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Leith diffusion model for homogeneous anisotropic turbulence

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ABSTRACT

A new spectral closure model for homogeneous anisotropic turbulence is proposed. The systematic development begins by closing the third-order correlation describing nonlinear interactions by an anisotropic generalization of the Leith diffusion model for isotropic turbulence. The correlation tensor is then decomposed into a tensorially isotropic part, or directional anisotropy, and a trace-free remainder, or polarization anisotropy. The directional and polarization components are then decomposed using irreducible representations of the SO(3) symmetry group. Under the ansatz that the decomposition is truncated at quadratic order, evolution equations are derived for the directional and polarization pieces of the correlation tensor. Numerical simulation of the model equations for a freely decaying anisotropic flow illustrate the non-trivial effects of spectral dependencies on the different return-to-isotropy rates of the directional and polarization contributions.

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1. Introduction

The Leith diffusion model for energy transfer in homogeneous isotropic turbulence [1] offers a significant improvement over other elementary closures such as the Kovasznay and Heisenberg models [2]. To cite only one advantage, at any scale k at which the energy spectrum satisfies $E(k) = 0$, the Kovasznay and Heisenberg models both give $\dot{E}(k) = 0$; this guarantees realizability but it also implies that energy cannot be transferred to an unexcited scale by nonlinearity alone. On the contrary, the diffusive character of the Leith model ensures that the energy at such a scale will eventually be nonzero provided the energy spectrum does not vanish identically. Some further comparisons demonstrating the capabilities of the Leith model appear in [3,4]. The analytical simplicity of the Leith model also makes it an attractive way to approach problems of greater physical complexity involving coupled fluctuating fields [5] or resonant wave interactions [6].

On several occasions, Kraichnan suggested treating turbulent energy transfer as a diffusion process in wavenumber space [7,8]. But it was Weinstock [9] who first made an explicit connection between a diffusion model and an analytical closure. Briefly, the connection is that so-called 'local' nonlinear interactions – triad interactions among modes with wavenumbers k, p, and q in which p or (q) is small and $q \approx k$ (or $p \approx k$)– are found to be approximately described by diffusion in k -space in the Eddy Damped Quasi-Normal Markovianized (EDQNM) closure [10]. This may seem plausible, or even unremarkable on general grounds if we think of a mode with wavevector $\boldsymbol{k}$ being 'scattered' into another with a nearby wavevector $\boldsymbol{k}+\boldsymbol{q}$. However, the analysis is nontrivial and depends significantly on the geometry of nonlinear interactions in the EDQNM closure [4]. In any case, this type of result is important, because it shows that the Leith diffusion model is not merely a phenomenological guess, but a limit of a rational closure theory. For a similar treatment of the Heisenberg model, refer to [11,12].

While providing conceptual support to the Leith model, this analysis also suggests a significant modification, because its outcome is not a pure diffusion model but an integro-differential model in which the effective 'diffusivity' at wavenumber k depends on an integral over scales $p \leq k$. The result can be described as a 'pair-interaction' model in the sense of [13]. Only if the integral is replaced by some function of k alone is a purely diffusive model recovered. An integro-differential closure of this type, the 'LWN' (for Local Wave-Number) model, was proposed as a refinement of the original Leith model and explored in detail in [14].

Experience with the LWN and similar simplified spectral closure models [3,12] has shown that they can recover nontrivial properties of far more complex closures based on the triad interactions of the exact Navier-Stokes nonlinearity. This observation suggests a practical role for simplified closures, because it is evident that addressing the challenges of turbulence modeling with even as

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simple a model of triad interactions as EDQNM is impractical: roughly speaking, it would require solving three-dimensional EDQNM at every point in a flow region. From this viewpoint, simplified closures for the locally homogeneous part of the problem seem both theoretically natural and practically indispensible. The BHRZ model [15] is a good example of an attempt to apply the Leith model to practical problems.

The simplified models discussed above have either been restricted to isotropic statistics, as in the energy transfer models described in [2], or have treated anisotropy in a rudimentary manner: for example, the BHRZ model [15] simply adds anisotropic terms suggested by single-point models to the isotropic Leith diffusion model. Although the result is useful, it seems incomplete, because logically, single-point models should be deduced from two-point models, not the other way around. It therefore seemed reasonable to investigate whether a diffusion model similar to the Leith model could apply to anisotropic turbulence. Some heuristic anisotropic diffusion models have already been proposed and analyzed in [5,16,17]. The purpose of this paper is to generalize the Leith model to anisotropic statistics more systematically by carrying out the arguments of [4] without assuming isotropy.

The analysis will be based on the directional-polarization decomposition of the correlation tensor due to Cambon [18]: this is a generalization of the trace-deviator decomposition for solenoidal tensors. We will apply the approximate analysis of local interactions following [4] to derive diffusion models for the directional and polarization components of the correlation tensor. We then substitute a low-order model for these components using the SO(3) decomposition [19] as developed in [20]; in the terminology of [20] this is the 'spin two' part of the correlation, which alone determines the Reynolds stresses. The results will be applied to single-point modeling, with a brief assessment of the Rotta linear relaxation model [21] from the viewpoint of the present approximate two-point theory.

