Moving grids for problems of gas dynamics

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Objectives

Assumptions:

- The time integration scheme is explicit.
- The adaptive mesh should be smooth (example).
- The mesh topology is not changed during adaptation (r-adaptation).

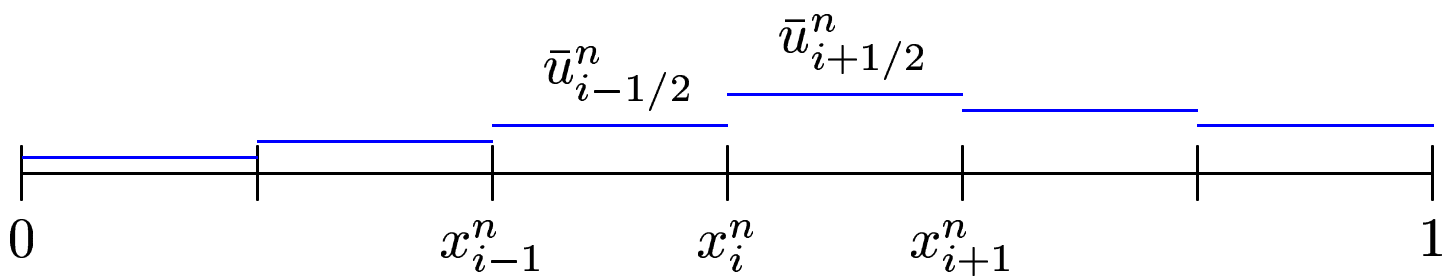
The look-ahead strategy:

- Assume that the data at $t = t^n$ are given and exact.
- Derive an error functionals at $t = t^{n+1}$.
- Build an adapted grid minimizing the error functional.
- Interpolate data to the adapted grid and perform one time step.

Model hyperbolic problem

Consider the Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = u_0(x).$$



The conservative discretization is

$$\bar{u}^{n+1}_{i+1/2} = \bar{u}^n_{i+1/2} - \frac{\Delta t^n}{h^n_{i+1/2}}(f^n_{i+1} - f^n_i) + \frac{\varepsilon\Delta t^n}{h^n_{i+1/2}}\left(\left[\frac{\delta u^n}{\delta x}\right]_{i+1} - \left[\frac{\delta u^n}{\delta x}\right]_i\right)$$

where

$$\bar{u}^n_{i+1/2} \approx \frac{1}{h^n_{i+1/2}}\int_{x^n_i}^{x^n_{i+1}} u(x,t^n)\mathrm{d}x.$$

Error functional (1/8)

Total error at $t = t^{n+1}$

Error of space discretization

Error of time integration

Error of data interpolation

Assumptions:

- mesh is smooth;
- interpolation operator is exact for linear functions;
- the CFL condition holds.

Total error at $t = t^{n+1}$ $\rightarrow$ Error of space discretization at $t = t^n$

The same statement holds for higher order time integration schemes.

Error functional (2/8)

The error of space discretization is

$$F_{ex}(\{\tilde{x}_i^n\}) = \sum_{i=0}^{M} \int_{\tilde{x}_i^n}^{\tilde{x}_{i+1}^n} \left(u(x,t^n) - \bar{\tilde{u}}_{i+1/2}^n\right)^2 \mathrm{d}x$$

where

$$\bar{\tilde{u}}_{i+1/2}^n \approx \frac{1}{\tilde{h}_{i+1/2}^n} \int_{\tilde{x}_i^n}^{\tilde{x}_{i+1}^n} u(x,t^n)\mathrm{d}x.$$

Thus, we have to solve the minimization problem

$$\min_{\tilde{x}_1^n,\ldots,\tilde{x}_M^n} F_{ex}(\{\tilde{x}_i^n\}), \qquad F_{ex}(\{\tilde{x}_i^n\}) = \frac{1}{12}\sum_{i=0}^{M} \left(\left.\frac{\partial u}{\partial x}\right|_{x_{i+1/2}^*}\right)^2 \left[\tilde{h}_{i+1/2}^n\right]^3$$

where $x_{i+1/2}^*$ is a point from interval $(\tilde{x}_i^n,\ \tilde{x}_{i+1}^n)$.

