathbf{u}_L \times \mathbf{u}'_T) + (\mathbf{u}_T \times \mathbf{u}'_L)) = -\frac{1}{30} hr^2, \quad (23)$$

where the velocity vector has been separated into its longitudinal (along the separation vector  $\mathbf{r}$ ) and transverse components as  $\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T$ . The unprimed velocities denote their value at  $\mathbf{x}$  and primed velocities denote their value at  $\mathbf{x} + \mathbf{r}$ . The first equality of Eq. (23) is equivalent to the so-called “2/15ths law” derived by L’vov et al. [4] (see Appendix A for more details).

### 3.2. The scaling behavior of higher-order correlation functions

The antisymmetrized correlation functions may be thought of as newly defined structure functions appropriate for helical flows. In the second- and third-order ( $R_{\alpha\beta}^A$  and  $S_{\alpha\gamma,\beta}^A$ , respectively) we have shown that the antisymmetrized correlation functions are Galilean invariant, so that sweeping effects are eliminated as in the case of the symmetric structure functions, and we may hope for universal properties for small  $r$ . For the third-order correlation, we have seen that the scaling in the inertial range is  $\sim r^2$  (Eq. (21)). If we now make the K41 assumption of self-similarity such that  $S_{\alpha\gamma,\beta}^A \sim (R_{\alpha\beta}^A)^{3/2}$ , we obtain the inertial range behavior for  $R_{\alpha\beta}^A \sim r^{4/3}$ . This corresponds to an inertial range scaling of the (helicity) cospectrum  $\tilde{E}_{12}(k_3) \sim k^{-7/3}$ . The  $k_3$  denotes the wavenumber component in the direction mutually orthogonal to  $\alpha = 1$ , and  $\beta = 2$ . This estimate for the scaling of the cospectrum coincides with the Lumley estimate [16] for the (anisotropic sector  $j = 2$ ) shear-stress (Reynolds) cospectrum  $\tilde{E}_{12}(k_1)$ . However, the present dimensional estimate for the cospectrum  $\tilde{E}_{12}(k_3)$  is due to the reflection-symmetry breaking, not due to the rotational symmetry breaking.

If we construct  $n$ th-order antisymmetrized correlation functions with scaling exponents  $\xi_n$  in the inertial range, the self-similarity argument dictates that  $\xi_n = 2n/3$ . This would be the helical counterpart to the K41 scaling prediction for the structure functions which says that the  $n$ th-order structure functions have scaling exponents  $\zeta_n = n/3$ . It is not at all clear that self-similarity is a reasonable assumption to make even in the case of low-order helical statistics [17]. This conjecture may only hold in the case of the maximal helical cascade in which there is no joint cascade of energy [18].

## 4. Conclusion

The understanding of helicity dynamics in three-dimensional flows is still evolving. It has been known for some time that helicity is conserved in the fluid equations in the inviscid limit [19]. The simultaneous existence of both helicity and energy cascades to the high-wavenumbers was first considered by Brissaud et al. [20]. In that work, the scenario for a pure helicity or maximally helical cascade was also proposed, in which energy cascade to the small scales is blocked, giving rise to an energy spectrum  $E(k) \sim k^{-7/3}$ . Kraichnan [21] showed, based on physical considerations, that the scenario of joint energy and helicity cascades to the high-wavenumbers, with recovery of the Kolmogorov energy spectrum  $E(k) \sim k^{-5/3}$  is more likely. This joint-cascade picture has subsequently been strengthened by observations in numerical simulations [22] from which it seems likely that the helicity injected at the large scales cascades downscale, more or less passively transported by the energy cascade. More recently, Ditlevsen and Giuliani [23] show, both theoretically and using shell-model calculations [24], that at high-Reynolds numbers a joint-cascade of energy and helicity must exist in some range of wavenumbers. They argue that for wavenumbers larger than this range the reflection-symmetry is restored by the dominant helicity dissipation term. Chen et al. [25] have shown by means of helical-wave decomposition of the velocity field, that the detailed transfer of energy (and helicity) between helical-wave modes of opposite parity is consistent with the existence of a joint-cascade, with  $-5/3$  scaling for both energy and helicity spectra. They also confirm their theoretical predictions using numerical simulations. Their analysis disagrees with [23,24] over the precise range wavenumber over which these cascades exist, but nonetheless, both works agree that for high-Reynolds numbers, a joint-cascade of energy and helicity will coexist for some range of wavenumbers, with parity restoration at sufficiently large wavenumbers. The present analysis is also consistent with the joint-cascade at high-Reynolds numbers. The original Kármán–Howarth result and the helical version derived here are not mutually exclusive. The former picks out the reflection-symmetric part of the flow, while the latter picks out the reflection-antisymmetric part. The two contributions are measured by different quantities which allow for the *separation* of the parity and symmetry properties of the flow. It is not clear,