2. Development of the general closure model

The spectral evolution equation for the correlation tensor $U_{ij}(\boldsymbol{k})$ in homogeneous anisotropic turbulence has the form

$$\dot{U}_{ij}(\boldsymbol{k}) = \int d\boldsymbol{p}d\boldsymbol{q}\ \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \tag{1}$$

where $S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ is the third-order correlation describing nonlinear interactions. The term $\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})$ states that Fourier modes interact in *triads* satisfying $\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}$. The viscous and production terms are not written in Eq. (1) because they have no role in this paper. The EDQNM closure for $S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ has the form

$$\begin{aligned} S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = {} & P_{imn}(\boldsymbol{k})P_{jrs}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})U_{ns}(\boldsymbol{q})U_{mr}(\boldsymbol{p}) \\ & - P_{imn}(\boldsymbol{k})P_{mrs}(\boldsymbol{p})\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})U_{ns}(\boldsymbol{q})U_{rj}(\boldsymbol{k}) \\ & - P_{imn}(\boldsymbol{k})P_{mrs}(\boldsymbol{q})\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})U_{ns}(\boldsymbol{p})U_{rj}(\boldsymbol{k}) \end{aligned} \tag{2}$$

where

$$P_{imn}(\boldsymbol{k}) = \frac{1}{2}\left(k_m P_{in}(\boldsymbol{k}) + k_n P_{im}(\boldsymbol{k})\right) \tag{3}$$

is the Navier-Stokes nonlinearity including the pressure term, and

$$P_{im}(\boldsymbol{k}) = \delta_{ij} - k^{-2}k_i k_j \tag{4}$$

is the transverse projection operator arising from the solenoidal constraint $k_i U_{ij}(\boldsymbol{k}) = k_j U_{ij}(\boldsymbol{k}) = 0$ [2].

In Eq. (2), $\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ is a relaxation time for triad interactions [22]. In EDQNM, it is a phenomenological term for which a variety of forms have been proposed and applied [3,22]. We will leave it unspecified; the only important constraint is that to ensure energy conservation, it must be invariant to interchange of the wavevector arguments $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$. But it should be noted that treating the triad interaction time as a scalar (following Cambon: see [23]) is a significant simplification: in Kraichnan's Test-field Model [24], it is a rank six tensor related to the product of three response tensors. Its replacement by a scalar is based on the expectation, or perhaps hope, that the main effects of anisotropy occur through the correlation tensor alone.

Ref. [4] treated the local interaction limit $0 \approx q \ll p \approx k$, arguing that in this limit, the last term in Eq. (2) can be ignored on the basis that in a power-law intertial range, for example, $U(q) \gg U(p)$. Then

$$\begin{aligned} S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \approx {} & P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})U_{ns}(\boldsymbol{q}) \\ & \times \left(P_{jrs}(\boldsymbol{k})U_{mr}(\boldsymbol{p}) - P_{mrs}(\boldsymbol{p})U_{rj}(\boldsymbol{k})\right) \end{aligned} \tag{5}$$

Strictly speaking, we should add the terms coming from the complementary limit $0 \approx p \ll q \approx k$ to ensure the required (i, j) index symmetry. But rather than lengthening all the formulas by writing these terms every time, we will understand index symmetrization, and write it explicitly in the final, if not in every intermediate result.

Next, develop the term in parentheses in Eq. (5) as

$$\begin{aligned} & P_{jrs}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) - P_{mrs}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) \\ & \quad = q_r P_{js}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) + k_s P_{jr}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) + q_r P_{ms}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) \\ & \qquad - k_s P_{mr}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) + q_s P_{mr}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) \\ & \quad = q_r P_{js}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) + q_r P_{ms}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) \\ & \qquad + k_s\left(P_{jr}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) - P_{mr}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k})\right) \end{aligned} \tag{6}$$

Note that we ignore derivatives of the relaxation time Θ in this analysis. The term proportional to q_s vanishes on contraction with $U_{ns}(\boldsymbol{q})$. Otherwise, ignore terms proportional to $\boldsymbol{q}$ following [4]. The last line in Eq. (6) will lead to diffusion. Taylor expansion in $\boldsymbol{q}$ to second order gives

$$\begin{aligned} & P_{jr}(\boldsymbol{k})U_{mr}(\boldsymbol{k}-q) - P_{mr}(\boldsymbol{k}-q)U_{rj}(\boldsymbol{k}) \\ & \quad = P_{jr}(\boldsymbol{k})\left(-\frac{\partial U_{mr}(\boldsymbol{k})}{\partial k_\nu}q_\nu + \frac{1}{2}\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}q_\nu q_\mu\right) \\ & \qquad - U_{jr}(\boldsymbol{k})\left(-\frac{\partial P_{mr}(\boldsymbol{k})}{\partial k_\nu}q_\nu + \frac{1}{2}\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}q_\nu q_\mu\right) \\ & \quad = \left(U_{jr}(\boldsymbol{k})\frac{\partial P_{mr}(\boldsymbol{k})}{\partial k_\nu} - P_{jr}(\boldsymbol{k})\frac{\partial U_{mr}(\boldsymbol{k})}{\partial k_\nu}\right)q_\nu \\ & \qquad - \frac{1}{2}\left(U_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} - P_{jr}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}\right)q_\nu q_\mu \end{aligned} \tag{7}$$