Error functional (3/8)

In order to approximate $\partial u/\partial x|_{x^*_{i+1/2}}$ we proceed as follows:

1. compute a piecewise constant function $\{[\delta u^n/\delta x]_{i+1/2}\}$ on the initial grid $\{x^n_{i+1/2}\}$ using the 2nd order approximation:

$$\left[\frac{\delta u^n}{\delta x}\right]_{i+1/2} = \frac{1}{\hat{h}^n_{i+1/2}}\left(\left[\frac{\delta u^n}{\delta x}\right]_{i+1}\left(\frac{h^n_{i+1/2}}{6}+\frac{h^n_{i-1/2}}{3}\right)\right.$$
$$\left.+\left[\frac{\delta u^n}{\delta x}\right]_{i}\left(\frac{h^n_{i+1/2}}{6}+\frac{h^n_{i+3/2}}{3}\right)\right) \approx \left.\frac{\partial u}{\partial x}\right|_{x^*_{i+1/2}}$$

where

$$\hat{h}^n_{i+1/2} = \frac{1}{3}(h^n_{i-1/2}+h^n_{i+1/2}+h^n_{i+3/2}), \qquad \left[\frac{\delta u^n}{\delta x}\right]_i = \frac{\bar{u}^n_{i+1/2}-\bar{u}^n_{i-1/2}}{(h^n_{i+1/2}+h^n_{i-1/2})/2}.$$

2. interpolate the piecewise constant function $\{[\delta u^n/\delta x]_{i+1/2}\}$ on the mesh $\{\tilde{x}^n_{i+1/2}\}$.

Error functional (4/8)

The approximate minimization problem is

$$\min_{\tilde{x}_1^n,\ldots,\tilde{x}_M^n} F_{ap}(\{\tilde{x}_i^n\}), \qquad F_{ap}(\{\tilde{x}_i^n\}) = \frac{1}{12}\sum_{i=0}^{M}\left[\frac{\delta\tilde{u}^n}{\delta x}\right]_{i+1/2}^2 \left[\tilde{h}_{i+1/2}^n\right]^3.$$

- Functional F_{ap} depends only on the input discrete data.
- $F_{ap} \to F_{ex}$ as $M \to \infty$.
- Solutions of both minimization problems are not unique.

Error functional (5/8)

Let us consider a test function $u(x)$,

$$u(x) = 1 - \frac{9\,r_1 + 5\,r_1^5}{10(r_1 + r_1^5 + r_2)}, \quad r_1 = \exp\frac{1/2 - x}{20\varepsilon}, \quad r_2 = \exp\frac{3/8 - x}{2\varepsilon},$$

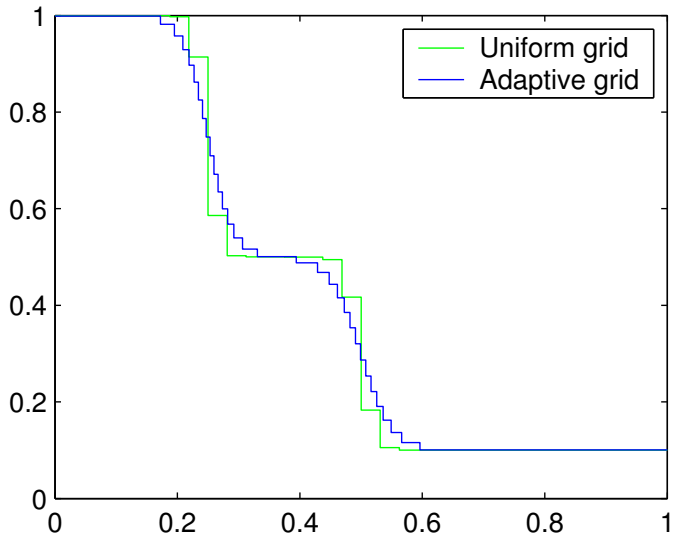
where $\varepsilon = 0.005$.