at least to this author, whether the present formulation predicts that energy and helicity cascades will coexist for *all* scales (consistent with [25]) or for only a certain range of scales (consistent with [23]) at Reynolds numbers high enough. This might have to be left to empirical tests using experimental and direct numerical simulations data. Thus far, only the shell-model simulations of Ditlevsen and Giuliani [24] and Biferale et al. [26] exist to guide our intuition as to the scaling ranges of the cascades.

From the analysis of [25] it appears that if one of the helical modes is blocked, which can easily be done in simulations and shell-model calculations, but may not be possible in real flows, a pure helicity cascade will develop in the remaining mode which blocks the energy cascade down to small scales, and yields an energy spectrum  $E(k) \sim k^{-7/3}$ . In this sense, the dimensional (self-similarity) argument of Section 3 for the scaling exponent of the cospectrum  $\tilde{E}_{12}(k_3) \sim k_3^{-7/3}$  is consistent with the scenario of a pure helicity cascade. Without speculating on the feasibility of physically achieving such a purely helical cascade, we remark that the *cospectrum*  $\tilde{E}_{12}(k_3)$ , of the two orthogonal components along the third orthogonal direction, is a fundamentally different object than the energy spectrum and may well display entirely different functional behavior. The helicity cospectrum, which should be identically zero for homogeneous, isotropic, reflection-symmetric turbulence is a sensitive measure of reflection-symmetry breaking [27] and the presence of helicity. The present work has shown that it arises from precisely that contribution to the second-order correlation which was excluded in the original isotropic, homogeneous, reflection-symmetric Kármán–Howarth equation.

A further new possibility suggested by this work is the construction of antisymmetric higher-order (greater than 3) correlation functions. Assuming each pair of indices of an  $n$ th-order, two-point velocity correlation function can be appropriately antisymmetrized as has been done here for the second- and third-order cases, we may have a new series of objects, which we will call helical structure functions. These, along with the usual structure functions familiar from studies of isotropic, reflection-symmetric flows, would form a complete set of statistical objects with which to investigate statistical turbulence theories which are not necessarily confined to reflection-symmetric configurations. All the usual issues such as scaling exponent values, intermittency and anomaly could be studied for the helical structure functions. This work provides a dimensional argument for what their scaling exponents could be. It is of interest to see how these behave relative to our predictions and to the anomalous scaling known for the usual structure functions.

In conclusion, the present approach taken to derive the antisymmetric, or helical Kármán–Howarth equation is not inconsistent with other recent work [3–5]. However, our derivation directly studies the dynamics of precisely those components of the second-order correlation functions that were omitted in the Kármán–Howarth equation because of assumed non-helicity of the flow. The information about helicity of the flow is then obtained from velocity correlations instead of velocity–vorticity correlations or other correlations involving velocity derivatives. In a high-Reynolds number experimental flow, measuring the second-order antisymmetric velocity correlation function  $R_{12}^A(r)$  in a coordinate system chosen such that  $\mathbf{r} = r\hat{\mathbf{k}}$ , would give information about the mean helicity in the flow, while measurement of  $S_{132}^A(r)$  would give information about the helicity flux. Such objects are ideal candidates for detecting parity violation in flows without having to resort to direct measurement of helicity. We intend to present the related analysis of numerical and experimental data in a future work.

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self-similarity and scaling laws. The Erwin Schrödinger Institute of Mathematical Physics (Program on Developed Turbulence 2002), Vienna, provided partial support during this work.