Taking the local interaction limit has led to a simplified 'pair interaction' model [13] between modes with wavevectors $\boldsymbol{k}$ and $\boldsymbol{q}$, thus effectively replacing $S_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ by $S_{ij}(\boldsymbol{k},\boldsymbol{q})$. The term following the second equality sign in Eq. (7) will vanish on integration over q. Thus, restoring the (i, j) index symmetry,

$$\begin{aligned} S_{ij}(\boldsymbol{k},\boldsymbol{q}) = {} & P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ & \times q_\nu q_\mu\left(-U_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{jr}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}\right) \\ & + P_{jmn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ & \times q_\nu q_\mu\left(-U_{ir}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{ir}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}\right) \end{aligned} \tag{8}$$

where we have replaced the triad relaxation time $\Theta(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ by a 'pair' relaxation time $\Theta(\boldsymbol{k},\boldsymbol{q})$ and absorbed the factors of 1/2 into $\Theta(\boldsymbol{k},\boldsymbol{q})$. Although these formal manipulations only apply in the limit $q \approx 0$, we will borrow from [4] the idea of applying Eq. (8) to the entire region $q \le k$. Then instead of the actual energy transfer relation Eq. (1) based on triad interactions, we have a new pair

interaction model,

$$\begin{aligned}\dot{U}_{ij}(\boldsymbol{k}) &= \int_{q\le k} d\boldsymbol{q}\; S_{ij}(\boldsymbol{k},\boldsymbol{q}) \\ &= \int_{q\le k} d\boldsymbol{q}\; \Big\{ P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ &\quad \times q_\nu q_\mu \left(-U_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{jr}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}\right) \\ &\quad + P_{jmn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ &\quad \times q_\nu q_\mu \left(-U_{ir}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{ir}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu}\right)\Big\} \end{aligned} \tag{9}$$

We could stop here, but it is very convenient at this point to introduce the directional-polarization decomposition of Cambon [18]. It consists in the orthogonal decomposition

$$U_{ij}(\boldsymbol{k}) = U^{dir}(\boldsymbol{k})P_{ij}(\boldsymbol{k}) + U^{pol}_{ij}(\boldsymbol{k}) \tag{10}$$

where

$$U^{dir}(\boldsymbol{k}) = \frac{1}{2}P_{ij}(\boldsymbol{k})U_{ij}(\boldsymbol{k}) \tag{11}$$

and

$$U^{pol}_{ij}(\boldsymbol{k}) = U_{ij}(\boldsymbol{k}) - U^{dir}(\boldsymbol{k})P_{ij}(\boldsymbol{k}) \tag{12}$$

is simply the 'remainder' that satisfies Eq. (10). Note that the energy is given by

$$\frac{1}{2}\delta_{ij}\int d\boldsymbol{k}\; U_{ij}(\boldsymbol{k}) = \int d\boldsymbol{k}\; U^{dir}(\boldsymbol{k}) \tag{13}$$

and that $U^{pol}_{ij}(\boldsymbol{k})$ is trace-free.

Substitute the decomposition Eq. (10) in the first term in parentheses in Eq. (9). Taking account of an obvious cancellation of two terms proportional to U^{dir},

$$\begin{aligned} &-U_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{jr}(\boldsymbol{k})\frac{\partial^2 U_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \\ &\quad = -U^{pol}_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + 2P_{jr}(\boldsymbol{k})\frac{\partial U^{dir}(\boldsymbol{k})}{\partial k_\nu}\frac{\partial P_{mr}(\boldsymbol{k})}{\partial k_\mu} \\ &\qquad + P_{jr}(\boldsymbol{k})P_{mr}(\boldsymbol{k})\frac{\partial^2 U^{dir}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} + P_{jr}(\boldsymbol{k})\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \end{aligned} \tag{14}$$

Then

$$S_{ij}(\boldsymbol{k},\boldsymbol{q}) = S^a_{ij}(\boldsymbol{k},\boldsymbol{q}) + S^b_{ij}(\boldsymbol{k},\boldsymbol{q}) + S^c_{ij}(\boldsymbol{k},\boldsymbol{q}) \tag{15}$$

where

$$\begin{aligned} S^a_{ij}(\boldsymbol{k},\boldsymbol{q}) &= P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ &\quad \times q_\nu q_\mu P_{jr}(\boldsymbol{k})P_{mr}(\boldsymbol{k})\frac{\partial^2 U^{dir}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \end{aligned} \tag{16}$$

$$\begin{aligned} S^b_{ij}(\boldsymbol{k},\boldsymbol{q}) &= -P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ &\quad \times q_\nu q_\mu U^{pol}_{jr}(\boldsymbol{k})\frac{\partial^2 P_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \end{aligned} \tag{17}$$

$$\begin{aligned} S^c_{ij}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) &= P_{imn}(\boldsymbol{k})\Theta(\boldsymbol{k},\boldsymbol{q})U_{ns}(\boldsymbol{q})k_s \\ &\quad \times q_\nu q_\mu P_{jr}(\boldsymbol{k})\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \end{aligned} \tag{18}$$