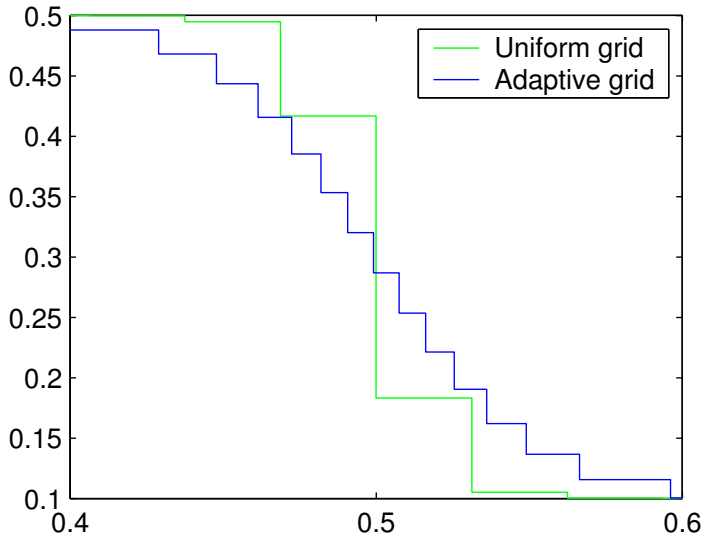
Define

$$E(\{\tilde{x}_i\}) = \sqrt{F_{ex}(\{\tilde{x}_i\})}$$

Error functional (6/8)

The test function $u(x)$.