## Appendix A

The antisymmetric tensors  $R_{\alpha\beta}^A$  and  $S_{\alpha\gamma,\beta}^A$  are directly related to helicity dissipation and fluxes as we will now demonstrate. Let us consider a particular geometry in which the separation vector  $r$  is along the  $z(3)$ -axis. Then the only non-zero components are  $R_{12}^A = -R_{21}^A = \langle u_1(\mathbf{x})u_2(\mathbf{x} + \mathbf{r}) - u_2(\mathbf{x})u_1(\mathbf{x} + \mathbf{r}) \rangle$ . This particular object is the correlation of the two components of the velocity orthogonal to the vector  $\mathbf{r}$ . In the usual convention, it is the correlation of the  $v$  and  $w$  components relative to the separation vector. The only non-zero contribution to the isotropic, antisymmetric velocity correlation tensor comes from the velocity components orthogonal to the separation vector.

We contract the tensor  $R_{\alpha\beta}^A$  with the antisymmetric tensor  $\epsilon_{\alpha\beta\gamma}$  [15].

$$\begin{aligned} \epsilon_{\alpha\beta\gamma} R_{\alpha\beta}^A(\mathbf{r}) &= \epsilon_{\alpha\beta\gamma} H(r) \epsilon_{\alpha\beta\gamma} \frac{r_\nu}{r} \frac{\epsilon_{\alpha\beta\gamma} \langle u_\alpha(\mathbf{x})u_\beta(\mathbf{x} + \mathbf{r}) \rangle + \epsilon_{\beta\alpha\gamma} \langle u_\beta(\mathbf{x})u_\alpha(\mathbf{x} + \mathbf{r}) \rangle}{2} \\ &= 2H(r) \delta_{\gamma\nu} \frac{r_\nu}{r} \langle \mathbf{u}(\mathbf{x}) \times \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle_\gamma = 2H(r) \frac{r_\gamma}{r}. \end{aligned} \quad (\text{A.1})$$

The (pseudo)scalar function  $H(r)$  is the mean cross-product of the velocities at two points separated by the vector length scale  $\mathbf{r}$ . To choose a particular coordinate system, if the separation vector  $\mathbf{r}$  lies along the  $z$ -axis, then  $H(r)$  is the  $z$ -component of the cross-product. It vanishes as  $|r| \rightarrow 0$ . The physical picture is depicted in the cartoon of Fig. 1. This result is to be compared with the corresponding result for the symmetric contribution  $R_{\alpha\beta}^S$  contracted with  $\delta_{\alpha\beta}$ . In that case, what is obtained is  $\sim \langle u_\alpha(x)u_\alpha(x+r) \rangle = \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle$ , the mean scalar (dot) product of the velocities at two points separated by the scale  $\mathbf{r}$ ; as  $r \rightarrow 0$ , we recover the non-zero mean energy  $\sim \langle u^2 \rangle$ .

The function  $R_{12}^A(r_3)$  may be thought of as a measure of momentum transfer between two orthogonal components of velocity along the direction perpendicular to both of them. If we take the curl of  $R_{\alpha\beta}^A$ , we have

$$\epsilon_{\alpha\beta\nu} \partial_\nu R_{\alpha\beta}^A(\mathbf{r}) = \epsilon_{\alpha\beta\nu} \partial_\nu \left( H(r) \epsilon_{\alpha\beta\gamma} \frac{r_\gamma}{r} \right), \quad \langle \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x} + \mathbf{r}) \rangle = 2 \frac{\partial H(r)}{\partial r} + 4 \frac{H(r)}{r}. \quad (\text{A.2})$$

Taking the limit as  $r \rightarrow 0$ , the left-hand side reduces to  $\langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle = 2\mathcal{H}$ , where  $\mathcal{H}$  is the mean helicity of the flow, and we can solve for what must be the leading-order behavior of  $H(r)$ :

$$H(r) = \frac{1}{3} \mathcal{H} r + \dots \quad (\text{A.3})$$

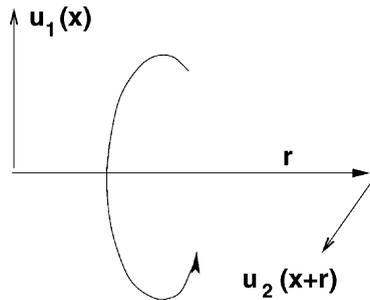


Fig. 1. Caricature of the type of correlation functions  $R_{\alpha\beta}^A$  which are non-zero in flows that are not reflection-symmetric. The curved arrow indicates the “handedness” of the function.