Evaluating the products of projection operators in Eq. (16),

$$S^a_{ij}(\boldsymbol{k},\boldsymbol{q}) = q_\nu q_\mu U_{ns}(\boldsymbol{q})\Theta(\boldsymbol{k},\boldsymbol{q})k_s k_n P_{ij}(\boldsymbol{k})\frac{\partial^2 U^{dir}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \tag{19}$$

Proportionality to $P_{ij}(\boldsymbol{k})$ implies that this term contributes only to the evolution equation for $U^{dir}(\boldsymbol{k})$; suppose then that

$$\dot{U}^{dir}(\boldsymbol{k}) = \left(\int_{q\le k} d\boldsymbol{q}\; q_\nu q_\mu U_{ns}(\boldsymbol{q})\Theta(\boldsymbol{k},\boldsymbol{q})\right)k_n k_s \frac{\partial^2 U^{dir}(\boldsymbol{k})}{\partial k_\nu \partial k_\mu} \tag{20}$$

Following [4], in order to insure energy conservation we modify Eq. (20) to read

$$\dot{U}^{dir}(\boldsymbol{k}) = \frac{\partial}{\partial k_\nu}\left\{\left(\int_{q\le k} d\boldsymbol{q}\; q_\nu q_\mu U_{ns}(\boldsymbol{q})\Theta(\boldsymbol{k},\boldsymbol{q})\right)k_n k_s \frac{\partial}{\partial k_\mu}U^{dir}(\boldsymbol{k})\right\} \tag{21}$$

The divergence form ensures that energy conservation is satisfied identically under plausible conditions on $U^{dir}(k)$ when $k = 0$ and $k \to \infty$.

Next look at the contributions to Eq. (14) dependent on U^{pol}_{ij}. Straightforward evaluation shows that

$$S^b_{ij}(\boldsymbol{k},\boldsymbol{q}) = -2q_\nu q_\mu U_{ns}(\boldsymbol{q})k^{-2}\left(P_{in}(\boldsymbol{k})k_s k_\nu - P_{i\nu}(\boldsymbol{k})k_s k_n\right)U^{pol}_{j\mu}(\boldsymbol{k}) \tag{22}$$

and that further simplification of $S^c_{ij}(\boldsymbol{k},\boldsymbol{q})$ is not possible.

At this point, we simplify the model by restricting attention to the isotropic contribution to the damping integral in Eq. (21) by setting

$$U_{ns}(\boldsymbol{q}) = U(q)P_{ns}(\boldsymbol{q}) \quad \Theta(\boldsymbol{k},\boldsymbol{q}) = \Theta(k,q) \tag{23}$$

The more general case of general tensor damping, which allows the damping to be direction-dependent, is left to future research. For the isotropic damping described by Eq. (23),

$$\begin{aligned} &\int_{q\le k} d\boldsymbol{q}\; q_\nu q_\mu P_{ns}(\boldsymbol{q})U(q)\Theta(\boldsymbol{k},\boldsymbol{q}) \\ &\quad = \frac{1}{15}\left(2\delta_{\mu\nu}\delta_{ns} - \delta_{\mu n}\delta_{\nu s} - \delta_{\mu s}\delta_{\nu n}\right)\eta(k) \end{aligned} \tag{24}$$

where

$$\eta(k) = \int_{q\le k} dq\; q^4\Theta(k,q)U(q) \tag{25}$$

Using Eqs. (21), (22), and (18) with the isotropic damping formula Eqs. (24) and (25) results in the model

$$\dot{U}^{dir}(\boldsymbol{k}) = \frac{4}{15}\frac{\partial}{\partial k_\nu}\eta(k)k^2\frac{\partial}{\partial k_\nu}U^{dir}(\boldsymbol{k}) - \frac{2}{15}\frac{\partial}{\partial k_\nu}\eta(k)k_\nu k_\mu\frac{\partial}{\partial k_\mu}U^{dir}(\boldsymbol{k}) \tag{26}$$

$$\begin{aligned} \dot{U}^{pol}_{ij}(\boldsymbol{k}) &= \frac{2}{3}\eta(k)U^{pol}_{ij}(\boldsymbol{k}) + \frac{2}{15}k^2 P_{imjr}(\boldsymbol{k})\eta(k)\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\nu} \\ &\quad - \frac{1}{15}k_s k_n P_{imjr}(\boldsymbol{k})\eta(k)\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_n \partial k_s} \\ &\quad - \frac{1}{15}k_s k_m P_{injr}(\boldsymbol{k})\eta(k)\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_n \partial k_s} \end{aligned} \tag{27}$$

where

$$P_{imjr}(\boldsymbol{k}) = P_{im}(\boldsymbol{k})P_{jr}(\boldsymbol{k}) + P_{jm}(\boldsymbol{k})P_{ir}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k})P_{rm}(\boldsymbol{k}) \tag{28}$$

The appearance of an integral in the damping factor $\eta(k)$ of Eq. (25) distinguishes the present diffusion model from Leith's original model [1]; compare also [14] where such a modification was first suggested. Eqs. (26) and (27) implicitly contain an arbitrary constant through the phenomenological term $\Theta(k, q)$ in Eq. (25). We refer to [3] for the choice of this constant, and to [12] for some of the possible models for $\Theta(k, q)$.