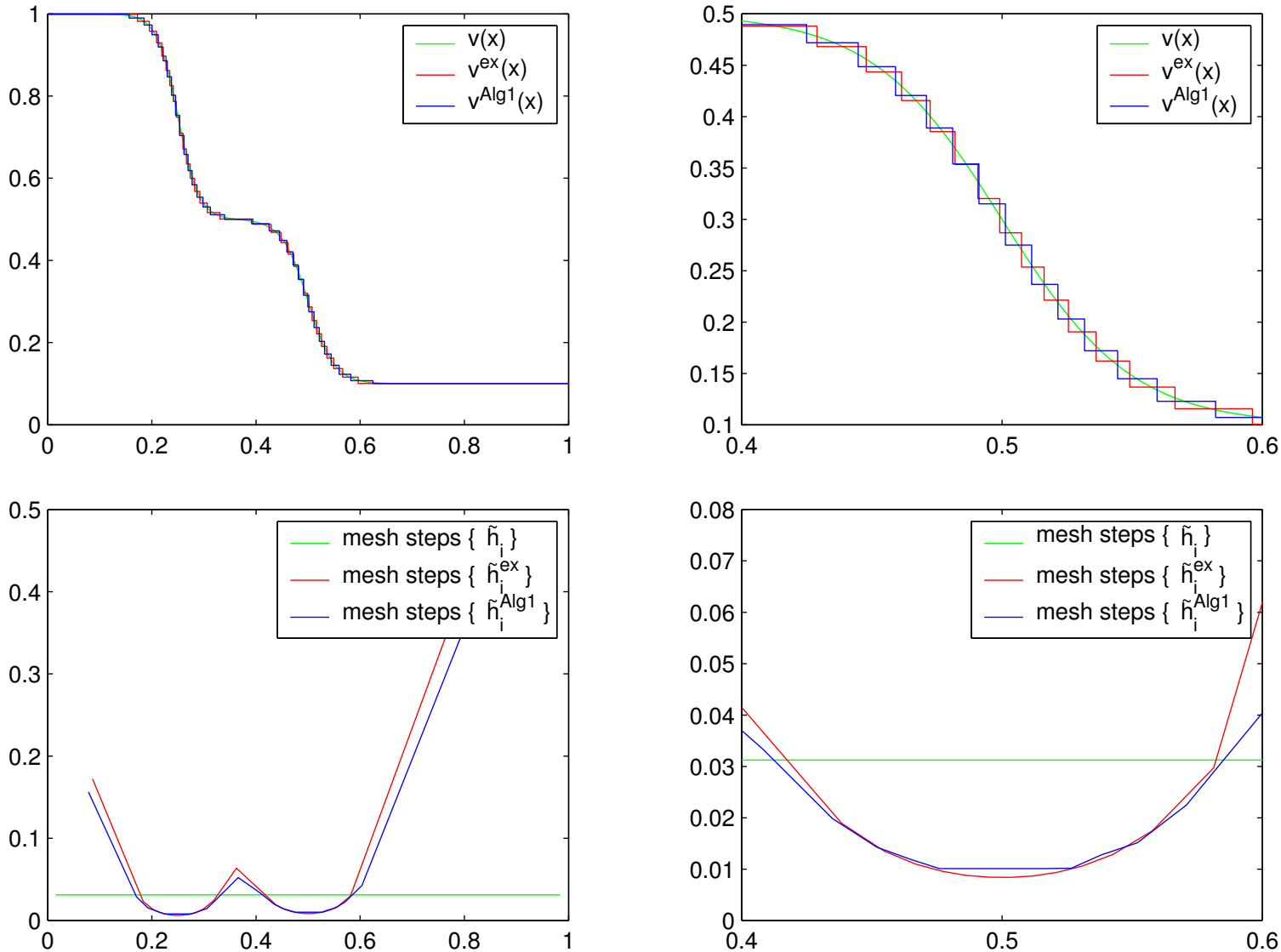




Accuracy analysis of algorithm of approximate minimization.

M	$E(\{x_i^{uniform}\})$	$E(\{\tilde{x}_i^{exact}\})$	$E(\{\tilde{x}_i^{approx}\})$
16	2.99e-2	1.01e-2	1.19e-2
32	1.59e-2	4.99e-3	5.18e-3
64	7.99e-3	2.48e-3	2.50e-3
128	4.00e-3	1.24e-3	1.24e-3

Error functional (7/8)



Error functional (8/8)

Algorithm 1. **Approximate optimization**

1. For the initial grid $\{x_i\}$, compute values $[\delta v/\delta x]_{i+1/2}$ and set $\{x_i^{(0)}\} = \{x_i\}$.
2. Choose a positive number K_{max}. Then for $k = 1, \ldots, K_{max}$ do

   2a. Perform one Gauss-Seidel sweep

$$\min_{x_i^{(k+1)}} \left\{ \left[\widehat{\frac{\delta v}{\delta x}}\right]^2_{i+1/2} (x_{i+1}^{(k)} - x_i^{(k+1)})^3 + \left[\widehat{\frac{\delta v}{\delta x}}\right]^2_{i-1/2} (x_i^{(k+1)} - x_{i-1}^{(k+1)})^3 \right\},$$

   where $i = 1, \ldots, M$ and $\{\widehat{[\delta v/\delta x]}_{i+1/2}\}$ is the conservative interpolant of $\{[\delta v/\delta x]_{i+1/2}\}$ from the initial grid to the intermediate grid

$$0 = x_0^{(k+1)} < \ldots < x_i^{(k+1)} < x_{i+1}^{(k)} < \ldots < x_M^{(k)} = 1.$$

   2b. Stop iterations if

$$\max_i \frac{|x_i^{(k)} - x_i^{(k+1)}|}{x_{i+1}^{(k)} - x_{i-1}^{(k)}} \leq TOL.$$

Grid smoothing (1/3)

The smooth mesh should satisfy:

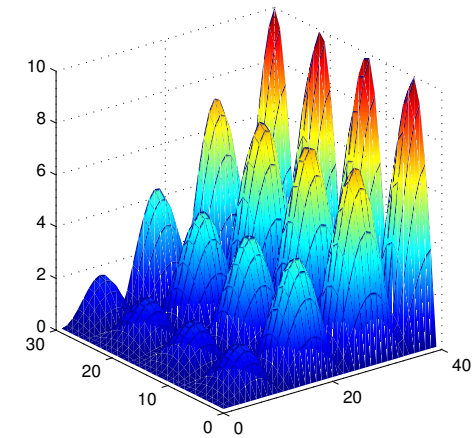
$$\frac{\alpha}{\alpha+1} \leq \frac{h_{cell}}{h_{neib}} \leq \frac{\alpha+1}{\alpha}$$

- smoothing of the mesh is equivalent to smoothing of the error function F_{ap} by solving an elliptic equation