The scalar coefficient  $H(r)$  of the antisymmetric tensor  $R_{\alpha\beta}^A(\mathbf{r})$  is, in the leading-order, a direct measure of the mean helicity in the flow. If we consider the particular coordinate system of Fig. 1 we see that  $R_{12}^A(r) = H(r)$  which is a leading-order measure of the mean helicity of the flow. We note again the advantage of this formulation which allows one to measure mean helicity using only velocity correlations, without having to measure any local gradients.

We perform a similar analysis for the third-order object with the contraction

$$\begin{aligned} \epsilon_{\alpha\beta\mu} S_{\alpha\gamma\beta}^A &= \epsilon_{\alpha\beta\mu} \frac{S(r)}{2} \left( 2\epsilon_{\alpha\beta\nu} \frac{r_\nu r_\gamma}{r^2} + \epsilon_{\gamma\beta\nu} \frac{r_\nu r_\alpha}{r^2} - \epsilon_{\gamma\alpha\nu} \frac{r_\nu r_\beta}{r^2} \right), \\ \langle u_\gamma(x) (\mathbf{u}(\mathbf{x}) \times \mathbf{u}(\mathbf{x} + \mathbf{r}))_\mu \rangle &= S(r) \left( 3 \frac{r_\gamma r_\mu}{r^2} - \delta_{\gamma\mu} \right). \end{aligned} \quad (\text{A.4})$$

If we now proceed to write, as in [4], the velocity vector as the sum of its longitudinal (along  $\mathbf{r}$ ) and transverse components such that  $\mathbf{u}(\mathbf{x}) = \mathbf{u}_L(\mathbf{x}) + \mathbf{u}_T(\mathbf{x})$ , we have

$$\langle (u_L + u_T)_\gamma ((\mathbf{u}_L + \mathbf{u}_T) \times (\mathbf{u}'_L + \mathbf{u}'_T))_\mu \rangle = S(r) \left( 3 \frac{r_\gamma r_\mu}{r^2} - \delta_{\gamma\mu} \right), \quad (\text{A.5})$$

where the unprimed velocities denote measurement at  $\mathbf{x}$  and the primed velocities denote measurement at  $\mathbf{x} + \mathbf{r}$ . It is clear that both the left- and right-side vanish for  $\gamma = \mu$ . But we would like to examine the detailed balance in terms of the longitudinal and transverse components on the left-hand side:

$$M_{\gamma\mu} = \langle (u_L + u_T)_\gamma ((\mathbf{u}_L \times \mathbf{u}'_L)_\mu + (\mathbf{u}_L \times \mathbf{u}'_T)_\mu + (\mathbf{u}_T \times \mathbf{u}'_L)_\mu + (\mathbf{u}_T \times \mathbf{u}'_T)_\mu) \rangle = S(r) \left( 3 \frac{r_\gamma r_\mu}{r^2} - \delta_{\gamma\mu} \right). \quad (\text{A.6})$$

Since this must be true for any choice  $\mathbf{r}$ , we can, without loss of generality, choose  $\mathbf{r} = r\hat{\mathbf{i}}$  so that it lies along the  $x$ -axis. The matrix of Eq. (A.6) is then diagonal and traceless, and we see, using Eq. (21), that

$$\begin{aligned} M_{11} &= \mathbf{u}_L \cdot (\mathbf{u}_T \times \mathbf{u}'_T) = 2S(r) = \frac{1}{15}hr^2, \\ M_{22} = M_{33} &= \mathbf{u}_T \cdot ((\mathbf{u}_L + \mathbf{u}_T) \times (\mathbf{u}'_L + \mathbf{u}'_T)) = -S(r) = -\frac{1}{30}hr^2. \end{aligned} \quad (\text{A.7})$$

Eq. (A.7) is a form of the so-called “2/15ths law” derived by L’vov et al. [4]. The exact result of that work is obtained by computing  $(\mathbf{u}_L - \mathbf{u}'_L) \cdot (\mathbf{u}_T \times \mathbf{u}'_T)$  which is equal, by homogeneity to  $2\mathbf{u}_L \cdot (\mathbf{u}_T \times \mathbf{u}'_T) = 4S(r) = (2/15)hr^2$ .

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