3. Evolution equations for spin two

Our next step is to substitute the SO(3) decompositions [19,20] of U^{dir} and U^{pol}_{ij} in Eqs. (26) and (27). In this paper, the SO(3) decomposition will be truncated at quadratic order, so that we assume [20]

$$U^{dir}(\boldsymbol{k}) = \frac{1}{2}U(k) + H_{mn}(k)k^{-2}k_m k_n \tag{29}$$

and

$$U^{pol}_{ij}(\boldsymbol{k}) = J_{ij}(k) - k^{-2}k_i k_a J_{aj}(k) - k^{-2}k_j k_a J_{ai}(k) + \frac{1}{2}\left(\delta_{ij} + k^{-2}k_i k_j\right)k^{-2}J_{mn}(k)k_m k_n \tag{30}$$

where $H_{ij}(k)$ and $J_{ij}(k)$ are both symmetric trace-free second rank tensors. These are the only terms in the SO(3) decomposition that contribute directly to the Reynolds stress [20]. Simple calculations [18] show that the Reynolds stress deviator, given by

$$R_{ij} = \int d\boldsymbol{k}\ \left((H_{mn}(k)k^{-2}k_m k_n)P_{ij}(\boldsymbol{k}) + U^{pol}_{ij}(\boldsymbol{k})\right) \tag{31}$$

is related to the single-point moments

$$H_{ij} = 4\pi \int_0^\infty dk\ k^2 H_{ij}(k) \tag{32}$$

$$J_{ij} = 4\pi \int_0^\infty dk\ k^2 J_{ij}(k) \tag{33}$$

by

$$R_{ij} = \frac{15}{2}H_{ij} + \frac{5}{2}J_{ij} \tag{34}$$

For connections to the decomposition of the stress by 'structure tensors,' we refer to [25].

3.1. Directional anisotropy

The equation of motion for $U(k)$ is found by integrating Eq. (26) over a sphere of radius k:

$$\oint d\Omega\ \dot{U}^{dir}(\boldsymbol{k}) = 4\pi k^2 \dot{U}(k) = \oint d\Omega \left[\frac{4}{15}\frac{\partial}{\partial k_\nu}\left(\eta(k)k^2\frac{\partial}{\partial k_\nu}U(k)\right) - \frac{2}{15}\frac{\partial}{\partial k_\nu}\left(\eta(k)k_\nu k_\mu \frac{\partial}{\partial k_\mu}U(k)\right)\right] \tag{35}$$

Use

$$\frac{\partial}{\partial k_\nu}\left(\eta(k)k^2\frac{\partial}{\partial k_\nu}U(k)\right) = \frac{\partial}{\partial k_\nu}\left(\eta(k)kk_\nu U'(k)\right) = 3\eta(k)kU'(k) + k\left(\eta(k)kU'(k)\right)' \tag{36}$$

where the primes denote derivatives with respect to k. Then

$$\oint d\Omega\ \frac{\partial}{\partial k_\nu}\left(\eta(k)k^2\frac{\partial}{\partial k_\nu}U(k)\right) = 3\eta(k)k^3U'(k) + k^3\left(\eta(k)kU'(k)\right)' = \left(k^3\eta(k)kU'(k)\right)' \tag{37}$$

If $\eta(k)$ is defined by the elementary closure

$$\eta(k) = \mathrm{C}\sqrt{k^3E(k)} \tag{38}$$

for some phenomenological constant C, then the result is simply the isotropic Leith model in its original form [1]. Because Eq. (37) is an exact derivative, we recover energy conservation

$$\int_0^\infty dk\ k^2\dot{U}(k) = \int_0^\infty dk\ \left(k^3\eta(k)kU'(k)\right)' = 0 \tag{39}$$

The equation for $H_{ij}(k)$ follows by multiplying Eq. (26) by $k_i k_j$ and integrating over a sphere of radius k. The result is

$$\dot{H}_{ij}(k) = \frac{2}{15}\eta(k)k^2H''_{ij}(k) + \frac{2}{15}(\eta'(k)k^2 + 4k\eta(k))H'_{ij}(k) - \frac{24}{15}\eta(k)H_{ij}(k) \tag{40}$$

It is interesting to compare this model to the simple linear relaxation model

$$\dot{H}_{ij}(k) = -\frac{24}{15}\eta(k)H_{ij}(k) \tag{41}$$

This model damps anisotropy, and is thus consistent with intuitive notions of 'return to isotropy;' however, it clearly cannot create anisotropy at any initially isotropic scale of motion. Thus, if the large scales are forced anisotropically, Eq. (41) predicts that there can be no anisotropic excitation beyond the forced scales. In contrast, the derivative terms in Eq. (40) can propagate the anisotropy to all scales of motion.