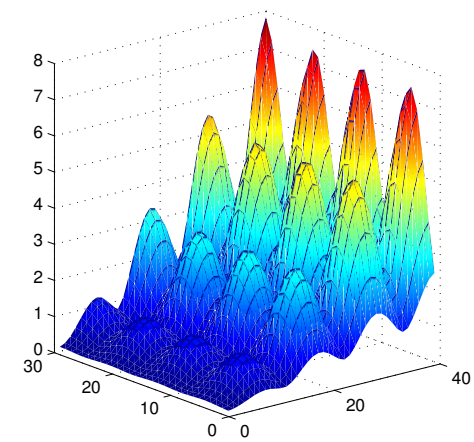
$$(I - \alpha(\alpha+1)\Delta)\tilde{\omega} = \omega$$

  for the coefficients of F_{ap}

- smoothing of the error function can be done via smoothing of the input data
- smoothing preserves main features of the input data but results in increasing of the total error (less than 5% in 1D experiments)



$$2.7 \cdot 10^{-6} \leq \frac{\omega_{cell}}{\omega_{neib}} \leq 3.7 \cdot 10^{5}$$



$$0.57 \leq \frac{\tilde{\omega}_{cell}}{\tilde{\omega}_{neib}} \leq 1.8$$

Grid smoothing (2/3)

The new minimization problem is

$$\min_{\tilde{x}_1^n,\ldots,\tilde{x}_M^n} F_{sm}(\{\tilde{x}_i^n\}), \qquad F_{sm}(\{\tilde{x}_i^n\}) = \frac{1}{12}\sum_{i=0}^{M}\left(\mathcal{S}\left[\frac{\delta\tilde{u}^n}{\delta x}\right]_{i+1/2}\right)^2 \left[\tilde{h}_{i+1/2}^n\right]^3.$$

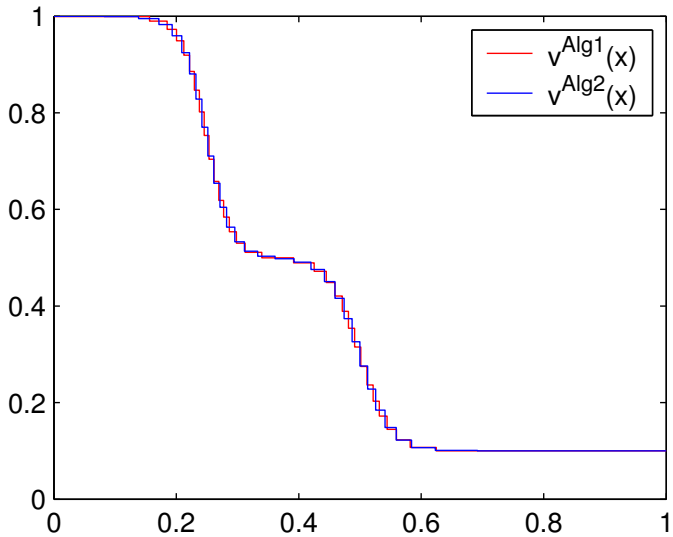
Recall that

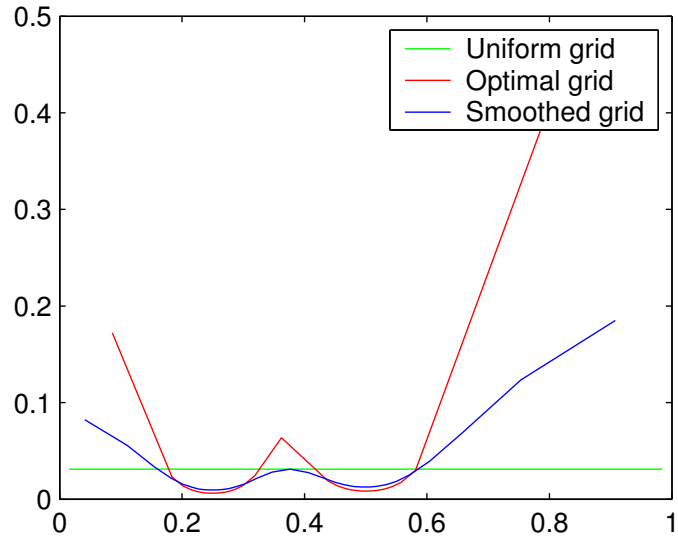
$$E(\{\tilde{x}_i\}) = \sqrt{F_{ex}(\{\tilde{x}_i\})}.$$