3.2. Polarization anisotropy

To derive the equation satisfied by $J_{ij}(k)$, substitute Eq. (30) in Eq. (27). Use the results

$$\begin{aligned}\nabla^2 J^{pol}_{ij}(\boldsymbol{k}) = &\left(P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k}) + P_{in}(\boldsymbol{k})P_{jm}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k})P_{mn}(\boldsymbol{k})\right)\left(J''_{mn}(k) + \frac{2}{k}J'_{mn}(k)\right)\\ &+4\Big(-P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k}) - P_{in}(\boldsymbol{k})P_{jm}(\boldsymbol{k}) + P_{ij}(\boldsymbol{k})P_{mn}(\boldsymbol{k})\\ &+k_ik_mP_{jn}(\boldsymbol{k}) + k_ik_nP_{jm}(\boldsymbol{k}) + k_jk_mP_{in}(\boldsymbol{k})\\ &+k_jk_nP_{im}(\boldsymbol{k})\Big)k^{-2}J_{mn}(k)\end{aligned} \tag{42}$$

$$P_{imjr}(\boldsymbol{k})\frac{\partial^2 U^{pol}_{mr}(\boldsymbol{k})}{\partial k_\nu \partial k_\nu} = 2P_{iajb}(\boldsymbol{k})\left(J''_{ab}(k) + \frac{2}{k}J'_{ab}(k) - 4k^{-2}J_{ab}(k)\right) \tag{43}$$

$$k_nk_s\frac{\partial}{\partial k_s}\frac{\partial}{\partial k_n}J^{pol}_{mr}(\boldsymbol{k}) = P_{marb}(\boldsymbol{k})k^2J''_{ab}(k) \tag{44}$$

$$\begin{aligned}k_nk_s\frac{\partial}{\partial k_m}\frac{\partial}{\partial k_s}J^{pol}_{nr}(\boldsymbol{k}) &= k_s\left(\frac{\partial}{\partial k_m}k_n\frac{\partial}{\partial k_s} - \delta_{mn}\frac{\partial}{\partial k_s}\right)J^{pol}_{nr}(\boldsymbol{k})\\ &= k_s\frac{\partial}{\partial k_m}\frac{\partial}{\partial k_s}k_nJ^{pol}_{nr}(\boldsymbol{k}) - k_n\frac{\partial}{\partial k_m}J^{pol}_{nr}(\boldsymbol{k}) - k_s\frac{\partial}{\partial k_s}J^{pol}_{mr}(\boldsymbol{k})\\ &= P_{marb}(\boldsymbol{k})\left(J_{ab}(k) - kJ'_{ab}(k)\right)\end{aligned} \tag{45}$$

Integrating Eq. (27) over a sphere of radius k gives the equation of motion

$$\dot{J}_{ij}(k) = \frac{2}{15}\eta(k)\left(k^2J''_{ij}(k) + 5kJ'_{ij}(k) - 4J_{ij}(k)\right) \tag{46}$$

4. Application to the return to isotropy

An obvious application of an anisotropic model is the 'return to isotropy,' in which initially anisotropic turbulence decays freely in time. There is a related notion of the recovery of statistical isotropy of the small scales in high Reynolds number turbulence. This latter problem partly motivated the use of the SO(3) decomposition of tensor correlations in [19] and subsequent studies on the scaling exponents and decay rates (in scale) of the higher order rotational modes or spins. The wider significance of that problem and its connection to the K41 theory are discussed in [26]. The present emphasis will instead be on the decay of overall anisotropy in time.

In turbulence modeling, the return to isotropy is described by the decay of the Reynolds stress deviator through the action of

the 'slow pressure-strain' correlation. One such model is the *Rotta model* [21]

$$\dot{R}_{ij} = -C_R \frac{\epsilon}{k} R_{ij} \tag{47}$$

where ϵ is the dissipation rate, k is the turbulent kinetic energy, and C_R is an empirical constant called the *Rotta constant*. Here, however, we must consider the relaxation of both tensors H_{ij} and J_{ij}, which are related to the stress deviator by Eq. (34). In this sense, Eqs. (40) and (46) can be considered a model for the slow pressure-strain correlation, albeit one of a nonstandard form.

A relaxation equation for H_{ij} follows from rewriting Eq. (40) as

$$k^2 \dot{H}_{ij}(k) = \frac{2}{15}\left(k^4 \eta(k) H'_{ij}(k)\right)' - \frac{24}{15}\eta(k) k^2 H_{ij}(k) \tag{48}$$

and integrating over all wavenumbers k to obtain,

$$\dot{H}_{ij} = -\frac{8}{5}\int_0^\infty dk\ \eta(k) k^2 H_{ij}(k) \tag{49}$$

The result is not a rotta model of the form Eq. (47), because the right side is not a multiple of H_{ij}, and it can only be so written after making additional assumptions about both $H_{ij}(k)$ and $\eta(k)$ at all scales of motion.

The analogous relaxation equation for J_{ij} follows from rewriting Eq. (46) as

$$\begin{aligned} k^2 \dot{J}_{ij}(k) = \frac{2}{15}\{&(\eta(k)k^4 J'_{ij}(k))' + 5(\eta(k)k^3 J_{ij}(k))' - ((\eta(k)k^4)' J_{ij}(k))' \\ &+ ((\eta(k)k^4)'' - 5(\eta(k)k^3)' - 4\eta(k)k^2) J_{ij}(k)\} \end{aligned} \tag{50}$$

and again integrating over all k to obtain,

$$\dot{J}_{ij} = \frac{2}{15}\int_0^\infty dk\ \left((\eta(k)k^4)'' - 5(\eta(k)k^3)' - 4\eta(k)k^2\right) J_{ij}(k) \tag{51}$$

This result is even farther from a rotta model than Eq. (49). Comparing Eqs. (49) and (51) suggests that it is very unlikely that the relaxation of H_{ij} and J_{ij} could be described even approximately by models of the form Eq. (47) with the same relaxation rate C_R.

To illustrate the evolution of anisotropy according to this model, we consider a problem in which isotropic forcing leads to a steady state for $U(k, t)$ (it will be useful to indicate the time variable explicitly) while the anisotropy tensors $H_{ij}(k, t)$ and $J_{ij}(k, t)$ decay freely following Eqs. (40) and (46). Initially,

$$U(k, t = 0) = U_0 k \exp\left(-\frac{3}{2}(k/k_0)^2\right) \tag{52}$$

As all components $H_{ij}(k, t)$ and all components $J_{ij}(k, t)$ decay freely the same way, it suffices to consider the evolution of any one component of each tensor, which we will denote by $H(k, t)$ and $J(k, t)$ respectively. These are initialized similarly as

$$H(k, t = 32) = H_0 k \exp\left(-\frac{3}{2}(k/k_0)^2\right) \tag{53}$$

$$J(k, t = 32) = J_0 k \exp\left(-\frac{3}{2}(k/k_0)^2\right) \tag{54}$$

so that the single point moments

$$H(t) = 4\pi \int_0^\infty dk\ k^2 H(k, t) \tag{55}$$

$$J(t) = 4\pi \int_0^\infty dk\ k^2 J(k, t) \tag{56}$$

are equal at the time ($t = 32$) they are initiated.

The form of the U equation solved is

$$\dot{U}(k) = C_U\ \left(k^3 \eta(k) k U'(k)\right)' = 0, \tag{57}$$

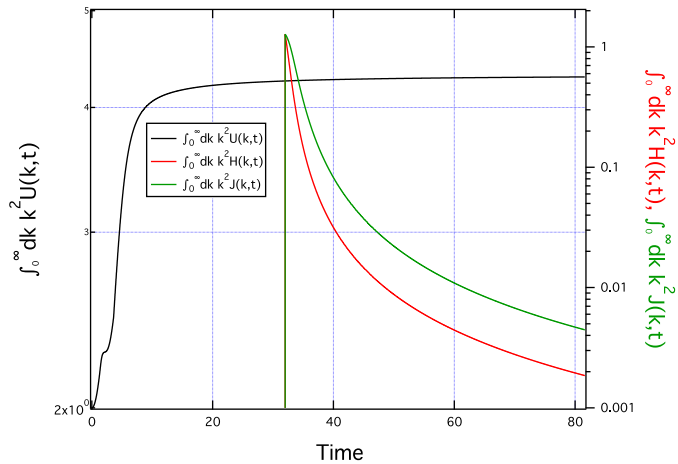

Fig. 1. Time evolution of total energy (black) from $t = 0$. Anisotropic components $H(t)$ (red) and $J(t)$ (green) initiated at $t = 32$.

where the constant C_U was chosen so that Kolmogorov's 1941 result, $E(k) = C_K \epsilon^{2/3} k^{-5/3}$ is satisfied in the inertial range using a value of $C_k = 1.5$ [4,14]. For purposes of computing, we have also introduced analogous constants in the H- and J-equations (C_H, and C_J), which multiply the right-hand-side of the respective equations. We have not yet determined a "correct" value for C_H and C_J but have found that setting them to

$$\frac{2}{15}C_H = \frac{2}{15}C_J = C_U, \tag{58}$$

yields realizable results for the case of simple anisotropic decay.

The model equations were solved using an iterative-implicit Crank-Nicolson temporal discretization and a second-order spatial discretization, to achieve second-order temporal and spatial accuracy. The iterations were performed to a convergence yielding an averaged absolute error between iterations of $< 10^{-9}$ over the mesh at each time step. In order to resolve both high and low wavenumbers, we have made the transformation $k = \exp(z)$. The actual variables solved for are $k^3 U(k)$, $k^3 H(k)$, and $k^3 J(k)$. The high-z and low- z boundaries are treated by extrapolating a ghost-cell based on an arbitrary power-law in k–which becomes a simple process on the z-mesh. This treatment of the low- and high- k boundaries permits a self-consistent flux condition at the boundaries, permitting the U, H and J to flux off the numerical mesh. This allows the calculations to be conducted with zero viscosity (infinite Reynolds number) in which case the dissipation is accounted for by the flux off the mesh. The grid spacing was selected so that the time-decay was independent of the mesh size. The mesh was uniform in z, with $\Delta z = 1.5 \times 10^2$, and spanning $z = [-20, +10]$. The time-steps were held constant at $\Delta t = 5 \times 10^{-5}$. Convergence tests have indicated that this gives the current results approximately six digits of accuracy.