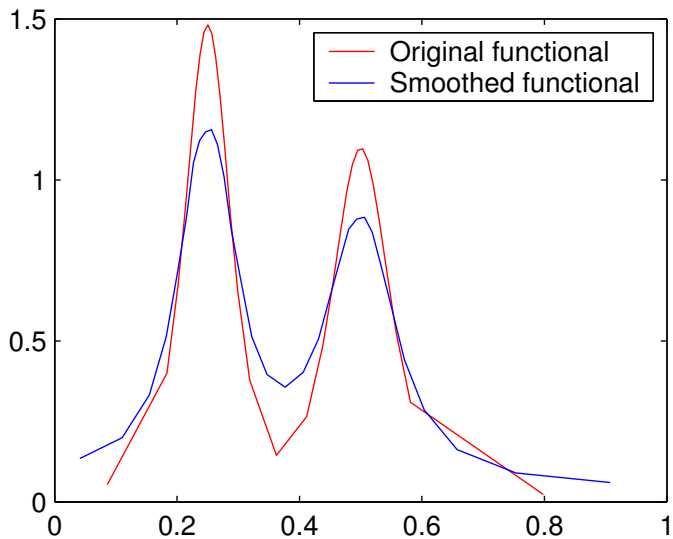
Accuracy analysis of smoothed error functional.

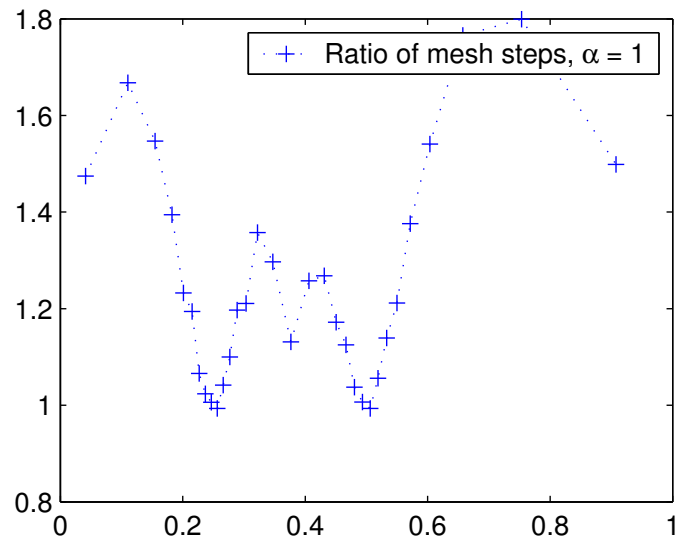
M	$E(\{\tilde{x}_i^{exact}\})$	$E(\{\tilde{x}_i^{approx}\})$	$E(\{\tilde{x}_i^{smooth}\})$
16	1.01e-2	1.19e-2	1.75e-2
32	4.99e-3	5.18e-3	6.28e-3
64	2.48e-3	2.50e-3	2.70e-3
128	1.24e-3	1.24e-3	1.28e-3

Grid smoothing (3/3)









Minimization vs equidistribution

The minimum of the error functional is achived when

$$\left[\frac{\delta u^n}{\delta x}\right]^2_{i-1/2}\left[h^n_{i-1/2}\right]^3 = \left[\frac{\delta u^n}{\delta x}\right]^2_{i+1/2}\left[h^n_{i+1/2}\right]^3$$

which may be rewritten as follows:

$$\omega_{i+1/2}(x^n_{i+1} - x^n_i) - \omega_{i-1/2}(x^n_i - x^n_{i-1}) = 0.$$

It is a discretization of the non-linear elliptic equation

$$\frac{\partial}{\partial \xi}\left(\omega(x^n)\frac{\partial x^n}{\partial \xi}\right) = 0, \qquad x(0) = 0, \quad x(1) = 1,$$

on a reference grid with the coefficient $\omega(x)$ given by

$$\omega(x) = \left|\frac{\partial u}{\partial x}\right|^{2/3}.$$

Minimization vs equidistribution

- The minimization of the error functional F_{ap} is equivalent to solving the nonlinear elliptic equation only in the 1D case.
- In higher dimensions, we regularize the error functional in order to enforce the shape-regularity of quadrilateral and hexahedral mesh cells.
- To summarize, we have the following sequence of modifications:

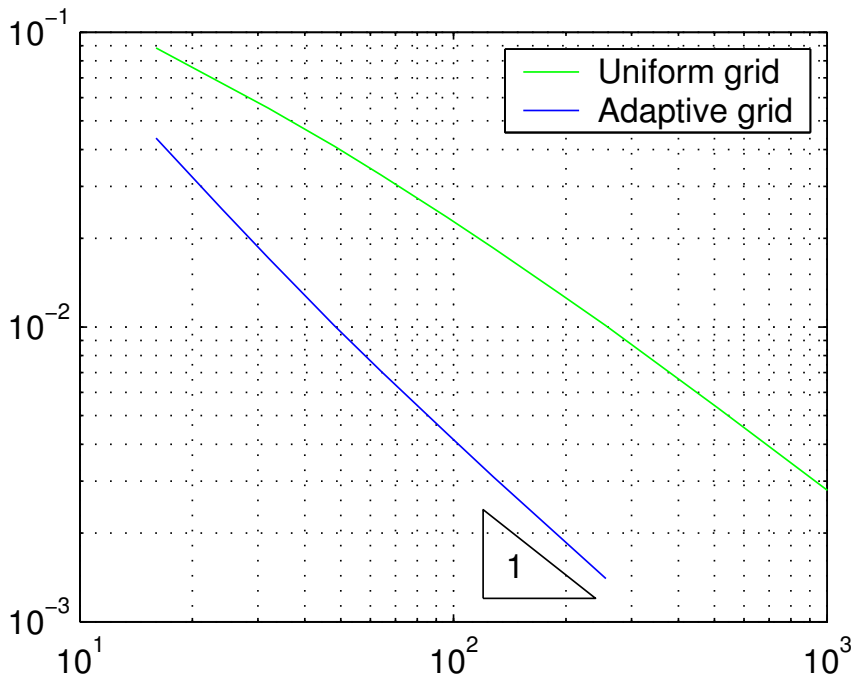
$$F_{ex} \to F_{ap} \to F_{sm} \to F_{reg}.$$

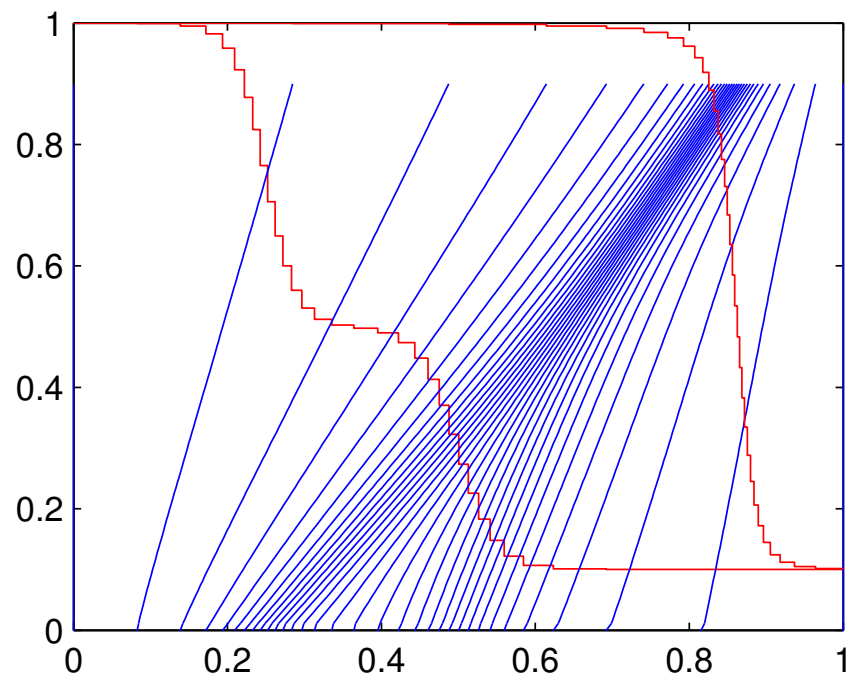
Example of 2D nonlinear waves.