The calculations were conducted by first driving U towards a steady state with a constant forcing given by

$$F(k) = F_0 k^2 \exp\left(-\frac{3}{2}(k/k_0)^2\right), \tag{59}$$

where F_0 is chosen so that

$$F = \int_0^\infty dk\ k^2 F(k) = 1. \tag{60}$$

At a time of $t = 32$, the anisotropy was initiated. This strategy was adopted so that the decay dynamics of H and of J could be isolated from the temporal dynamics of U, since the dynamical behavior of U has already been studied [4].

Fig. 1 shows the evolution of the total energy $E(t)$ from $t = 0$ and the single point moments $H(t)$ and $J(t)$ from $t = 32$. The energy

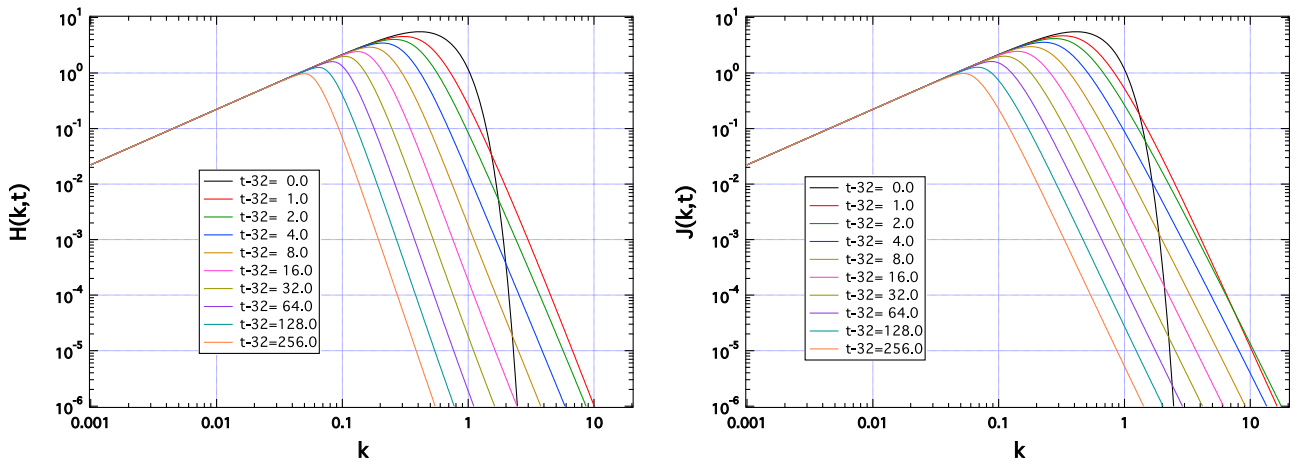

Fig. 2. Spectra of directional contribution $H(k, t)$ (left) and polarization contribution $J(k, t)$ (right) evolving in time.

reaches a steady value, while the anisotropic moments decay from their initial values.

The evolution of the spectra $H(k, t)$ and $J(k, t)$ at identical times after initialization are shown in Fig. 2. It is evident that the diffusive terms generate smaller anisotropic scales, whilst the spectrum as a whole decays; note the shift of the peak to lower values as the evolution proceeds. The spectral contribution of J in the high wavenumbers persists longer than that of H given identical initializations. These results illustrate two basic properties of this model: diffusive creation of anisotropic small scales, and the absence of any simple relation between $H(t)$ and $J(t)$ during decay.

These calculations also demonstrate the emergence of a power-law scaling at high wavenumbers (small scales) for $H(k, t)$ and $J(k, t)$. The exponents of these power-laws can be analytically computed for the case of a steady-state turbulence. In this situation, the model can be shown to admit scalings for

$$H(k,t) = H_\infty k^{\gamma_H} \tag{61}$$

with

$$\gamma_H = -[11 + \sqrt{553}]/6 \approx -5.752\ldots, \tag{62}$$

and for

$$J(k,t) = J_\infty k^{\gamma_J} \tag{63}$$

with

$$\gamma_J = -2[1 + \sqrt{2}] \approx -4.828\ldots \tag{64}$$

The values computed numerically for a decaying anisotropic turbulence are, for $H(k, t)$

$$\bar{\gamma}_H \approx -5.750\ldots \tag{65}$$

and

$$\bar{\gamma}_J \approx -4.652\ldots \tag{66}$$

Although the numerically computed exponents correspond to freely decaying anisotropic turbulence, they appear to be in reasonable agreement with the analytical results for steady-state turbulence, thus lending support to our numerical calculations and boundary treatment. We will forbear attempts to justify these exponents on physical grounds, but simply note that different models may give different results, (see for example [17,27]).

5. Conclusions

We have developed a diffusion model for anisotropic turbulence analogous to the Leith model for isotropic turbulence by considering local interactions in the EDQNM closure following [4,9]. A preliminary conclusion from the analysis of the return to isotropy is that single point modeling of this problem is not straightforward because spectral properties have an important role.

It would be unrealistic to expect that local interactions could give a complete account of the dynamics. Even in the isotropic case, the Leith model seems to treat energy transfer at any scale as random straining by larger scales, somewhat at variance with the usual eddy viscosity heuristics. Thus, an anisotropic analysis of the complementary distant interaction limit $k \ll p \approx q$, which is related to the Heisenberg model [12], provides an interesting perspective for future research.

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