Viscous Burgers equation

Let $T = 0.9$, $\varepsilon = 0.005$ and

$$E(\{x_i^N\}) = \left[\sum_{i=0}^{M} \int_{x_i^N}^{x_{i+1}^N} (u(x,T) - \bar{\tilde{u}}_{i+1/2}^N)^2 \mathrm{d}x \right]^{1/2} .$$

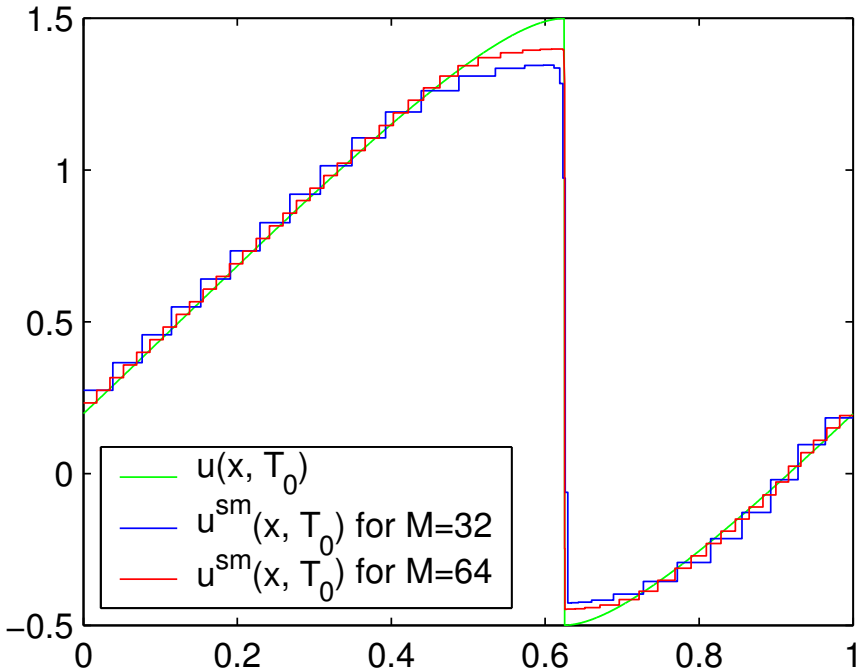


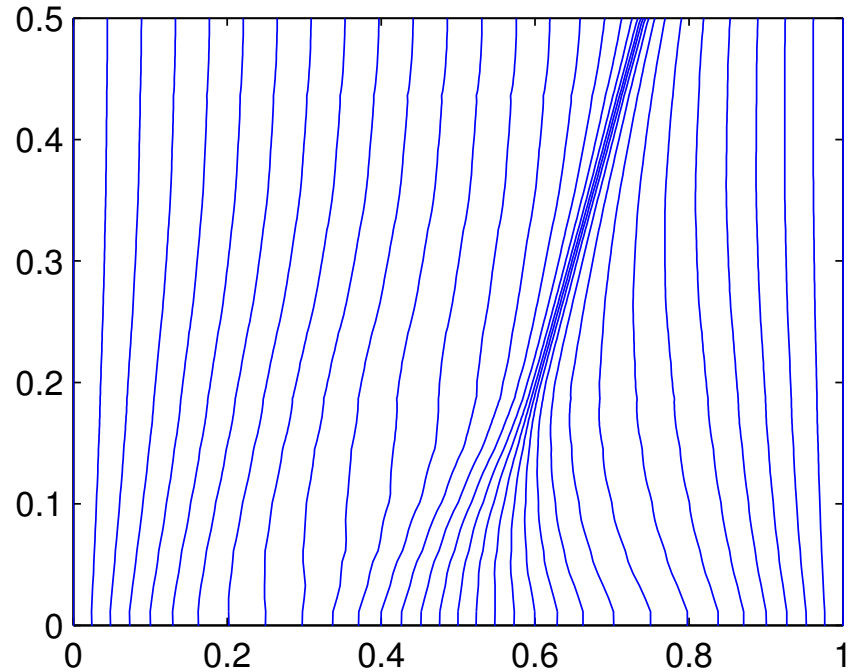


Inviscous Burgers equation

Let $T = 0.5$, $\varepsilon = 0$ and the initial condition be the periodic function

$$u_0(x) = 0.5 + \sin(2\pi x).$$





Conclusions and future work

- The error introduced by the time integration can be ignored even for lower order schemes.
- The error introduced by the numerical interpolation can be ignored when the interpolation operator is one order more accurate than the discretization.
- Necessity of a grid smoothing has been observed in many 1D and 2D numerical experiments.

Future work:

- Extend the error analysis to equations of 1D gas dynamics.
- Complete the error analysis for regularized functionals in 2D.